On the irrationality of Cantor and Ahmes series

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To the memory of Béla Brindza

Abstract

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of integers with $b_n > 1$ for all n. We derive criteria for the (ir)rationality of the sum $\sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n}$ in terms of the sequence $(\frac{b_n}{a_n})_{n=1}^{\infty}$. We present refinements of criteria of Oppenheim, Erdős and Straus, and Tijdeman and Yuan. Furthermore we make some remarks on a similar approach to determine the (ir)rationality of sums $\sum_{n=1}^{\infty} \frac{1}{a_n}$.

1 Introduction

In Sections 2 and 3 we consider series $\sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n}$ where $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences of integers with $a_n > 1$ for all n. We assume that (a_n) and (b_n) are explicitly given and that $b_n = o(a_{n-1}a_n)$ as $n \to \infty$. We wonder how we can use the behaviour of the sequence $(r_n)_{n=1}^{\infty}$, where $r_n = b_n/a_n$ for all n, to decide whether $\alpha := \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n}$ is rational. Theorem 1 is a sharpening of results of Oppenheim [15] and Tijdeman and Yuan [19]. Theorem 2 is a simplification of a theorem of Erdős and Straus [11]. Together they make a useful test.

In Section 4 we wonder whether a similar approach works for convergent sequences $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$. Theorem 3 shows that even in case $b_n = 1$ for all n the fact that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ is an irrational number θ does not guarantee the irrationality of the sum $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$. We wonder whether the more severe condition $a_{n+1} - \theta a_n \to 0$ for some irrational number θ implies irrationality, but unfortunately we cannot answer the question.

The history of the developments is given in the various sections.

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2 Limit points in case of Cantor series

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of integers with $a_n > 1$ for all n such that $\alpha := \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n}$ converges. Put

$$r_n = \frac{b_n}{a_n}, \quad R_n = \sum_{i=0}^{\infty} \frac{b_{n+i}}{a_n \cdots a_{n+i}} \quad (n = 1, 2, \ldots).$$

It is an easy observation that α is rational if $\frac{b_n}{a_n - 1}$ is ultimately constant. In 1869 Cantor [6] showed that if $0 \le b_n < a_n$ and for every positive integer q there is a positive integer n such that q divides $a_1 a_2 \cdots a_n$, then α is irrational if and only if $b_n > 0$ infinitely often and $b_n < a_n - 1$ infinitely often. In 1954 Oppenheim [15] extended this result to sequences satisfying $|b_n| < a_n$ for all n. It also follows from his results that if α is rational, then all the limit points of (r_n) are rational, and that, if $\alpha \in \mathbb{Q}$ and (r_n) has an integer limit point t (hence $t \in \{-1, 0, 1\}$), then $b_n = t(a_n - 1)$ for all n larger than some n_0. Cantor and Oppenheim found the results in connection with their studies of expansions of real numbers as sums of infinite series of rational numbers which now bear their names. A Cantor expansion of α is a series $\alpha := \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n}$ with $0 \le b_n < a_n$ for all n.

In 2002 Tijdeman and Yuan [19] showed that $\alpha \notin \mathbb{Q}$ under the more general condition that $b_n = O(a_n)$ as $n \to \infty$ and (r_n) has an irrational limit point. In the present paper we prove the following improvement. By a denominator of a rational number α we mean the smallest positive integer m such that $m\alpha$ is an integer. Furthermore $x(\text{mod } 1)$ denotes the number y with $y - x \in \mathbb{Z}$ and $0 \le y < 1$.

Theorem 2.1 *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of integers with $a_n > 1$ for all n and $b_n = o(a_{n-1}a_n)$ as $n \to \infty$. Put $\alpha := \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n}$, $r_n = \frac{b_n}{a_n}$, $R_n = \sum_{i=0}^{\infty} \frac{b_{n+i}}{a_n \cdots a_{n+i}}$ as $n = 1, 2, \ldots$. If α is rational, then all the limit points of $(r_n(\text{mod } 1))_{n=1}^{\infty}$ are rational numbers having the same denominator u.*

It follows immediately from Theorem 2.1 that if α is rational then the sequence $(r_n(\text{mod } 1))_{n=1}^{\infty}$ has only finitely many limit points. The following Cantor expansion shows that it can happen that all the possible values modulo 1 occur as limit points:

$$\frac{1}{5} = \frac{1}{7} + \frac{4}{7 \cdot 12} + \frac{13}{7 \cdot 12 \cdot 17} + \frac{13}{7 \cdot 12 \cdot 17 \cdot 22} + \frac{5}{7 \cdot 12 \cdot 17 \cdot 22 \cdot 27} + \ldots$$

where $\frac{b_{4n}}{a_{4n}} \to \frac{3}{5}$, $\frac{b_{4n+1}}{a_{4n+1}} \to \frac{1}{5}$, $\frac{b_{4n+2}}{a_{4n+2}} \to \frac{2}{5}$, $\frac{b_{4n+3}}{a_{4n+3}} \to \frac{4}{5}$ as $n \to \infty$. More generally, if p is an odd prime and g is a primitive root of p, then the Cantor expansion of $\frac{1}{p}$ as

$$\frac{1}{p} = \sum_{n=1}^{\infty} \frac{b_n}{\prod_{i=1}^{n}(ip + g)}$$

has the property that $(r_n)_{n=1}^{\infty}$ has $\frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{p}$ as limit points. On the other hand, the examples

$$\frac{1}{5} = \frac{1}{6} + \frac{2}{6 \cdot 11} + \frac{3}{6 \cdot 11 \cdot 16} + \frac{4}{6 \cdot 11 \cdot 16 \cdot 21} + \frac{5}{6 \cdot 11 \cdot 16 \cdot 21 \cdot 26} + \cdots$$

and

$$\frac{1}{6} = \frac{1}{8} + \frac{4}{8 \cdot 11} - \frac{5}{8 \cdot 11 \cdot 14} + \frac{6}{8 \cdot 11 \cdot 14 \cdot 17} - \frac{7}{8 \cdot 11 \cdot 14 \cdot 17 \cdot 20} + \cdots$$

show that it can also happen that $(r_n (\mathrm{mod}\ 1))$ does not assume all possible values and that u does not equal the denominator of α. In the latter case $q = 6, u = 3$.

Proof. of Theorem 2.1. Suppose α is rational with denominator q. Then for $n = 1, 2, \ldots$

$$qR_n = q \sum_{i=0}^{\infty} \frac{b_{n+i}}{a_n \cdots a_{n+i}} = q\alpha a_1 \cdots a_{n-1} - q \sum_{i=1}^{n-1} b_i a_{i+1} \cdots a_{n-1} \in \mathbb{Z}. \qquad (1)$$

By applying this argument to R_n in place of α, we find that if q_n denotes the denominator of R_n for $n = 1, 2, \ldots$, then $q_m \mid q_n$ whenever $n < m$. Hence $(a_n)_{n=1}^{\infty}$ is a non-increasing sequence of integers with limit u, say.

Let $0 < \varepsilon < 1/(4q)$. Let n_0 be so large that $uR_n \in \mathbb{Z}$ and $|\frac{b_n}{a_{n-1}a_n}| < \varepsilon$ for $n > n_0$. Then, for $n > n_0$, u is the denominator of R_n and

$$|R_n - \frac{b_n}{a_n}| < \varepsilon + \frac{\varepsilon}{a_n} + \frac{\varepsilon}{a_n a_{n+1}} \cdots \leq 2\varepsilon < \frac{1}{2q}. \qquad (2)$$

Thus all the limit points of the sequence $(r_n (\mathrm{mod}\ 1))_{n=1}^{\infty}$ are of the form t/u where t is an integer coprime to u.

□

Remarks 1. The condition $b_n = o(a_{n-1}a_n)$ in Theorem 2.1 can be replaced with the condition $b_n = O(a_n)$ as $n \to \infty$. The proof requires a slight modification of formula (2).
2. If $d_n := \gcd(q_n, a_{n+1})$ then $q_{n+1} = q_n/d_n$. Therefore we have

$$u = q/(\lim_{n\to\infty} \gcd(q, a_1 a_2 \cdots a_n)).$$

In particular, if for every integer q there exists an n such that $q \mid a_1 a_2 \cdots a_n$ then we are certain that $u = 1$, hence that all the limit points of the sequence $(r_n)_{n=1}^{\infty}$ are integers. This special case has been investigated thoroughly by Cantor [6] and Oppenheim [15].

3 The order in which limit points are visited

For the rationality of α it does not suffice that all the limit points of the sequence $(r_n(\text{mod }1))_{n=1}^{\infty}$ are rationals with the same denominator. For example $e = \sum_{n=1}^{\infty} \frac{1}{n!}$ is irrational but the only limit point of $(r_n(\text{mod }1))_{n=1}^{\infty}$ is 0. Erdős and Straus [11] Lemma 2.29 = Erdős and Straus [12] Theorem 2.1 gave the following criterion for the rationality of α: *there exists a positive integer B and a sequence of integers $(t_n)_{n=1}^{\infty}$ so that for all large n we have*

$$Bb_n = t_n a_n - t_{n+1}, \qquad \mid t_{n+1} \mid < a_n/2. \tag{3}$$

Suppose α is rational with denominator q. It follows from their proof that one may choose $B = q$ and t_n as the integer nearest to $q\frac{b_n}{a_n}$. According to (2) the integer t_n equals qR_n for $n > n_0$. Hence the recurrence relation $Bb_n = t_n a_n - t_{n+1}$ becomes $b_n = R_n a_n - R_{n+1}$, but this relation follows immediately from the definition of $(R_n)_{n=1}^{\infty}$. The theorem of Erdős and Straus implies that the relation is crucial for the rationality of α. The condition $\mid t_{n+1} \mid < a_n/2$ is optional. We specify their theorem to a more practical criterion, omit the optional condition and adjust their proof.

Theorem 3.1 *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of integers with $a_n > 1$ for all n and $b_n = o(a_{n-1}a_n)$ as $n \to \infty$. Then*

$$\sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n} \tag{4}$$

is rational if and only if for some positive integer q and all large n the integers t_n nearest to $q\frac{b_n}{a_n}$ satisfy

$$qb_n = t_n a_n - t_{n+1}. \tag{5}$$

Proof. Assume that (5) holds for $n \geq \mathbb{N}$. Then

$$qa_1 \cdots a_{N-1} \sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n} \in \mathbb{Z} + \sum_{n=N}^{\infty} \frac{t_n a_n - t_{n+1}}{a_N \cdots a_n} = \mathbb{Z} + t_N = \mathbb{Z}.$$

So condition (5) is sufficient for the rationality of the series (4).

To prove the necessity we recall from the proof of Theorem 2.1 that $|R_n - \frac{b_n}{a_n}| < \frac{1}{2q}$ for $n > n_0$. From the definition of R_n we immediately see that $a_n R_n = b_n + R_{n+1}$ for all n. Choosing t_n as the integer nearest to $q\frac{b_n}{a_n}$ we obtain that $t_n = qR_n$ and that (5) holds for $n \geq n_0$. □

Theorems 2.1 and 3.1 provide the following test for the rationality of α given the values of $b_n/a_n = r_n$.

1. Determine the limit points of $(r_n(\text{mod }1))_{n=1}^{\infty}$ and check whether they all have

the same denominator u. If not, then $\alpha \notin \mathbb{Q}$.
2. Let s_n denote the integer nearest to ur_n for all n. Check whether $ub_n = a_n s_n - s_{n+1}$ for all large n. If not, then $\alpha \notin \mathbb{Q}$. Otherwise $\alpha \in \mathbb{Q}$.

In the test s_n/u is the simplified fraction t_n/q for all large n.

Application 1 Suppose a, b, c, d, e, f are integers with $a > 0, b > 0$ such that $a_n = an^2 + cn + e, b_n = bn^2 + dn + f$ for all n. We wonder when $\alpha \in \mathbb{Q}$. Since $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{b}{a}$, we can choose $u = a$. The nearest integer to $a\frac{b_n}{a_n}$ is b for large n. Hence $\alpha \in \mathbb{Q}$ if and only if $a(bn^2 + dn + f) = b(an^2 + cn + e) - b$ for all large n, that is, if $bc = ad$ and $b(e - 1) = af$.

Application 2 Suppose a, b, c, d are integers with $a > 0, b > 0$ such that $a_n = an^2 + c, b_n = bn^3 + d$ for all n. Since $\frac{b_n}{a_n} = \frac{bn}{a} + o(1)$, the limit points of $(r_n (\mathrm{mod}\ 1))$ are the multiples of $\frac{b}{a}$ considered modulo 1. If $a > 1$, then not all denominators are equal and α is irrational. If $a = 1$, then $s_n = bn$ for all large n. Hence $\alpha \in \mathbb{Q}$ if and only if $bn^3 + d = bn(n^2 + c) - b(n + 1)$ for all large n, that is, if $c = 1$ and $d = -b$.
Remark If $b_n = o(a_{n-1}a_n)$ is not satisfied, then the above method may still be applicable. If there exist integers c_n such that $R_n = \frac{c_n}{a_n} + o(1)$, then it suffices to replace b_n in the test with c_n and r_n with $\frac{c_n}{a_n}$ for all n. See Hančl and Tijdeman [13], [14] and Tijdeman and Yuan [19] for conditions in case $b_n = o(a_{n-1}a_n)$ is not satisfied.

4 Limits of a_{n+1}/a_n for Ahmes series

One may wonder whether the limit point approach is also applicable for series of the form $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$. To simplify matters we shall study the so-called Ahmes series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ where the a_n's are positive integers such that the series converges. There are several irrationality results in case (a_n) grows doubly exponential, see e.g. Erdős and Straus [10], Sándor [18], Badea [2], and Duverney [7], [8]. We would like to have irrationality results in case of simple exponential growth. Such results have been given for special sequences, e.g. by André-Jeannin [1], Borwein [3], [4], Borwein and Zhou [5], Duverney, Nishioka, Nishioka and Shiokawa [9] and Prévost [16], [17].

We wondered whether $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \theta$ with $\theta \notin \mathbb{Q}$ would imply irrationality of α. The opposite is true as is shown by the following theorem.

Theorem 4.1 *Let $\alpha, \theta \in \mathbb{R}, \alpha > 0, \theta > 1$. Then there exists an increasing sequence of positive integers $a_1, a_2, \ldots$ such that $\sum_{n=1}^{\infty} \frac{1}{a_n} = \alpha$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \theta$.*

Proof. Put $\Delta_0 = \alpha$, $a_1 = [\frac{1}{\alpha}] + 1$, $\Delta_1 = \alpha - \frac{1}{a_1}$. Then $0 < \Delta_1 < \Delta_0$. We proceed by induction. Let a_n and Δ_n be defined with $\Delta_n > 0$. Put

$$a_{n+1} = \max([\frac{1}{\Delta_n}] + 1, a_n + 1), \quad \Delta_{n+1} = \Delta_n - \frac{1}{a_{n+1}}.$$

Then $0 < \Delta_{n+1} < \Delta_n$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, there are infinitely many integers n with $a_{n+1} - a_n > 1$. Then $a_{n+1} - 1 \leq \frac{1}{\Delta_n} < a_{n+1}$ and $\Delta_n \leq 1$, hence

$$\Delta_{n+1} = \Delta_n - \frac{1}{a_{n+1}} \leq \Delta_n - \frac{\Delta_n}{\Delta_n + 1} = \Delta_n \frac{\Delta_n}{\Delta_n + 1} \leq \frac{1}{2}\Delta_n.$$

Thus there is an N with $\Delta_N < \min(\frac{1}{\theta - 1}, 1)$.

From N on we change the choice of a_n by choosing $a_{n+1} = [\frac{\theta}{\Delta_n(\theta-1)}]$, $\Delta_{n+1} = \Delta_n - \frac{1}{a_{n+1}}$. Since $\Delta_n < \frac{1}{\theta-1}$, we have $\frac{\theta}{\Delta_n(\theta-1)} - 1 > \frac{1}{\Delta_n}$ whence $a_{n+1} > \frac{1}{\Delta_n}$ and $\alpha - \sum_{j=1}^{n+1} \frac{1}{\alpha_j} = \Delta_{n+1} > 0$. Therefore a_{n+2} is a well-defined integer and $\sum_{n=1}^{\infty} \frac{1}{a_n} \leq \alpha$.

Next we show that $\sum_{n=1}^{\infty} \frac{1}{a_n} = \alpha$. We have, for $n \geq N$,

$$\Delta_{n+1} = \Delta_n - \frac{1}{a_{n+1}} \leq \Delta_n - \frac{\Delta_n(\theta - 1)}{\theta} = \frac{\Delta_n}{\theta}. \tag{6}$$

Since $\theta > 1$, this shows that $\alpha - \sum_{j=1}^{n} \frac{1}{a_j} = \Delta_n \to 0$ as $n \to \infty$ and therefore $\sum_{n=1}^{\infty} \frac{1}{a_n} = \alpha$.

We now check that $a_{n+1} > a_n$ for $n \geq N$. It suffices to show that $\frac{\theta}{\Delta_n(\theta-1)} - \frac{\theta}{\Delta_{n-1}(\theta-1)} \geq 1$. Indeed we have, by (6) and $\Delta_n \leq 1$,

$$\frac{1}{\Delta_n} - \frac{1}{\Delta_{n-1}} \geq \frac{1}{\Delta_n} - \frac{1}{\theta\Delta_n} \geq \frac{\theta - 1}{\theta}.$$

Finally we prove that $\frac{\Delta_{n+1}}{\Delta_n} \to \theta$ as $n \to \infty$. We know that $\Delta_n \to 0$ as $n \to \infty$. Hence

$$\frac{a_{n+1}}{a_n} = \frac{\frac{\theta}{(\theta-1)\Delta_n} + O(1)}{\frac{\theta}{(\theta-1)\Delta_{n-1}} + O(1)} = \frac{\Delta_{n-1}(1 + o(1))}{\Delta_n(1 + o(1))} \geq \theta(1 + o(1)). \tag{7}$$

On the other hand,

$$\Delta_{n+1} = \Delta_n - \frac{1}{a_{n+1}} > \Delta_n - \frac{1}{\frac{\theta}{\Delta_n(\theta-1)} - 1} = \Delta_n - \frac{\Delta_n(\theta - 1)}{\theta}(1 + o(1)) = \frac{\Delta_n}{\theta}(1 + o(1))$$

so that

$$\frac{a_{n+1}}{a_n} = \frac{\Delta_{n-1}}{\Delta_n}(1 + o(1)) \leq \theta(1 + o(1)).$$

Thus $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \theta$.

$\square$

A much more restrictive requirement also leading to simple exponential growth is that $a_{n+1} - \theta a_n \to 0$ as $n \to \infty$ for some irrational number θ.

Open problem: *Is it true that if $(a_n)_{n=1}^{\infty}$ is a sequence of positive integers such that $\alpha := \sum_{n=1}^{\infty} \frac{1}{a_n}$ is a rational number and $a_{n+1} - \theta a_n \to 0$ as $n \to \infty$ for some number θ, then θ is an integer and $a_{n+1} = \theta a_n$ for all large n?*

It is obvious that $\alpha \in \mathbb{Q}$ if $\frac{a_{n+1}}{a_n} = \theta \in \mathbb{Z}_{>1}$ for all large n. On the other hand there are many numbers θ for which no sequence of positive integers $(a_n)_{n=1}^{\infty}$ exists such that $a_{n+1} - \theta a_n \to 0$ (independently of the arithmetic character of α). If $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$, then such a sequence cannot exist, since not all a_n's can be integers, and if θa_n is not an integer, it cannot be close to an integer. Hence, also roots of rational numbers are excluded, even roots of integers, such as $\sqrt{2}$ and $\sqrt[3]{3}$.

On the other hand, there exist algebraic numbers θ which admit such sequences. The set of numbers θ admitting an integer sequence $(a_n)_{n=1}^{\infty}$ with $a_{n+1} - \theta a_n \to 0$ as $n \to \infty$ comprises the Pisot numbers. A Pisot number is an algebraic integer $\gamma = \gamma_1$ all of whose conjugates $\gamma_2, \ldots, \gamma_k$ are less than 1 in absolute value. For example, by the theorem on symmetric functions the numbers $a_n := \gamma_1^n + \gamma_2^n + \ldots \gamma_k^n$ are rational integers. Furthermore, $a_{n+1} - \gamma a_n = O(\max_{i>1}|\gamma_i|^n)$ which tends to 0 exponentially fast. The most famous sequences with a Pisot number limit ratio are the Fibonacci sequence (F_n) and the Lucas sequence (L_n). One has $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \frac{1}{2} + \frac{1}{2}\sqrt{5}$. André-Jeannin [1] showed in 1989 that $\sum_{n=1}^{\infty} \frac{1}{F_n}$ and $\sum_{n=1}^{\infty} \frac{1}{L_n}$ are irrational. Duverney, Nishioka, Nishioka and Shiokawa [9] proved by another method that $\sum_{n=1}^{\infty} \frac{1}{F_n^{2s}}$ and $\sum_{n=1}^{\infty} \frac{1}{L_n^{2s}}$ are irrational for any positive integer s. In fact, they proved more general results on binary recurrence sequences. The result of André-Jeannin was further generalised by Prévost [16] and [17]. However, all these results concern only binary recurrence sequences with integer coefficients and therefore only quadratic irrational θ. What about Pisot numbers of degree > 2? The situation for transcendental numbers θ is totally obscure for us.

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