CUBES IN PRODUCTS OF TERMS IN ARITHMETIC PROGRESSION

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Abstract. Euler proved that the product of four positive integers in arithmetic progression is not a square. Győry, using a result of Darmon and Merel showed that the product of three coprime positive integers in arithmetic progression cannot be an l-th power for $l \geq 3$. There is an extensive literature on longer arithmetic progressions such that the product of the terms is an (almost) power. In this paper we prove that the product of k coprime integers in arithmetic progression cannot be a cube when $2 < k < 39$. We prove a similar result for almost cubes.

1. Introduction

In this paper we consider the problem of almost cubes in arithmetic progressions. This problem is closely related to the Diophantine equation

$$n(n+d)\dots(n+(k-1)d) = by^l \tag{1}$$

in positive integers n, d, k, b, y, l with $l \geq 2$, $k \geq 2$, $\gcd(n,d) = 1$, $P(b) \leq k$, where for $u \in \mathbb{Z}$ with $|u| > 1$, $P(u)$ denotes the greatest prime factor of u, and $P(\pm 1) = 1$.

This equation has a long history, with an extensive literature. Here (beside the papers referred to later on in the introduction) we only refer to the research and survey papers [3], [10], [11], [14], [16], [18], [19], [20], [23], [25], [26], [28], [29], [31], [32], [33], [34], [35], [36], [37], [38], [40], [41], and the references given there.

Now we concentrate on results where all solutions of (1) have been determined, under some assumptions for the unknowns. We start with results concerning squares, so in this paragraph we assume that $l = 2$. Already Euler proved that in this case equation (1) has no solutions with $k = 4$ and $b = 1$ (see [7] pp. 440 and 635). Obláth [21] extended this result to the case $k = 5$. Erdős [8] and Rigge [22] independently

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proved that equation (1) has no solutions with $b = d = 1$. Saradha and Shorey [27] proved that (1) has no solutions with $b = 1$, $k \geq 4$, provided that d is a power of a prime number. Later, Laishram and Shorey [19] extended this result to the case where either $d \leq 10^{10}$, or d has at most six prime divisors. Finally, most importantly from the viewpoint of the present paper, Hirata-Kohno, Laishram, Shorey and Tijdeman [17] completely solved (1) with $3 \leq k < 110$ for $b = 1$. Combining their result with Tengely [39] all solutions of (1) with $3 \leq k \leq 100$, $P(b) < k$ are determined.

Now assume for this paragraph that $l \geq 3$. Erdős and Selfridge [9] proved the celebrated result that equation (1) has no solutions if $b = d = 1$. In the general case $P(b) \leq k$ but still with $d = 1$, Saradha [24] for $k \geq 4$ and Győry [12] using a result of Darmon and Merel [6] for $k = 2, 3$ proved that (1) has no solutions with $P(y) > k$. For general d, Győry [13] showed that equation (1) has no solutions with $k = 3$, provided that $P(b) \leq 2$. Later, this result has been extended to the case $k < 12$ under certain assumptions on $P(b)$, see Győry, Hajdu, Saradha [15] for $k < 6$ and Bennett, Bruin, Győry, Hajdu [1] for $k < 12$.

In this paper we consider the problem for cubes, that is equation (1) with $l = 3$. The reason for doing so is that in this case one can use special methods which are not available for general exponents. We solve equation (1) nearly up to $k = 40$. In the proofs of our results we combine the approach of [17] with results of Selmer [30] and some new ideas.

2. Notation and results

As we are interested in cubes in arithmetic progressions, we take $l = 3$ in (1). That is, we consider the Diophantine equation

$$n(n+d)\dots(n+(k-1)d) = by^3 \tag{2}$$

in integers n, d, k, b, y where $k \geq 3$, $d > 0$, $\gcd(n, d) = 1$, $P(b) \leq k$, $y \neq 0$. (Note that similarly as e.g. in [1] we allow $n < 0$, as well.)

In the standard way, by our assumptions we can write

$$n + id = a_i x_i^3 \quad (i = 0, 1, \dots, k-1) \tag{3}$$

with $P(a_i) \leq k$, a_i is cube-free. Note that (3) also means that in fact $n + id$ $(i = 0, 1, \dots, k-1)$ is an arithmetic progression of almost cubes.

In case of $b = 1$ we prove the following result.

Theorem 2.1. *Suppose that (n, d, k, y) is a solution to equation (2) with $b = 1$ and $k < 39$. Then we have*

$$(n, d, k, y) = (-4, 3, 3, 2), (-2, 3, 3, -2), (-9, 5, 4, 6), (-6, 5, 4, 6).$$

We shall deduce Theorem 2.1 from the following theorem.

Theorem 2.2. *Suppose that* (n, d, k, b, y) *is a solution to equation* (2) *with* $k < 32$ *and that* $P(b) < k$ *for* $k = 3$ *and for* $k \geq 13$. *Then* (n, d, k) *belongs to the following list:*

$$(n, 1, k) \text{ with } -30 \leq n \leq -4 \text{ or } 1 \leq n \leq 5,$$
$$(n, 2, k) \text{ with } -29 \leq n \leq -3,$$
$$(-10, 3, 7), (-8, 3, 7), (-8, 3, 5), (-4, 3, 5), (-4, 3, 3), (-2, 3, 3),$$
$$(-9, 5, 4), (-6, 5, 4), (-16, 7, 5), (-12, 7, 5).$$

Note that the above theorem for $k < 12$ follows from Theorem 1.1 of Bennett, Bruin, Győry, Hajdu [1] under the condition $P(b) \leq P_k$ with $P_3 = 2$, $P_4 = P_5 = 3$, $P_6 = P_7 = P_8 = P_9 = P_{10} = P_{11} = 5$.

3. Lemmas and auxiliary results

We need some results of Selmer [30] on cubic equations.

Lemma 3.1. *The equations*

$$x^3 + y^3 = cz^3, \quad c \in \{1, 2, 4, 5, 10, 25, 45, 60, 100, 150, 225, 300\},$$
$$ax^3 + by^3 = z^3, \ (a, b) \in \{(2, 9), (4, 9), (4, 25), (4, 45), (12, 25)\}$$

have no solution in non-zero integers x, y, z.

As a lot of work will be done modulo 13, the following lemma will be very useful. Before giving that, we need to introduce a new notation. For $u, v, m \in \mathbb{Z}$, $m > 1$ by $u \overset{c}{\equiv} v \pmod{m}$ we mean that $uw^3 \equiv v \pmod{m}$ holds with some integer w where $\gcd(m, w) = 1$. We shall use this notation throughout the paper, without any further reference.

Lemma 3.2. *Let* n, d *be integers. Suppose that for five* $i \in \{0, 1, ..., 12\}$ *we have* $n + id \overset{c}{\equiv} 1 \pmod{13}$. *Then* $13 \mid d$, *and* $n + id \overset{c}{\equiv} 1 \pmod{13}$ *for all* $i = 0, 1, \ldots, 12$.

Proof. Suppose that $13 \nmid d$. Then there is an integer r such that $n \equiv rd \pmod{13}$. Consequently, $n + id \equiv (r + i)d \pmod{13}$. A simple calculation yields that the cubic residues of the numbers $(r + i)d$ ($i = 0, 1, \ldots, 12$) modulo 13 are given by a cyclic permutation of one of the sequences

$$0, 1, 2, 2, 4, 1, 4, 4, 1, 4, 2, 2, 1,$$
$$0, 2, 4, 4, 1, 2, 1, 1, 2, 1, 4, 4, 2,$$
$$0, 4, 1, 1, 2, 4, 2, 2, 4, 2, 1, 1, 4.$$

Thus the statement follows. $\square$

Lemma 3.3. *Let $\alpha = \sqrt[3]{2}$ and $\beta = \sqrt[3]{3}$. Put $K = \mathbb{Q}(\alpha)$ and $L = \mathbb{Q}(\beta)$. Then the only solution of the equation*

$$\mathcal{C}_1: \quad X^3 - (\alpha+1)X^2 + (\alpha+1)X - \alpha = (-3\alpha+6)Y^3$$

in $X \in \mathbb{Q}$ and $Y \in K$ is $(X,Y) = (2,1)$. Further, the equation

$$\mathcal{C}_2: \quad 4X^3 - (4\beta+2)X^2 + (2\beta+1)X - \beta = (-3\beta+3)Y^3$$

has the single solution $(X,Y) = (1,1)$ in $X \in \mathbb{Q}$ and $Y \in L$.

Proof. Using the point $(2,1)$ we can transform the genus 1 curve $\mathcal{C}_1$ to Weierstrass form

$$E_1: \quad y^2 + (\alpha^2+\alpha)y = x^3 + (26\alpha^2 - 5\alpha - 37).$$

We have $E_1(K) \simeq \mathbb{Z}$ as an Abelian group and $(x,y) = (-\alpha^2 - \alpha + 3, -\alpha^2 - 3\alpha + 4)$ is a non-torsion point on this curve. Applying elliptic Chabauty (cf. [4], [5]), in particular the procedure "Chabauty" of MAGMA (see [2]) with $p = 5$, we obtain that the only point on $\mathcal{C}_1$ with $X \in \mathbb{Q}$ is $(2,1)$.

Now we turn to the second equation $\mathcal{C}_2$. We can transform this equation to an elliptic one using its point $(1,1)$. We get

$$E_2: \quad y^2 = x^3 + \beta^2 x^2 + \beta x + (41\beta^2 - 58\beta - 4).$$

We find that $E_2(L) \simeq \mathbb{Z}$ and $(x,y) = (4\beta - 2, -2\beta^2 + \beta + 12)$ is a non-torsion point on E_2. Applying elliptic Chabauty (as above) with $p = 11$, we get that the only point on $\mathcal{C}_2$ with $X \in \mathbb{Q}$ is $(1,1)$. □

4. Proofs

In this section we provide the proofs of our results. As Theorem 2.1 follows from Theorem 2.2 by a simple inductive argument, first we give the proof of the latter result.

Proof of Theorem 2.2. As we mentioned, for $k = 3,4$ the statement follows from Theorem 1.1 of [1]. We give the proof of the theorem for each $k > 4$ separately. However, it turns out that the statement for every

$$k \in \{6,8,9,10,12,13,15,16,17,19,21,22,23,25,26,27,28,29,31\}$$

is a simple consequence of the result obtained for some smaller value of k. Indeed, for any such k let p_k denote the largest prime with $p_k < k$. Observe that in case of $k \le 13$ $P(a_0a_1\dots a_{p_k-1}) \le p_k$ holds, and for $k > 13$ we have $P(a_0a_1\dots a_{p_k}) < p_k + 1$. Hence, noting that we assume $P(b) \le k$ for $3 < k \le 11$ and $P(b) < k$ otherwise, the theorem follows inductively from the case of p_k-term products and p_k+1-term products, respectively. Hence in the sequel we deal only with the remaining values of k.

We discuss the cases $k = 5, 7$ separately from the others, because here the method of our proof is different from the one used for the larger values of k. Here already the "brute force" method suffices to treat all the cases. Furthermore, here there are some cases left which apparently cannot be handled with any classical tool. At these points we shall make use of the elliptic Chabauty method (see [4], [5]).

The case $k = 5$. In this case a very simple algorithm works already. Note that in view of Theorem 1.1 of [1], by symmetry it is sufficient to assume that $5 \mid a_2a_3$. We look at all the possible distribution of the primes $2, 3, 5$ in the coefficients a_i $(i = 0, \dots, 4)$ one-by-one. Checking the progressions modulo 7 and 9 (using that x^3 is congruent to ± 1 or 0 both $\pmod 7$ and $\pmod 9$ for any integer x), almost all possibilities can be excluded. For example,

$$(a_0, a_1, a_2, a_3, a_4) = (1, 1, 1, 10, 1)$$

is impossible modulo 7, while

$$(a_0, a_1, a_2, a_3, a_4) = (1, 1, 15, 1, 1)$$

is impossible modulo 9. (Note that the first choice of the a_i cannot be excluded modulo 9, and the second one cannot be excluded modulo 7.)

In case of the remaining possibilities, taking the linear combinations of three appropriately chosen terms of the arithmetic progression on the left hand side of (2) we get all solutions by Lemma 3.1. For example,

$$(a_0, a_1, a_2, a_3, a_4) = (2, 3, 4, 5, 6)$$

obviously survives the above tests modulo 7 and modulo 9. However, in this case using the identity $4(n+d)-3n = n+4d$, Lemma 3.1 implies that the only corresponding solution is given by $n = 2$ and $d = 1$.

After that we are left with the single possibility

$$(a_0, a_1, a_2, a_3, a_4) = (2, 9, 2, 5, 12).$$

Then we have

$$x_0^3 + x_2^3 = 9x_1^3 \text{ and } x_0^3 - 2x_2^3 = -6x_4^3. \tag{4}$$

Factorizing the first equation of (4), a simple consideration yields that $x_0^2 - x_0x_2 + x_2^2 = 3u^3$ holds for some integer u. Put $K = \mathbb{Q}(\alpha)$ with $\alpha = \sqrt[3]{2}$. Note that the ring O_K of integers of K is a unique factorization domain, $\alpha - 1$ is a fundamental unit and $1, \alpha, \alpha^2$ is an integral basis of K, and $3 = (\alpha - 1)(\alpha + 1)^3$, where $\alpha + 1$ is a prime in O_K. A simple calculation shows that $x_0 - \alpha x_2$ and $x_0^2 + \alpha x_0x_2 + \alpha^2 x_2^2$ can have only the prime divisors α and $\alpha+1$ in common. Hence checking the field norm of $x_0-\alpha x_2$, by the second equation of (4) we get that

$$x_0-\alpha x_2=(\alpha-1)^{\varepsilon}(\alpha^2+\alpha)y^3$$

with $y\in O_K$ and $\varepsilon\in\{0,1,2\}$. Expanding the right hand side, we deduce that $\varepsilon=0,2$ yields $3\mid x_0$, which is a contradiction. Thus we get that $\varepsilon=1$, and we obtain the equation

$$(x_0-\alpha x_2)(x_0^2-x_0x_2+x_2^2)=(-3\alpha+6)z^3$$

for some $z\in O_K$. Hence after dividing both sides of this equation by x_2^3, the theorem follows from Lemma 3.3 in this case.

The case $k=7$. In this case by a similar algorithm as for $k=5$, we get that the only possible exceptions for the theorem are given by

$$(a_0,a_1,a_2,a_3,a_4,a_5,a_6)=(4,5,6,7,1,9,10),(10,9,1,7,6,5,4).$$

By symmetry it is sufficient to deal with the first case. Then we have

$$x_1^3+8x_6^3=9x_5^3 \text{ and } x_6^3-3x_1^3=-2x_0^3. \tag{5}$$

Factorizing the first equation of (5), just as in case of $k=5$, a simple consideration gives that $4x_6^2-2x_1x_6+x_1^2=3u^3$ holds for some integer u. Let $L=\mathbb{Q}(\beta)$ with $\beta=\sqrt[3]{3}$. As is well-known, the ring O_L of integers of L is a unique factorization domain, $2-\beta^2$ is a fundamental unit and $1,\beta,\beta^2$ is an integral basis of L. Further, $2=(\beta-1)(\beta^2+\beta+1)$, where $\beta-1$ and $\beta^2+\beta+1$ are primes in O_L, with field norms 2 and 4, respectively. A simple calculation yields that $x_6-\beta x_1$ and $x_6^2+\beta x_1x_6+\beta^2x_1^2$ are relatively prime in O_L. Moreover, as $\gcd(n,d)=1$ and x_4 is even, x_0 should be odd. Hence as the field norm of $\beta^2+\beta+1$ is 4, checking the field norm of $x_6-\beta x_1$, the second equation of (5) yields

$$x_6-\beta x_1=(2-\beta^2)^{\varepsilon}(1-\beta)y^3$$

for some $y\in O_L$ and $\varepsilon\in\{0,1,2\}$. Expanding the right hand side, a simple computation shows that $\varepsilon=1,2$ yields $3\mid x_6$, which is a contradiction. Thus we get that $\varepsilon=0$, and we obtain the equation

$$(x_6-\beta x_1)(4x_6^2-2x_1x_6+x_1^2)=(-3\beta+3)z^3$$

for some $z\in O_L$. Hence after dividing both sides of this equation by x_1^3, the theorem follows from Lemma 3.3 in this case.

Description of the general method. So far we have considered all the possible distributions of the prime factors $\leq k$ among the coefficients a_i. However, for larger values of k this approach would be extremely time consuming, so more efficient procedure is needed. Note that it is not worth to use this procedure for $k \leq 7$. Now we outline the main ideas of our approach. Note that in [17] a similar method is applied.

In the main argument of our method we assume $3 \nmid d$, $7 \nmid d$ and $13 \nmid d$. We explain this case first.

The case $\gcd(3 \cdot 7 \cdot 13, d) = 1$**.** Suppose we have a solution to equation (2) with $k \geq 11$ and $\gcd(3 \cdot 7, d) = 1$. Then there exist integers r_7 and r_9 such that $n \equiv r_7 d \pmod 7$ and $n \equiv r_9 d \pmod 9$. Further, we can choose the integers r_7 and r_9 to be equal; put $r := r_7 = r_9$. Then $n + id \equiv (r+i)d \pmod q$ holds for $q \in \{7, 9\}$ and $i = 0, 1 \ldots, k-1$. In particular, we have $r + i \overset{c}{\equiv} a_i s_q \pmod q$, where $q \in \{7, 9\}$ and s_q is the inverse of d modulo q.

We may assume that the values $r + i$ $(i = 0, \ldots, k-1)$ belong to the torus $-31 \leq r + i < 32$ modulo 63. Since we are interested only in the residues of every $r + i$ modulo 7 and 9 up to cubes, we make a table containing these data. In fact we make a full table, but here we indicate only part of the torus, with $0 \leq r + i < 11$.

$r+i$	0	1	2	3	4	5	6	7	8	9	10
mod 7	0	1	2	4	4	2	1	0	1	2	4
mod 9	0	1	2	3	4	4	3	2	1	0	1

In the first row of the table we give the values of $r + i$, while in the second and third rows the corresponding residues of $r+i$ modulo 7 and modulo 9 up to cubes, respectively. Note that the classes of the relation $\overset{c}{\equiv}$ are represented by $0, 1, 2, 4$ modulo 7, and by $0, 1, 2, 3, 4$ modulo 9.

Let $a_{i_1}, \ldots, a_{i_t}$ be the coefficients in (3) which do not have prime divisors greater than 2. Put

$$E = \{(u_{i_j}, v_{i_j}) : r+i_j \overset{c}{\equiv} u_{i_j} \pmod 7, r+i_j \overset{c}{\equiv} v_{i_j} \pmod 9, 1 \leq j \leq t\}$$

and observe that E is contained in one of the sets

$$E_1 := \{(1,1), (2,2), (4,4)\},\ E_2 := \{(1,2), (2,4), (4,1)\},$$
$$E_3 := \{(2,1), (4,2), (1,4)\}.$$

Based upon this observation, we introduce certain filtering procedures. We start with explaining how these procedures work, then we illustrate the whole method on an example.

In what follows, we always assume k and r to be fixed. Further, in our method we apply these procedures in the order below.

Class cover. Let $u_i \overset{c}{\equiv} r+i \pmod 7$ and $v_i \overset{c}{\equiv} r+i \pmod 9$ ($i = 0,1,\dots,k-1$). For $l=1,2,3$ put

$$C_l = \{i \ : \ (u_i, v_i) \in E_l, \ i = 0,1,\dots,k-1\}.$$

Check whether the sets $C_1 \cup C_2$, $C_1 \cup C_3$, $C_2 \cup C_3$ can be covered by the multiples of the primes p with $p < k$, $p \neq 2,3,7$. If this is not possible for $C_{l_1} \cup C_{l_2}$, then we know that $E \subseteq E_{l_3}$ is impossible. (Here $\{l_1, l_2, l_3\} = \{1,2,3\}$). In the opposite case the pairs (u_i, v_i) belonging to the a_i with $P(a_i) \leq 2$ may be contained in E_{l_3} and we need to execute the following tests.

The forthcoming procedures are applied separately for each case where $E \subseteq E_l$ remains possible for some l. From this point on we also assume that the odd prime factors of the a_i are fixed.

Parity. Suppose that $E \subseteq E_l$ for some $l \in \{1,2,3\}$ remains possible after the test *Class cover.* Define the sets

$$I_e = \{(u_i, v_i) \in E_l \ : \ r+i \text{ is even}, \ P(a_i) \leq 2\},$$

$$I_o = \{(u_i, v_i) \in E_l \ : \ r+i \text{ is odd}, \ P(a_i) \leq 2\}.$$

As the only odd power of 2 is 1, $\min(|I_e|, |I_o|) \leq 1$ must be valid. If this condition does not hold, then the corresponding case can be excluded.

Test modulo 13. Assume that $E \subseteq E_l$ with fixed $l \in \{1,2,3\}$. Further, suppose that based upon the previous tests we can decide whether a_i can be even for the even or the odd values of i. For $t = 0,1,2$ put

$$U_t = \{i \ : \ a_i = \pm 2^t, \ i \in \{0,1,\dots,k-1\}\}$$

and let

$$U_3 = \{i \ : \ a_i = \pm 5^\gamma, \ i \in \{0,1,\dots,k-1\}, \gamma \in \{0,1,2\}\}.$$

Assume that $13 \mid n + i_0 d$ with some i_0. Then as $13 \nmid d$ and $5 \overset{c}{\equiv} 1 \pmod{13}$, if $i,j \in U_t$ for any $t \in \{0,1,2,3\}$, then $i - i_0 \overset{c}{\equiv} j - i_0 \pmod{13}$ must hold. If this condition is not valid, then the corresponding fixed choice of the prime factors of the a_i can be excluded. Further, if $i - i_0 \overset{c}{\equiv} j - i_0 \pmod{13}$ holds for some $i \in U_{t_1}$, $j \in U_{t_2}$ with $0 \leq t_1 < t_2 \leq 2$, then we get the same conclusion.

Test modulo 7. Assume again that $E \subseteq E_l$ with fixed $l \in \{1,2,3\}$. Check whether the actual distribution of the prime divisors of the a_i

yields that for some i with $7 \nmid n+id$, both $a_i = \pm t$ and $|r+i| = t$ hold for some positive integer t with $7 \nmid t$. Then

$$t \stackrel{c}{\equiv} n+id \stackrel{c}{\equiv} (r+i)d \stackrel{c}{\equiv} td \pmod 7$$

implies that $d \stackrel{c}{\equiv} 1 \pmod 7$. Now consider the actual distribution of the prime factors of the coefficients a_i ($i = 0, 1, \dots, k-1$). If in any a_i we know the exponents of all primes with one exception, and this exceptional prime p has $p \stackrel{c}{\equiv} 2, 3, 4, 5 \pmod 7$, then we can fix the exponent of p using the above information for d. As an example, assume that $7 \mid n$, and $a_1 = \pm 5^\gamma$ with $\gamma \in \{0, 1, 2\}$. Then $d \stackrel{c}{\equiv} 1 \pmod 7$ immediately implies $\gamma = 0$.

We remark that we used this procedure for $0 \geq r \geq -k+1$, when in almost all cases it turned out that a_i is even for $r+i$ even. Further, we typically could prove that with $|r+i| = 1$ or 2 we have $a_i = \pm 1$ or ± 2, respectively, to conclude $d \stackrel{c}{\equiv} 1 \pmod 7$. The test is typically effective in case when r is "around" $-k/2$. The reason for this is that then in the sequence $r, r+1, \dots, -1, 0, 1, \dots, k-r-2, k-r-1$ several powers of 2 occur.

Induction. For fixed distribution of the prime divisors of the coefficients a_i, search for arithmetic sub-progressions of length l with $l \in \{3, 5, 7\}$ such that for the product Π of the terms of the sub-progression $P(\Pi) \leq L_l$ holds, with $L_3 = 2$, $L_5 = 5$, $L_7 = 7$. If there is such a sub-progression, then in view of Theorem 1.1 of [1], all such solutions can be determined.

An example. Now we illustrate how the above procedures work. For this purpose, take $k = 24$ and $r = -8$. Then, using the previous notation, we work with the following stripe (with $i \in \{0, 1, \dots, 23\}$):

$r+i$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
mod 7	1	0	1	2	4	4	2	1	0	1	2	4	4	2	1	0	1	2	4	4	2	1	0	1 .
mod 9	1	2	3	4	4	3	2	1	0	1	2	3	4	4	3	2	1	0	1	2	3	4	4	3

In the procedure *Class cover* we get the following classes:

$$C_1 = \{0, 4, 6, 7, 9, 10, 12, 16\}, \quad C_2 = \{3, 13, 18\}, \quad C_3 = \{19, 21\}.$$

For $p = 5, 11, 13, 17, 19, 23$ put

$$m_p = |\{i \ : \ i \in C_1 \cup C_2, \ p \mid n+id\}|,$$

respectively. Using the condition $\gcd(n, d) = 1$, one can easily check that

$$m_5 \leq 3, \ m_{11} \leq 2, \ m_{13} \leq 2, \ m_{17} \leq 1, \ m_{19} \leq 1, \ m_{23} \leq 1.$$

Hence, as $|C_1 \cup C_2| = 11$, we get that $E \subseteq E_3$ cannot be valid in this case. By a similar (but more sophisticated) calculation one gets that

$E \subseteq E_2$ is also impossible. So after the procedure *Class cover* only the case $E \subseteq E_1$ remains.

From this point on, the odd prime divisors of the coefficients a_i are fixed, and we look at each case one-by-one. We consider two possible distributions of the prime divisors. Note that here it is better to write $p \mid n + id$ rather than $p \mid a_i$. Indeed, if p divides $n + id$ exactly on the third power say, then of course $p \mid n + id$, however, $p \nmid a_i$. Further, certainly $p \mid n + id$ implies $p \mid n + jd$ whenever $i \equiv j \pmod p$.

Suppose first that we have

$$3 \mid n+2d,\ 5 \mid n+d,\ 7 \mid n+d,\ 11 \mid n+7d,\ 13 \mid n+7d,$$

$$17 \mid n+3d,\ 19 \mid n,\ 23 \mid n+13d.$$

Then by a simple consideration we get that in *Test modulo* 13 either

$$4 \in U_1 \text{ and } 10 \in U_2,$$

or

$$10 \in U_1 \text{ and } 4 \in U_2.$$

In the first case, using $13 \mid n + 7d$ we get

$$-3d \overset{c}{\equiv} 2 \pmod{13} \text{ and } 3d \overset{c}{\equiv} 4 \pmod{13},$$

which by $-3d \overset{c}{\equiv} 3d \pmod{13}$ yields a contradiction. In the second case we get a contradiction in a similar manner.

Consider now the case where

$$3 \mid n+2d,\ 5 \mid n+d,\ 7 \mid n+d,\ 11 \mid n+7d,\ 13 \mid n+8d,$$

$$17 \mid n+3d,\ 19 \mid n,\ 23 \mid n+13d.$$

Then *Test modulo* 13 does not work. However, using the strategy explained in *Test modulo* 7, we can fix several a_i. Indeed, one can easily check that if a_i is even then i is even, which yields $a_9 = \pm 1$. This immediately gives $d \overset{c}{\equiv} 1 \pmod 7$. Further, we have $a_7 = \pm 11^{\varepsilon_7}$ with $\varepsilon_7 \in \{0, 1, 2\}$. Hence we get that

$$\pm 11^{\varepsilon_7} \overset{c}{\equiv} n + 7d \overset{c}{\equiv} d \overset{c}{\equiv} 1 \pmod 7.$$

This gives $\varepsilon_7 = 0$, thus $a_7 = \pm 1$. However, then $P(a_4 a_7 a_{10}) \leq 2$, and the theorem follows by *Induction* in this case. Note that in some cases it happens that already at *Test modulo* 7 we get a contradiction, implied by the fact that there is no valid choice for an a_i for some i.

The case $\gcd(3\cdot 7\cdot 13, d) \neq 1$**.** In this case in our argument we shall use the fact that almost the half of the coefficients are odd. With a slight abuse of notation, when $k > 11$ we shall assume that the coefficients $a_1, a_3, \dots, a_{k-1}$ are odd, and the other coefficients are given either by $a_0, a_2, \dots, a_{k-2}$ or by $a_2, a_4, \dots, a_k$. Note that in view of $\gcd(n, d) = 1$ this can be done without loss of generality. We shall use this notation in the corresponding parts of our arguments without any further reference.

Now we continue the proof, considering the remaining cases $k \geq 11$.

The case $k = 11$**.** In the case where $\gcd(3 \cdot 7, d) = 1$, already the procedures *Class cover*, *Test modulo* 7 and *Induction* suffice. Hence we may suppose that $\gcd(3 \cdot 7, d) > 1$.

Assume first that $7 \mid d$. Observe that then one of $P(a_0a_1 \dots a_4) \leq 5$, $P(a_5a_6 \dots a_9) \leq 5$ is always valid. Hence the statement follows from the part $k = 5$ in this case.

Suppose next that $3 \mid d$. Observe that if $11 \nmid a_4a_5a_6$ then one of $P(a_0a_1 \dots a_6) \leq 7$, $P(a_4a_5 \dots a_{10}) \leq 7$ is valid. Hence by induction and symmetry we may assume that $11 \mid a_5a_6$. Assume first that $11 \mid a_6$. If $7 \mid a_0a_6$ then we have $P(a_1a_2a_3a_4a_5) \leq 5$. Further, in case of $7 \mid a_5$ we have $P(a_0a_1a_2a_3a_4) \leq 5$. Thus by induction we may suppose that $7 \mid a_1a_2a_3a_4$. If $7 \mid a_1a_2a_4$ then in case of $5 \nmid n$ we have $P(a_0a_5a_{10}) \leq 2$, whence using the identity $n + (n + 10d) = 2(n + 5d)$ by Lemma 3.1 we get all solutions of the equation (2) in this case. Assume next that $7 \mid a_1a_2a_4$ and $5 \mid n$. Hence we deduce that one among $P(a_2a_3a_4) \leq 2$, $P(a_1a_4a_7) \leq 2$, $P(a_1a_2a_3) \leq 2$ is valid, and the statement follows in each case in a similar manner as above. If $7 \mid a_3$, then a simple calculation yields that one among $P(a_0a_1a_2) \leq 2$, $P(a_0a_4a_8) \leq 2$, $P(a_1a_4a_7) \leq 2$ is always valid, and we are done. Finally, assume that $11 \mid a_5$. Then by symmetry we may suppose that $7 \mid a_0a_1a_4a_5$. If $7 \mid a_4a_5$ then $P(a_6a_7a_8a_9a_{10}) \leq 5$, and the statement follows by induction. If $7 \mid a_0$ then we have $P(a_2a_4a_6a_8a_{10}) \leq 5$, and we are done again. In case of $7 \mid a_1$ one among $P(a_0a_2a_4) \leq 2$, $P(a_2a_3a_4) \leq 2$, $P(a_0a_3a_6) \leq 2$ always holds, hence the theorem follows.

The case $k = 14$**.** Note that without loss of generality we may assume that $13 \mid a_i$ with $3 \leq i \leq 10$, otherwise the statement follows by induction from the case $k = 11$. Then, in particular we have $13 \nmid d$.

In case of $\gcd(3\cdot 7\cdot 13, d) = 1$ by the help of the procedures described in the previous section all cases can be excluded. Assume now that $\gcd(3 \cdot 7 \cdot 13, d) > 1$ (but recall that $13 \nmid d$).

Suppose first that $7 \mid d$. Among the odd coefficients $a_1, a_3, \dots, a_{13}$ there are at most three multiples of 3, two multiples of 5 and one multiple of 11. As $13 \stackrel{c}{\equiv} 1 \pmod 7$, this shows that at least for one of these a_i-s we have $a_i \stackrel{c}{\equiv} 1 \pmod 7$. Hence $a_i \stackrel{c}{\equiv} 1 \pmod 7$ for every $i = 1, 3, \dots, 13$. Further, as none of $3, 5, 11$ is a cube modulo 7, we deduce that if i is odd, then either $\gcd(3 \cdot 5 \cdot 11, a_i) = 1$ or a_i must be divisible by at least two out of $3, 5, 11$. Noting that $13 \nmid d$, by Lemma 3.2 there can be at most four $a_i = \pm 1$ among $a_1, a_3, \dots, a_{13}$. Moreover, $\gcd(n, d) = 1$ implies that $15 \mid a_i$ can be valid for at most one $i \in \{0, 1, \dots, k-1\}$. Hence among the coefficients with odd indices there is exactly one multiple of 11, exactly one multiple of 15, and exactly one multiple of 13, respectively. Moreover, the multiple of 11 in question is also divisible either by 3 or by 5. In view of the proof of Lemma 3.2 a simple calculation yields that the cubic residues of $a_1, a_3, \dots, a_{13}$ modulo 13 must be given by $1, 1, 4, 0, 4, 1, 1$, in this order. Looking at the spots where 4 occurs in this sequence, we get that either $3 \mid a_5, a_9$ or $5 \mid a_5, a_9$ is valid. However, this contradicts the assumption $\gcd(n, d) = 1$.

Assume now that $3 \mid d$, but $7 \nmid d$. Then among the odd coefficients $a_1, a_3, \dots, a_{13}$ there are at most two multiples of 5 and one multiple of 7, 11 and 13 each. Lemma 3.2 together with $5 \stackrel{c}{\equiv} 1 \pmod{13}$ yields that there must be exactly four odd i-s with $a_i \stackrel{c}{\equiv} 1 \pmod{13}$, and further, another odd i such that a_i is divisible by 13. Hence as above, the proof of Lemma 3.2 shows that the a_i-s with odd indices are $\stackrel{c}{\equiv} 1, 1, 4, 0, 4, 1, 1 \pmod{13}$, in this order. As the prime 11 should divide an a_i with odd i and $a_i \stackrel{c}{\equiv} 4 \pmod{13}$, this yields that $11 \mid a_5 a_9$. However, as above, this immediately yields that $P(a_0 a_2 \dots a_{12}) \le 7$ (or $P(a_2 a_4 \dots a_{14}) \le 7$), and the theorem follows by induction again.

The case $k = 18$. In case of $\gcd(3 \cdot 7 \cdot 13, d) = 1$ by the help of the procedures described in the previous section all cases can be excluded. So we may assume $\gcd(3 \cdot 7 \cdot 13, d) > 1$.

Suppose first that $7 \mid d$. Among $a_1, a_3, \dots, a_{17}$ there are at most three multiples of 3, two multiples of 5 and one multiple of 11, 13 and 17 each. Hence at least for one odd i we have $a_i = \pm 1$. Thus all of $a_1, a_3, \dots, a_{17}$ are $\stackrel{c}{\equiv} 1 \pmod 7$. Among the primes $3, 5, 11, 13, 17$ only $13 \stackrel{c}{\equiv} 1 \pmod 7$, so the other primes cannot occur alone. Hence we get that $a_i = \pm 1$ for at least five out of $a_1, a_3, \dots, a_{17}$. However, by Lemma 3.2 this is possible only if $13 \mid d$. In that case $a_i = \pm 1$ holds at least for six coefficients with i odd. Now a simple calculation shows that among them three are in arithmetic progression. This leads to an equation of the shape $X^3 + Y^3 = 2Z^3$, and by Lemma 3.1 the theorem is proved in this case.

Assume next that $13 \mid d$, but $7 \nmid d$. Among the odd coefficients $a_1, a_3, \dots a_{17}$ there are at most three multiples of 3, two multiples of 5 and 7 each, and one multiple of 11 and 17 each. Hence, by $5 \stackrel{c}{\equiv} 1 \pmod{13}$ there are at least two $a_i \stackrel{c}{\equiv} 1 \pmod{13}$, whence all $a_i \stackrel{c}{\equiv} 1 \pmod{13}$. As from this list only the prime 5 is a cube modulo 13, we get that at least four out of the above nine odd a_i-s are equal to ± 1. Recall that $7 \nmid d$ and observe that the cubic residues modulo 7 of a seven-term arithmetic progression with common difference not divisible by 7 is a cyclic permutation of one of the sequences

$$0, 1, 2, 4, 4, 2, 1, \quad 0, 2, 4, 1, 1, 4, 2, \quad 0, 4, 1, 2, 2, 1, 4.$$

Hence remembering that for four odd i we have $a_i = \pm 1$, we get that the cubic residues of $a_1, a_3, \dots, a_{17}$ modulo 7 are given by $1, 1, 4, 2, 0, 2, 4, 1, 1$, in this order. In particular, we have exactly one multiple of 7 among them. Further, looking at the spots where $0, 2$ and 4 occur, we deduce that at most two of the a_i-s with odd indices can be multiples of 3. Switching back to modulo 13, this yields that $a_i = \pm 1$ for at least five a_i-s. However, this contradicts Lemma 3.2.

Finally, assume that $3 \mid d$. In view of what we have proved already, we may further suppose that $\gcd(7 \cdot 13, d) = 1$. Among the odd coefficients $a_1, a_3, \dots, a_{17}$ there are at most two multiples of 5 and 7 each, and one multiple of 11, 13 and 17 each. Hence as $7 \nmid d$ and $13 \stackrel{c}{\equiv} 1 \pmod 7$, we get that the cubic residues modulo 7 of the coefficients a_i with odd i are given by one of the sequences

$$1, 0, 1, 2, 4, 4, 2, 1, 0, \quad 0, 1, 2, 4, 4, 2, 1, 0, 1, \quad 1, 1, 2, 4, 0, 4, 2, 1, 1.$$

In view of the places of the 2 and 4 values, from this we see that it is not possible to appropriately distribute the prime divisors $5, 7, 11$ for the a_i-s with odd indices. Hence the theorem follows also in this case.

The case $k = 20$. In case of $\gcd(3 \cdot 7 \cdot 13, d) = 1$ by the help of the procedures described in the previous section all cases can be excluded. Assume now that $\gcd(3 \cdot 7 \cdot 13, d) > 1$.

We start with the case $7 \mid d$. Then among the odd coefficients $a_1, a_3, \dots, a_{19}$ there are at most four multiples of 3, two multiples of 5, and one multiple of 11, 13, 17 and 19 each. As $13 \stackrel{c}{\equiv} 1 \pmod 7$, this yields that $a_i \stackrel{c}{\equiv} 1 \pmod 7$ for all i. Hence the primes $3, 5, 11, 17, 19$ must occur at least in pairs in the a_i-s with odd indices, which yields that at least five such coefficients are equal to ± 1. Thus Lemma 3.2 gives $13 \mid d$, whence $a_i \stackrel{c}{\equiv} 1 \pmod{13}$ for all i. Hence we deduce that the prime 5 may be only a third prime divisor of the a_i-s with odd indices, and so at least seven out of $a_1, a_3, \dots, a_{19}$ equal ± 1. However, then there are three such coefficients which belong to an arithmetic progression. Thus by Lemma 3.1 we get all solutions in this case.

Assume next that $13 \mid d$. Without loss of generality we may further suppose that $7 \nmid d$. Then among the odd coefficients $a_1, a_3, \dots, a_{19}$ there are at most four multiples of 3, two multiples of 5 and 7 each, and one multiple of 11, 17 and 19 each. As $5 \overset{c}{\equiv} 1 \pmod{13}$ this implies $a_i \overset{c}{\equiv} 1 \pmod{13}$ for all i, whence the primes $3, 7, 11, 17, 19$ should occur at least in pairs in the a_i-s with odd i. Hence at least four of these coefficients are equal to ± 1. By a similar argument as in case of $k = 18$, we get that the cubic residues of $a_1, a_3, \dots, a_{19}$ modulo 7 are given by one of the sequences

$$1, 0, 1, 2, 4, 4, 2, 1, 0, 1, \quad 1, 1, 4, 2, 0, 2, 4, 1, 1, 4, \quad 4, 1, 1, 4, 2, 0, 2, 4, 1, 1.$$

In view of the positions of the $0, 2$ and 4 values, we get that at most two corresponding terms can be divisible by 3 in the first case, which modulo 13 yields that the number of odd i-s with $a_i = \pm 1$ is at least five. This is a contradiction modulo 7. Further, in the last two cases at most three terms can be divisible by 3, and exactly one term is a multiple of 7. This yields modulo 13 that the number of odd i-s with $a_i = \pm 1$ is at least five, which is a contradiction modulo 7 again.

Finally, suppose that $3 \mid d$. We may assume that $\gcd(7 \cdot 13, d) = 1$. Then among the odd coefficients $a_1, a_3, \dots, a_{19}$ there are at most two multiples of 5 and 7 each, and one multiple of 11, 13, 17 and 19 each. Hence Lemma 3.2 yields that exactly four of these coefficients should be $\overset{c}{\equiv} 1 \pmod{13}$, and exactly one of them must be a multiple of 13. Further, exactly two other of the a_i-s with odd indices are multiples of 7, and these a_i-s are divisible by none of $11, 13, 17, 19$. So in view of the proof of Lemma 3.2 a simple calculation gives that the cubic residues of $a_1, a_3, \dots, a_{19}$ modulo 13 are given by one of the sequences

$$0, 2, 4, 4, 1, 2, 1, 1, 2, 1, \quad 1, 2, 1, 1, 2, 1, 4, 4, 2, 0,$$

$$2, 4, 2, 1, 1, 4, 0, 4, 1, 1, \quad 1, 1, 4, 0, 4, 1, 1, 2, 4, 2.$$

In the first two cases we get that 7 divides two terms with $a_i \overset{c}{\equiv} 2 \pmod{13}$, whence the power of 7 should be 2 in both cases. However, this implies $7^2 \mid 14d$ and $7 \mid d$, a contradiction. As the latter two cases are symmetric, we may assume that the very last possibility occurs. In that case we have $7 \mid a_5$ and $7 \mid a_{19}$. We may assume that $11 \mid a_{17}$, otherwise $P(a_6 a_8 \dots a_{18}) \le 7$ and the statement follows by induction. Further, we also have $13 \mid a_7$, and $17 \mid a_9$ and $19 \mid a_{15}$ or vice versa. Hence either $P(a_3 a_8 a_{13}) \le 2$ or $P(a_4 a_{10} a_{16}) \le 2$, and the theorem follows by induction also in any case.

The case $k = 24$. In case of $\gcd(3 \cdot 7 \cdot 13, d) = 1$ by the help of the procedures described in the previous section all cases can be excluded. So we may assume that $\gcd(3 \cdot 7 \cdot 13, d) > 1$ is valid.

Suppose first that $7 \mid d$. Among the odd coefficients $a_1, a_3, \dots, a_{23}$ there are at most four multiples of 3, three multiples of 5, two multiples of 11, and one multiple of 13, 17, 19 and 23 each. We know that all a_i belong to the same cubic class modulo 7. As $3 \stackrel{c}{\equiv} 4 \pmod 7$, $5 \stackrel{c}{\equiv} 2 \pmod 7$ and among the coefficients $a_1, a_3, \dots, a_{23}$ there are at most two multiples of 3^2 and at most one multiple of 5^2, we get that these coefficients are all $\stackrel{c}{\equiv} 1 \pmod 7$. This yields that the primes $3, 5, 11, 17, 19, 23$ may occur only at least in pairs in the coefficients with odd indices. Thus we get that at least five out of $a_1, a_3, \dots, a_{23}$ are $\stackrel{c}{\equiv} 1 \pmod{13}$. Hence, by Lemma 3.2 we get that $13 \mid d$ and consequently $a_i \stackrel{c}{\equiv} 1 \pmod{13}$ for all i. This also shows that the 5-s can be at most third prime divisors of the a_i-s with odd indices. So we deduce that at least eight out of the odd coefficients $a_1, a_3, \dots, a_{23}$ are equal to ± 1. However, a simple calculation shows that from the eight corresponding terms we can always choose three forming an arithmetic progression. Hence the theorem follows from Lemma 3.1 in this case.

Assume next that $13 \mid d$, but $7 \nmid d$. Among the coefficients with odd indices there are at most four multiples of 3, three multiples of 5, two multiples of 7 and 11 each, and one multiple of 17, 19 and 23 each. Hence, by $5 \stackrel{c}{\equiv} 1 \pmod{13}$ we deduce $a_i \stackrel{c}{\equiv} 1 \pmod{13}$ for all i. As before, a simple calculation yields that at least for four of these odd coefficients $a_i = \pm 1$ hold. Hence looking at the possible cases modulo 7, one can easily see that we cannot have four 3-s at the places where $0, 2$ and 4 occur as cubic residues modulo 7. Hence in view of Lemma 3.2 we need to use two 11-s, which yields that $11 \mid a_1$ and $11 \mid a_{23}$ should be valid. Thus the only possibility for the cubic residues of $a_1, a_3, \dots, a_{23}$ modulo 7 is given by the sequence

$$2, 1, 0, 1, 2, 4, 4, 2, 1, 0, 1, 2.$$

However, then the positions of the 2-s and 4-s allow to use at most two primes 3 at places which are not divisible by 7. Hence switching back to modulo 13, we get that there are at least five $a_i = \pm 1$, a contradiction by Lemma 3.2.

Finally, assume that $3 \mid d$, and $\gcd(7 \cdot 13, d) = 1$. Then among $a_1, a_3, \dots, a_{23}$ there are at most three multiples of 5, two multiples of 7 and 11 each, and one multiple of 13, 17, 19 and 23 each. Hence by Lemma 3.2 we get that exactly four of the coefficients $a_1, a_3, \dots, a_{23}$ are $\stackrel{c}{\equiv} 1 \pmod{13}$, and another more is a multiple of 13. Further, all the listed primes (except the 5-s) should occur separately. Using that at most one of these coefficients can be divisible by 7^2 and 11^2, in view of the proof of Lemma 3.2 we get that the only possibilities for the cubic residues of these coefficients modulo 13 are given by one of the sequences

$$2,2,4,2,1,1,4,0,4,1,1,2, \quad 2,1,1,4,0,4,1,1,2,4,2,2.$$

By symmetry we may assume that the we have the first possibility. Then we have $7 \mid a_3$, $11 \mid a_1$, $13 \mid a_{15}$, and $17, 19, 23$ divide a_5, a_7, a_{13} in some order. Hence we may further assume that $5 \mid n + 4d$, otherwise $P(a_4 a_9 a_{14}) \leq 2$, and by induction we are done. However, then $P(a_{16} a_{18} a_{20}) \leq 2$, and the statement follows by induction.

The case $k = 30$**.** In case of $\gcd(3 \cdot 7 \cdot 13, d) = 1$ by the help of the procedures described in the previous section all cases can be excluded. Assume now that $\gcd(3 \cdot 7 \cdot 13, d) > 1$.

We start with the case $7 \mid d$. Then among the odd coefficients $a_1, a_3, \dots, a_{29}$ there are at most five multiples of 3, three multiples of 5, two multiples of 11 and 13 each, and one multiple of 17, 19, 23 and 29 each. As $13 \stackrel{c}{\equiv} 29 \stackrel{c}{\equiv} 1 \pmod 7$, this yields that $a_i \stackrel{c}{\equiv} 1 \pmod 7$ for all i. Hence the other primes must occur at least in pairs in the a_i-s with odd indices, which yields that at least six such coefficients are equal to ± 1. Further, we get that the number of such coefficients $\stackrel{c}{\equiv} 0, 1 \pmod{13}$ is at least eight. However, by Lemma 3.2 this is possible only if $13 \mid d$, whence $a_i \stackrel{c}{\equiv} 1 \pmod{13}$ for all i. Further, then 5 and 29 can be at most third prime divisors of the coefficients a_i-s with odd i-s. So a simple calculation gives that at least ten out of the odd coefficients $a_1, a_3, \dots, a_{29}$ are equal to ± 1. Hence there are three such coefficients in arithmetic progression, and the statement follows from Lemma 3.1 in this case.

Assume next that $13 \mid d$, but $7 \nmid d$. Then among the odd coefficients $a_1, a_3, \dots, a_{29}$ there are at most five multiples of 3, three multiples of 5 and 7 each, two multiples of 11, and one multiple of 17, 19, 23 and 29 each. From this we get that $a_i \stackrel{c}{\equiv} 1 \pmod{13}$ for all i. Hence the primes different from 5 should occur at least in pairs. We get that at least five out of the coefficients $a_1, a_3, \dots, a_{29}$ are equal to ± 1. Thus modulo 7 we get that it is impossible to have three terms divisible by 7. Then it follows modulo 13 that at least six a_i-s with odd indices are equal to ± 1. However, this is possible only if $7 \mid d$, which is a contradiction.

Finally, assume that $3 \mid d$, but $\gcd(7 \cdot 13, d) = 1$. Then among the odd coefficients $a_1, a_3, \dots, a_{29}$ there are at most three multiples of

5 and 7 each, two multiples of 11 and 13 each, and one multiple of 17, 19, 23 and 29 each. Further, modulo 7 we get that all primes $5, 11, 17, 19, 23$ must occur separately, and the number of odd i-s with $a_i \stackrel{c}{\equiv} 0, 1 \pmod 7$ is seven. However, checking all possibilities modulo 7, we get a contradiction. Hence the theorem follows also in this case. □

Proof of Theorem 2.1. Obviously, for $k < 32$ the statement is an immediate consequence of Theorem 2.2. Further, observe that $b = 1$ implies that for any k with $31 < k < 39$, one can always find an appropriate j with $0 \le j \le k-30$, such that $P(a_j a_{j+1} \dots a_{j+29}) \le 29$ is always valid. Hence the statement follows from Theorem 2.2 also in this case. □

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