On the irrationality of factorial series

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Abstract

In this paper we prove some general results which imply, for example, the irrationality of $\sum_{N=1}^{\infty} \frac{[N^\alpha]}{N!}$ $(\alpha \geq 0)$, $\sum_{N=1}^{\infty} \frac{[N \log^\beta N]}{N!}$ $(\beta \in \mathbb{R})$ and $\sum_{N=1}^{\infty} \frac{[N^\alpha \exp(\log^\beta N)]}{N!}$ $(\alpha \geq 0, 0 < \beta < 1)$. Moreover, we prove that such numbers are linearly independent over the rationals. The method of proof is an extension of the approach used in earlier works of the authors and Pingzhi Yuan.

1 Introduction

Let $(b_n)_{n=1}^{\infty}$ be a sequence of integers. In the present paper we study the irrationality of $R := \sum_{n=1}^{\infty} \frac{b_n}{n!}$ and, more generally, of $R^* := \sum_{N=1}^{\infty} \frac{b_N}{\prod_{n=1}^{N}(an+b)}$ where a and b are given positive integers. In 1766 Lambert [10] proved the irrationality of $e = \sum_{n=1}^{\infty} \frac{1}{n!}$. In 1873 Hermite [9] established the transcendence of e which implies the irrationality of $\sum_{n=1}^{\infty} \frac{m^n}{n!}$ for any nonzero integer m. In 1869 G. Cantor [2] showed that if $0 \leq b_n < n$, then R is rational if and only if $b_n > 0$ infinitely often and $b_n < n-1$ infinitely often. On the other hand, if $\frac{b_n}{n-1}$ is constant for n larger than some n_0, then $R \in \mathbb{Q}$. This is an exceptional case in many results. Oppenheim [12] showed that both the condition on $b_n > 0$ and the condition on $b_n < n-1$ can be relaxed. For example, it follows from his results that if $|b_n| < n$ for every n, then R is rational if and only if $\frac{b_n}{n-1}$ is ultimately a fixed integer. Thus if $|b_n| < n-1$ for every n and $R \in \mathbb{Q}$, then b_n is ultimately equal to 0. The results of Oppenheim were extended by the authors [8] who showed that if $n \nmid b_n$ for all n, $b_n = o(n^2)$ and $\liminf_{n\to\infty} \frac{|b_n|}{n} = 0$, then R is irrational. They further proved that R is irrational if $(b_n)_{n=1}^{\infty}$ is a monotonic sequence of positive integers such that $b_n = O(n^2)$ and $\gcd(b_n, n-1) = o(b_n)$. Tijdeman and Yuan [14] extended another result of Oppenheim by showing that R is irrational if $b_n = O(n)$ and the sequence $(\frac{b_n}{n})_{n=1}^{\infty}$ has an irrational limit point. See also Hančl [6] and [7].

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Erdős and Straus [5] started a series of results in which the size of the difference $b_{n+1} - b_n$ is a relevant factor. They used such results to establish the irrationality of R in case $(b_n)_{n=1}^{\infty}$ represents a multiplicative or other arithmetic function. It follows from their result that if $b_n > 0$ for all n, $b_{n+1} - b_n = o(n)$ and $\liminf_{n\to\infty} \frac{n}{b_n} = 0$, then R is irrational. The authors [8] showed that the condition $\liminf_{n\to\infty} \frac{n}{b_n} = 0$ can be replaced with the necessary condition that $\frac{b_n}{n-1}$ is not ultimately constant. Tijdeman and Yuan [14] showed that, moreover, the condition $b_n > 0$ for all n can be dropped: if $b_{n+1} - b_n = o(n)$, then $R \in \mathbb{Q}$ if and only if $\frac{b_n}{n-1}$ is ultimately a fixed integer. These results generalize Erdős' result [3] that $\sum_{n=1}^{\infty} \frac{p_n}{n!} \notin \mathbb{Q}$, where $\{p_n\}_{n=1}^{\infty}$ is the sequence of consecutive prime numbers. In fact Erdős claimed the irrationality of $\sum_{n=1}^{\infty} \frac{p_n^k}{n!} \notin \mathbb{Q}$ for $k = 1, 2, \ldots$, but unfortunately he proved only the case $k = 1$. Oppenheim [12] showed that $\sum_{n=1}^{\infty} \frac{\epsilon_n d_n}{n!}, \sum_{n=1}^{\infty} \frac{\epsilon_n \sigma_n}{n!}$ and $\sum_{n=1}^{\infty} \frac{\epsilon_n \phi_n}{n!}$ are irrational for all choices of $\epsilon_n \in \{-1, 1\}$, where $d(n), \sigma(n), \phi(n)$ denote the number of divisors, the sums of divisors, and the Euler function of n, respectively. A special case was treated by Erdős and Kac [4]. Erdős and Straus [5] proved that the numbers $1, \sum_{n=1}^{\infty} \frac{\sigma_n}{n!}, \sum_{n=1}^{\infty} \frac{\phi_n}{n!}$ and $\sum_{n=1}^{\infty} \frac{b_n}{n!}$, where $|b_n| < n^{\frac{1}{2}-\epsilon}$ for all large n and $b_n \neq 0$ infinitely often, are linearly independent over the rationals. Most of the mentioned results were stated in greater generality in the original papers than above.

Tijdeman and Yuan [14] started to compare second order differences (cf. the proof of their Theorem 4.3). In the present paper we pursue this idea by studying K-th order differences. For doing so we have to impose stronger regularity conditions on the numbers b_n. Nevertheless the results are valid for a wide class of sequences $(b_n)_{n=1}^{\infty}$. Corollary 3.1 precisely states for which polynomials $P(x)$ with integer coefficients $\sum_{n=1}^{\infty} \frac{P(n)}{n!}$ is rational. Section 3 further provides a method to establish the irrationality of a large class of numbers $\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N}(an+b)}$ where $f(N)$ is an integer-valued function satisfying $f(N) = (aN + b)F(N) + O(1)$ and F is a smooth function which does not grow faster than a polynomial. In particular it yields the irrationality of the following numbers:

$$\sum_{n=1}^{\infty} \frac{[\gamma n^{\alpha}]}{n!} \text{ for } \alpha \in \mathbb{R}_+, \alpha \notin \mathbb{Z}, \gamma \in \mathbb{R}_+, \quad \sum_{n=1}^{\infty} \frac{[\log n]}{n!}, \quad \sum_{n=1}^{\infty} \frac{[\exp(\log^{\frac{1}{2}} n)]}{n!}.$$

In Section 4 the linear independence over the rationals of such numbers is treated. For example, linear independence is shown for the numbers

$$1, \ e \text{ and } \sum_{n=1}^{\infty} \frac{[n^{\alpha}]}{n!} \text{ for all } \alpha \in \mathbb{R}_+, \alpha \notin \mathbb{Z}.$$

The results remind us of the result by Loxton and van der Poorten [11] proved by Mahler's method that $\sum_{n=1}^{\infty} [n\alpha]\beta^n$ is transcendental for α irrational and β algebraic with $0 < |\beta| < 1$.

2 Basic lemmas

Let $a > 0$ and b be integers such that $an + b \neq 0$ for every positive integer n. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of integers. We investigate under which conditions

$$R^* = \sum_{N=1}^{\infty} \frac{b_N}{\prod_{n=1}^{N}(an+b)} \tag{1}$$

is irrational. Since all terms are rational, we may neglect the terms with $an+b < 0$ and assume without loss of generality that $b \geq 0$.

The following lemma dealing with the sum

$$R_N^* := \sum_{m=N}^{\infty} \frac{b_m}{\prod_{n=N}^{m}(an+b)} \tag{2}$$

is crucial.

Lemma 2.1 *If $R^* = \frac{p}{q}$ for some $p \in \mathbb{Z}$, $q \in \mathbb{N}$, then $qR_N^* \in \mathbb{Z}$ for all N.*

Proof. We have

$$p \prod_{n=1}^{N-1}(an+b) = q \sum_{m=1}^{N-1} b_m \prod_{n=m+1}^{N-1}(an+b) + q \sum_{m=N}^{\infty} \frac{b_m}{\prod_{n=N}^{\infty}(an+b)}$$

and the first two terms are integers. □

Remark. If q divides $\prod_{n=1}^{N-1}(an+b)$, then we need not multiply by q to obtain integers and can conclude that R_N^* itself is an integer. If q is coprime to a, this is the case for sufficiently large N. In particular, it is the case if $a = 1$, hence for sequences $\sum_{n=1}^{\infty} \frac{b_n}{n!}$.

The following consequence of a theorem of Oppenheim implies that R^* is irrational if $b_n = o(n)$, but not ultimately constant 0.

Lemma 2.2 (Oppenheim [12], Theorem 8). *If $|b_n| < an + b$ for all $n > n_0$ and $\liminf_{n\to\infty} \frac{|b_n|}{n} = 0$, then R is rational if and only if $b_n = 0$ for all $n > n_0$.*

The next lemma displays some well known properties of Stirling numbers of the second kind.

Lemma 2.3 *Let K be a nonnegative integer. Put*

$$S(r, K) = \frac{1}{K!} \sum_{j=0}^{K} (-1)^{K-j} \binom{K}{j} j^r.$$

Then $S(r, K) = 0$ if $r < K$, $S(r, K) = 1$ if $r = K$ and $S(r, K) \in \mathbb{N}$ if $r > K > 0$.

For a proof see [1] Section III.2.

The following lemma gives partial fractions of the denominator of R^*.

Lemma 2.4 *Let $a > 0$, b, N and $s \geq 0$ be integers such that $a(N+i)+b \neq 0$ for $i = 0, 1, \ldots, s$. Then*

$$\frac{1}{\prod_{i=0}^{s}(a(N+i)+b)} = \frac{1}{s!a^s}\sum_{i=0}^{s}(-1)^i\binom{s}{i}\frac{1}{a(N+i)+b}. \tag{3}$$

Proof. We prove the identity by mathematical induction on s.
For $s = 0$ identity (3) is trivial. Suppose that (3) holds for $s = n$. Then

$$\frac{1}{\prod_{i=0}^{n+1}(a(N+i)+b)} = \frac{1}{(n+1)a}\left(\frac{1}{\prod_{i=0}^{n}(a(N+i)+b)} - \frac{1}{\prod_{i=1}^{n+1}(a(N+i)+b)}\right) =$$

$$\frac{1}{(n+1)!a^{n+1}}\left(\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{1}{a(N+i)+b} - \sum_{i=1}^{n+1}(-1)^{i-1}\binom{n}{i-1}\frac{1}{a(N+i)+b}\right) =$$

$$\frac{1}{(n+1)!a^{n+1}}\left(\frac{1}{aN+b} + \frac{(-1)^{n+1}}{a(N+n+1)+b} + \sum_{i=1}^{n}(-1)^i\left(\binom{n}{i}+\binom{n}{i-1}\right)\frac{1}{a(N+i)+b}\right) =$$

$$\frac{1}{(n+1)!a^{n+1}}\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}\frac{1}{a(N+i)+b}.$$

This completes the induction step . $\square$

The last lemma of the section will be used for all theorems except for Theorem 3.1 and Theorem 3.2.

Lemma 2.5 *Let $K \geq 0$, $a > 0$ and b be given integers such that $an+b \neq 0$ for every $n \in \mathbb{N}$. Let $H : \mathbb{R}_+ \to \mathbb{R}_+$ be a function. Suppose we have*

$$H(N+j) = \sum_{r=0}^{K}\frac{H^{(r)}(N)}{r!}j^r + O\left(\frac{H(N)}{N^{K+1}}\right) \quad \textit{for } j = 0, 1, \ldots, K \textit{ as } N \to \infty \tag{4}$$

and

$$H^{(r)}(N) = O\left(\frac{H(N)}{N^r}\right) \quad \textit{for } r = 0, 1, \ldots, K \textit{ as } N \to \infty. \tag{5}$$

Put

$$R_K^*(N) = \sum_{j=0}^{K}(-1)^j\binom{K}{j}\sum_{s=0}^{\infty}\frac{H(N+j+s)}{\prod_{n=N+j}^{N+j+s-1}(an+b)}.$$

Then

$$R_K^*(N) = (-1)^K H^{(K)}(N) + O(H(N)N^{-K-1}) \textit{ as } N \to \infty. \tag{6}$$

Proof. Let N be sufficiently large. Note that for $j = 0, 1, \ldots, K$, by (4) and (5),

$$H(N+j) = O\left(H(N) \sum_{r=0}^{K+1} \frac{1}{r!} (\frac{j}{N})^r \right) = O(H(N)e^{j/N}) = O(H(N)) \text{ as } N \to \infty. \tag{7}$$

Write

$$R_K^*(N) = R_{K,1}^*(N) + R_{K,2}^*(N) + R_{K,3}^*(N) \tag{8}$$

where

$$R_{K,1}^*(N) = \sum_{j=0}^{K} (-1)^j \binom{K}{j} H(N+j),$$

$$R_{K,2}^*(N) = \sum_{j=0}^{K} (-1)^j \binom{K}{j} \sum_{s=1}^{K} \frac{H(N+j+s)}{\prod_{n=N+j}^{N+j+s-1}(an+b)},$$

$$R_{K,3}^*(N) = \sum_{j=0}^{K} (-1)^j \binom{K}{j} \sum_{s=K+1}^{\infty} \frac{H(N+j+s)}{\prod_{n=N+j}^{N+j+s-1}(an+b)}.$$

By Lemma 2.3 and (4) we have

$$R_{K,1}^*(N) = \sum_{j=0}^{K} (-1)^j \binom{K}{j} \sum_{r=0}^{K} \frac{H^{(r)}(N)}{r!} j^r + O(H(N)N^{-K-1}) =$$

$$\sum_{r=0}^{K} \frac{H^{(r)}(N)}{r!} \sum_{j=0}^{K} (-1)^j \binom{K}{j} j^r + O(H(N)N^{-K-1}) =$$

$$(-1)^K H^{(K)}(N) + O(H(N)N^{-K-1}) \text{ as } N \to \infty. \tag{9}$$

We now turn to $R_{K,2}^*(N)$. We have, by Lemma 2.4, (4), (7), Lemma 2.3, and (5),

$$R_{K,2}^*(N) = \sum_{j=0}^{K} (-1)^j \binom{K}{j} \sum_{s=1}^{K} \frac{H(N+s+j)}{(s-1)!a^{s-1}} \sum_{i=0}^{s-1} (-1)^i \binom{s-1}{i} \frac{1}{a(N+i+j)+b} =$$

$$\sum_{j=0}^{K} (-1)^j \binom{K}{j} \sum_{s=1}^{K} \sum_{r=0}^{K} \frac{H^{(r)}(N+s)j^r}{r!(s-1)!a^{s-1}} \sum_{i=0}^{s-1} (-1)^i \binom{s-1}{i} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}(aj)^{m-1}}{(a(N+i)+b)^m}$$

$$+O\left(\sum_{s=1}^{K} \frac{H(N+s)}{(N+s)^{K+1}} \right) =$$

$$\sum_{s=1}^{K} \sum_{r=0}^{K} \frac{H^{(r)}(N+s)}{r!(s-1)!a^{s-1}} \sum_{i=0}^{s-1} (-1)^i \binom{s-1}{i} \sum_{m=1}^{\infty} \frac{(-a)^{m-1}}{(a(N+i)+b)^m} \sum_{j=0}^{K} (-1)^j \binom{K}{j} j^{r+m-1}$$

$$+O\left(\sum_{s=1}^{K}\frac{H(N)}{N^{K+1}}\right)=$$

$$\sum_{s=1}^{K}\sum_{r=0}^{K}\frac{H^{(r)}(N+s)}{r!(s-1)!a^{s-1}}\sum_{i=0}^{s-1}(-1)^i\binom{s-1}{i}\sum_{m=K-r+1}^{\infty}\frac{(-a)^{m-1}}{(a(N+i)+b)^m}\sum_{j=0}^{K}(-1)^j\binom{K}{j}j^{r+m-1}$$

$$+O(\frac{H(N)}{N^{K+1}})=O\left(\sum_{r=0}^{K}\frac{H(N)}{N^r}\sum_{m=K-r+1}^{\infty}\left(\frac{K}{N}\right)^m\right)+O\left(\frac{H(N)}{N^{K+1}}\right).$$

Thus

$$R^*_{K,2}(N)=O\left(\frac{H(N)}{N^{K+1}}\right). \tag{10}$$

Finally we estimate $R^*_{K,3}(N)$. By (7) there exists a constant $c>1$ such that $H(n+1)<cH(n)$ for all sufficiently large n. Hence $H(N+s)<c^sH(N)$ for every positive integer s. It follows that

$$R^*_{K,3}(N)=O\left(\sum_{s=K+1}^{\infty}\frac{H(N+s)}{\prod_{n=N}^{N+s-1}(an+b)}\right)=O\left(\sum_{s=K+1}^{\infty}\frac{H(N)c^s}{N^s}\right).$$

Hence

$$R^*_{K,3}(N)=O\left(\frac{H(N)}{N^{K+1}}\right)\quad\text{as } N\to\infty. \tag{11}$$

The combination of (8), (9), (10) and (11) yields (6). $\square$

Remark In applications of Lemma 2.5 the integer K is usually chosen as the smallest nonnegative integer such that $H^{(K)}(N)\to 0$ as $N\to\infty$.

3 Irrationality

The next lemma implies that $\sum_{n=1}^{\infty}\frac{P(n)}{n!}$ for $P(x)\in\mathbb{Q}[x]$ is a linear combination over $\mathbb{Q}$ of 1 and e.

Lemma 3.1 *Let $a>0$ and b be integers such that $an+b\neq 0$ for every $n\in\mathbb{N}$. Suppose that $P(x)=\sum_{i=0}^{T}a_ix^i$ is a polynomial with rational coefficients. Let d be the least common denominator of $a_0,a_1,\dots,a_T$. Then there exist rational numbers Q_0 and Q_1 such that*

$$\sum_{N=1}^{\infty}\frac{P(N)}{\prod_{n=1}^{N}(an+b)}=Q_0+Q_1\sum_{N=1}^{\infty}\frac{1}{\prod_{n=1}^{N}(an+b)}$$

where

$$a^TdQ_0=\sum_{i=0}^{T}a_i\sum_{k=0}^{i}\sum_{h=k}^{i}\binom{i}{h}a^{T-i+h-k}(-b)^{i-h}S(h,k)\sum_{N=1}^{k}\prod_{n=0}^{k-N-1}(b-na) \tag{12}$$

and

$$a^T dQ_1 = \sum_{i=0}^{T} a_i \sum_{k=0}^{i} \sum_{h=k}^{i} \binom{i}{h} a^{T-i+h-k}(-b)^{i-h} S(h,k) \tag{13}$$

are integers.

Proof. We can write

$$x^i = \sum_{k=0}^{i} b_{i,k} \prod_{m=1}^{k} (a(x-m+1)+b). \tag{14}$$

On substituting $x = -\frac{b}{a} + r$ for $r = 0, 1, \ldots, i$ into equation (14) we get

$$(-\frac{b}{a}+r)^i = \sum_{k=0}^{i} b_{i,k} \prod_{m=1}^{k} \left(a(-\frac{b}{a}+r-m+1)+b\right) = \sum_{k=0}^{i} b_{i,k} a^k \frac{r!}{(r-k)!}. \tag{15}$$

We consider (15) as a system of $i+1$ equations with $i+1$ unknowns $b_{i,0}, b_{i,1}, \ldots, b_{i,i}$. Using that $r \leq i$ we find that the solution of this system for $k = 0, 1, \ldots, i$ is given by

$$b_{i,k} = \frac{1}{k!a^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (-\frac{b}{a}+k-j)^i. \tag{16}$$

We write

$$\sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N}(an+b)} = \sum_{i=0}^{T} a_i \sum_{N=1}^{\infty} \frac{N^i}{\prod_{n=1}^{N}(an+b)}.$$

From this, (14) and (16) we obtain

$$\sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N}(an+b)} = \sum_{i=0}^{T} a_i \sum_{k=0}^{i} b_{i,k} \sum_{N=1}^{\infty} \frac{\prod_{m=1}^{k}(a(N-m+1)+b)}{\prod_{n=1}^{N}(an+b)} =$$

$$\sum_{i=0}^{T} a_i \sum_{k=0}^{i} b_{i,k} \sum_{N=1}^{k} \prod_{n=0}^{k-N-1} (b-na) + \sum_{i=0}^{T} a_i \sum_{k=0}^{i} b_{i,k} \sum_{N=k+1}^{\infty} \frac{1}{\prod_{n=1}^{N-k}(an+b)}$$

$$=: Q_0 + Q_1 \sum_{N=1}^{\infty} \frac{1}{\prod_{n=1}^{N}(an+b)},$$

where Q_0 and Q_1 are rational numbers. We have, by (16) and Lemma 2.3,

$$b_{i,k} = \frac{1}{k!a^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{h=0}^{i} \binom{i}{h} \left(-\frac{b}{a}\right)^{i-h} (k-j)^h =$$

$$\frac{1}{k!a^k} \sum_{h=0}^{i} \binom{i}{h} \left(-\frac{b}{a}\right)^{i-h} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^h = \sum_{h=k}^{i} \binom{i}{h} \frac{1}{a^k} \left(-\frac{b}{a}\right)^{i-h} S(h,k)$$

where $S(h,k)$ is an integer. Note that the exponent of a in $\frac{1}{a^k}(\frac{-b}{a})^{i-h}$ has minimal value $-i$, viz. if $h=k$. Hence $a^i b_{i,k}$ is an integer for $i=0,1,\ldots,k$. We obtain that $da^T Q_0$ and $da^T Q_1$ are the integers given by (12) and (13), respectively. $\square$

We denote the set of the positive integers by $\mathbb{N}$.

Theorem 3.1 *Let $a>0$ and b be integers such that $an+b\neq 0$ for every $n\in\mathbb{N}$. Let $P(x)=\sum_{i=0}^T a_i x^i \in \mathbb{Z}[x]$. Then $R^* := \sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^N (an+b)}$ is rational if and only if*

$$Q_1 = \sum_{i=0}^T a_i \sum_{k=0}^i \frac{1}{k!a^k} \sum_{j=0}^k (-1)^j \binom{k}{j} \left(-\frac{b}{a}+k-j\right)^i = 0. \tag{17}$$

Proof. It is obvious that $\sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^N (an+b)}$ is absolutely convergent. It follows immediately from Lemma 3.1 that if (17) holds, then $R^* \in \mathbb{Q}$. On the other hand, suppose (17) does not hold. By Oppenheim's theorem (Lemma 2.2) we know that $\sum_{N=1}^{\infty} \frac{1}{\prod_{n=1}^N (an+b)}$ is irrational. Hence, by Lemma 3.1, R^* is irrational. $\square$

Corollary 3.1 *Let $P(x)=\sum_{i=0}^T a_i x^i \in \mathbb{Z}[x]$.*
Then $\sum_{N=1}^{\infty} \frac{P(N)}{N!}$ is rational if and only if $\sum_{i=0}^T a_i \sum_{k=0}^i S(i,k)=0$.
If $a_T>0$ and $a_i \geq 0$ for $i=0,1,\ldots,T-1$, then $\sum_{N=1}^{\infty} \frac{P(N)}{N!} \notin \mathbb{Q}$.

Proof. Apply Theorem 3.1 with $a=1, b=0$ and Lemma 2.3. $\square$

Remark We recall that $\sum_{n=1}^{\infty} \frac{n-1}{n!} = 1 \in \mathbb{Q}$. Hence the condition on the nonnegativeness of the coefficients in the second statement cannot be dropped.

Corollary 3.2 *Let $a>0$ and b be integers such that $an+b\neq 0$ for every $n\in\mathbb{N}$. Let $P(x)=\sum_{i=0}^T a_i x^i \in \mathbb{Z}[x]$. If $a \not| \, a_T(b-1)^T$ then $\sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^N (an+b)}$ is irrational.*

Proof. Observe that, by (13) and Lemma 2.3, the terms in $a^T Q_1$ are divisible by a unless $h=k, i=T$. Hence, by Lemma 2.3,

$$a^T Q_1 \equiv a_T \sum_{k=0}^T \binom{T}{k} (-b)^{T-k} S(k,k) = a_T(1-b)^T \pmod{a}.$$

$\square$

Theorem 3.2 *Let $a>0$ and b be integers such that $an+b\neq 0$ for every $n\in\mathbb{N}$. Let $P(x)=\sum_{i=0}^T a_i x^i \in \mathbb{Q}[x]$. Let $f:\mathbb{N}\to\mathbb{Z}$ be a sequence such that $f(N)=P(N)+o(N)$ as $N\to\infty$. Suppose $\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^N (an+b)} \in \mathbb{Q}$. Then*

$$f(N) = P(N) - Q_1 \qquad \textit{for all } N$$

where Q_1 is given by (13).

Proof. We have, by Lemma 3.1,

$$\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N}(an+b)} = \sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N}(an+b)} + \sum_{N=1}^{\infty} \frac{f(N)-P(N)}{\prod_{n=1}^{N}(an+b)}$$

$$= Q_0 + Q_1 \sum_{N=1}^{\infty} \frac{1}{\prod_{n=1}^{N}(an+b)} + \sum_{N=1}^{\infty} \frac{f(N)-P(N)}{\prod_{n=1}^{N}(an+b)}$$

$$= Q_0 + \sum_{N=1}^{\infty} \frac{Q_1 + (f(N)-P(N))}{\prod_{n=1}^{N}(an+b)}.$$

The numerator of the fraction is a rational number which is $o(N)$ as $N \to \infty$ and has a denominator which is independent of N. Hence, by Lemma 2.2, $Q_1 + f(N) - P(N) = 0 \quad for\ N \geq N_0$. It follows that $f(N) = P(N) - Q_1$. □

Corollary 3.3 *Under the conditions of Theorem 3.2 we have* $P(N) \equiv Q_1 \mathrm{mod} 1$ *for all* N *and therefore* $dQ_1 \in \mathbb{Z}$.

Theorem 3.3 *Let* $a > 0$ *and* b *be integers such that* $an + b \neq 0$ *for every* $n \in \mathbb{N}$. *Let* $P(x) = \sum_{i=0}^{T} a_i x^i \in \mathbb{R}[x]$. *Let* $f : \mathbb{N} \to \mathbb{Z}$ *be a sequence such that* $f(N) = (aN + b)P(N) + O(1)$ *as* $N \to \infty$. *Suppose* $\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N}(an+b)} \in \mathbb{Q}$. *Then* $a_T, a_{T-1}, \ldots, a_1, a_0 \in \mathbb{Q}$.

Proof. Suppose $\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N}(an+b)} = \frac{p}{q}$ where p and $q > 0$ are coprime integers. Let U be the largest index i with $a_i \notin \mathbb{Q}$. Let d be a common denominator of $a_{U+1}, \ldots, a_T$. Put $P_1(x) = \sum_{i=0}^{U} a_i x^i$ and $P_2(x) = (ax+b) \sum_{i=U+1}^{T} a_i x^i$. Then we have, by Lemma 3.1,

$$a^T dp = a^T dq \sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N}(an+b)} =$$

$$q \sum_{N=1}^{\infty} \frac{a^T d(f(N) - P_2(N) + P_2(N))}{\prod_{n=1}^{N}(an+b)} =$$

$$qa^T dQ_{2,0} + q \sum_{N=1}^{\infty} \frac{a^T d(f(N) - P_2(N)) + a^T dQ_{2,1}}{\prod_{n=1}^{N}(an+b)}$$

where $Q_{2,0}$ and $Q_{2,1}$ are rational numbers corresponding to P_2 according to Lemma 3.1. From this and Lemma 2.1 we obtain that for every positive integer N the number

$$R_N^* := qa^T d \sum_{s=0}^{\infty} \frac{f(N+s) - P_2(N+s) + Q_{2,1}}{\prod_{n=N}^{N+s}(an+b)}$$

is an integer. It follows that

$$R_N^* = qa^T d \sum_{s=0}^{\infty} \frac{P_1(N+s)}{\prod_{n=N}^{N+s-1}(an+b)} + O(\frac{1}{N}).$$

This combined with Lemma 2.5 applied to $H(X) = qa^T dP_1(X+j)$ implies that the number

$$R_U^*(N) := \sum_{j=0}^{U} (-1)^j \binom{U}{j} R_{N+j}^* = (-1)^U qa^T dP_1^{(U)}(N) + O(\frac{1}{N}) =$$

$$(-1)^U qa^T U! da_U + O(\frac{1}{N})$$

is an integer. This is a contradiction for a sufficiently large number N. □

Corollary 3.4 *Let* $P(x) = \sum_{i=0}^{T} a_i x^i \in \mathbb{R}[x]$ *with nonnegative coefficients and* $a^T > 0$. *Then* $\sum_{N=1}^{\infty} \frac{[P(N)]}{N!} \notin \mathbb{Q}$.

Proof. Suppose $\sum_{N=1}^{\infty} \frac{[P(N)]}{N!} \in \mathbb{Q}$. It follows from Theorem 3.3 that $a_T, \ldots, a_1, a_0$ are rational. By Theorem 3.2 $P(N) - [P(N)] = Q_1$ for all N where Q_1 is a rational number. Hence Q_1 equals $a_0 - [a_0]$. It follows that $[P(x)] = \sum_{i=1}^{T} A_i x^i + [a_0] \in \mathbb{Q}[x]$. Since $[P(x)]$ has nonnegative coefficients and $a_T > 0$, we obtain a contradiction by applying Corollary 3.1. □

Remark The Corollary implies that for any positive integer K the sum $\sum_{n=1}^{\infty} \frac{[\beta n^K]}{n!}$ is a monotonically strictly increasing function of $\beta > 0$ which does not attain rational values. This phenomenon will be met several times later on.

Theorem 3.4 *Let* $K \geq 0$, $a > 0$ *and* b *be given integers such that* $an + b \neq 0$ *for every* $n \in \mathbb{N}$. *Let* $F : \mathbb{R}_+ \to \mathbb{R}_+$ *be a function such that*

$$F(N+j) = \sum_{r=0}^{K} \frac{F^{(r)}(N)}{r!} j^r + O\left(\frac{F(N)}{N^{K+1}}\right) \quad for\ j = 0, 1, \ldots, K\ as\ N \to \infty, \tag{18}$$

$$F^{(r)}(N) = O\left(\frac{F(N)}{N^r}\right) \quad for\ r = 0, 1, \ldots, K\ as\ N \to \infty, \tag{19}$$

$$\lim_{N\to\infty} F^{(K)}(N) = 0,\ \lim_{N\to\infty} \frac{N^{K+1}|F^{(K)}(N)|}{F(N)} = \infty. \tag{20}$$

and

$$\limsup_{N\to\infty} N|F^{(K)}(N)| = \infty. \tag{21}$$

Let $f : \mathbb{N} \to \mathbb{Z}$ *be a sequence such that* $R^* := \sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N}(an+b)}$ *is absolutely convergent and* $f(N) = (aN+b)F(N) + O(1)$ *as* $N \to \infty$. *Then* R^* *is irrational.*

Proof. Let N be sufficiently large. Suppose $R^* = \frac{p}{q}$ where p and $q > 0$ are coprime integers. Put

$$R_K^*(N) = \sum_{j=0}^{K} (-1)^j \binom{K}{j} \sum_{s=0}^{\infty} \frac{f(N+j+s)}{\prod_{n=N+j}^{N+j+s}(an+b)}.$$

By Lemma 2.1, $qR_K^*(N)$ is an integer for every positive integer N. We have, by Lemma 2.5,

$$R_K^*(N) = \sum_{j=0}^{K} (-1)^j \binom{K}{j} \sum_{s=0}^{\infty} \frac{F(N+j+s)}{\prod_{n=N+j}^{N+j+s-1}(an+b)} + O(\frac{1}{N}) =$$

$$(-1)^K F^{(K)}(N) + O\left(\frac{F(N)}{N^{K+1}} + \frac{1}{N}\right) =$$

$$(-1)^K F^{(K)}(N) \left(1 + O\left(\frac{F(N)}{N^{K+1}|F^{(K)}(N)|}\right)\right) + O(\frac{1}{N}).$$

By (20) the right-hand side tends to 0 as $N \to \infty$. Since $qR_K^*(N)$ is an integer, we infer that $R_K^*(N) = 0$ for $N \geq N_0$. It follows that $0 = (1+o(1))NF^{(K)}(N) + O(1)$ for $N \geq N_0$ which contradicts (21). □

Corollary 3.5 *Let* $\alpha \in \mathbb{R}_{\geq 0}, \gamma \in \mathbb{R}_+$. *Then* $\sum_{N=1}^{\infty} \frac{[\gamma N^\alpha]}{N!} \notin \mathbb{Q}$.

Proof. If $\alpha \notin \mathbb{Z}$, then apply Theorem 3.4 with $a = 1, b = 0, K = [\alpha], f(N) = [\gamma N^\alpha], F(N) = \gamma N^{\alpha-1}$. If $\alpha \in \mathbb{Z}$, then apply Corollary 3.4 with $T = \alpha$. □

Remark It is remarkable that Corollary 3.5 holds for all $\alpha \geq 0$ and $\gamma > 0$ so that $\sum_{N=1}^{\infty} \frac{[\gamma N^\alpha]}{N!}$ is a strictly monotonic function of α and γ missing all rational values.

Corollary 3.6 *Let* $\alpha \in \mathbb{R}_{\geq 0} \setminus \mathbb{Z}, \gamma \in \mathbb{R}_+$. *Then* $\sum_{N=1}^{\infty} \frac{[\gamma N^\alpha \log N]}{N!} \notin \mathbb{Q}$.

Proof. Apply Theorem 3.4 with $a = 1, b = 0, K = [\alpha], f(N) = [\gamma N^\alpha \log N], F(N) = \gamma N^{\alpha-1} \log N$. □

Theorem 3.4 is not strong enough to prove the irrationality of $\sum_{N=1}^{\infty} \frac{[N \log N]}{N!}$. For such series we provide a variant of Theorem 3.4 where condition (21) becomes weaker, but conditions (18) and (19) become stronger.

Theorem 3.5 *Let $K \geq 1$, $a > 0$ and b be given integers such that $an + b \neq 0$ for every $n \in \mathbb{N}$. Let $F : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that*

$$F(N+x) = \sum_{r=0}^{\infty} \frac{F^{(r)}(N)}{r!} x^r \quad for\ x = o(N) \text{ as } N \to \infty, \tag{22}$$

$$F^{(r)}(N) = O\left(r!\frac{F(N)}{N^r}\right) \quad uniformly\ for\ r = 0, 1, \ldots \text{ as } N \to \infty, \tag{23}$$

$$\lim_{x\to\infty} F^{(K)}(x) = 0,\ \lim_{x\to\infty} \frac{x^{K+1}|F^{(K)}(x)|}{F(x)} = \infty \tag{24}$$

and

$$\lim_{x\to\infty} x^2|F^{(K)}(x)| = \infty. \tag{25}$$

Let $f : \mathbb{N} \to \mathbb{Z}$ be a sequence such that $R^* := \sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N}(an+b)}$ is absolutely convergent and $f(N) = (aN + b)F(N) + O(1)$ as $N \to \infty$. Then R^* is irrational.

Proof. Let N be sufficiently large. Suppose $R^* = \frac{p}{q}$ where p and $q > 0$ are coprime integers. Conditions (22) and (23) imply conditions (18) and (19). Hence, by Theorem 3.4, we may assume without loss of generality that

$$NF^{(K)}(N) = O(1). \tag{26}$$

Observe that by (23) and (22), for $x = o(N)$,

$$F(N + x) \leq \sum_{r=0}^{\infty} |F^{(r)}(N)|\frac{x^r}{r!} = O\left(F(N)\sum_{r=0}^{\infty}(\frac{x}{N})^r\right) = O(F(N)). \tag{27}$$

Put

$$R^*_{K-1}(N) = \sum_{j=0}^{K-1}(-1)^j\binom{K-1}{j}\sum_{s=0}^{\infty}\frac{f(N+j+s)}{\prod_{n=N+j}^{N+j+s}(an+b)}.$$

By Lemma 2.1, $qR^*_{K-1}(N)$ is an integer for every positive integer N. We have, by Lemma 2.5,

$$R^*_{K-1}(N) = \sum_{j=0}^{K-1}(-1)^j\binom{K-1}{j}\sum_{s=0}^{\infty}\frac{F(N+j+s)}{\prod_{n=N+j}^{N+j+s-1}(an+b)} + O(\frac{1}{N}) =$$

$$(-1)^{K-1}F^{(K-1)}(N) + O\left(\frac{F(N)}{N^K} + \frac{1}{N}\right) + O(\frac{1}{N}). \tag{28}$$

Put

$$t = 1 + \left[\frac{N}{\min_{x\in[N,2N]}(x^2|F^{(K)}(x)|)^{\frac{1}{2}}} + \frac{N}{\min_{x\in[N,2N]}(\frac{x^{K+1}|F^{(K)}(x)|}{F(x)})^{\frac{1}{2}}}\right]. \tag{29}$$

It follows from (25) and (24) that $t = o(N)$ as $N \to \infty$. Hence, by (27), similar to (28),

$$qR^*_{K-1}(N+t) = (-1)^{K-1}qF^{(K-1)}(N+t) + O\left(\frac{F(N)}{N^K} + \frac{1}{N}\right) \tag{30}$$

is an integer. We apply the Mean Value Theorem to (30) and (28). Hence there exists a real number τ with $0 < \tau < t$ such that

$$M(N) := qR^*_{K-1}(N+t) - qR^*_{K-1}(N) = (-1)^{K-1} qtF^{(K)}(N+\tau) + O\left(\frac{F(N)}{N^K} + \frac{1}{N}\right)$$

is an integer. It follows from (29), (27) and (24) that, for some positive constant c,

$$\frac{tN^K|F^{(K)}(N+\tau)|}{F(N)} \geq \frac{N^{K+1}|F^{(K)}(N+\tau)|}{F(N)\min_{x\in[N,2N]}(\frac{x^{K+1}|F^{(K)}(x)|}{F(x)})^{\frac{1}{2}}}$$

$$\geq c\left(\frac{(N+\tau)^{K+1}|F^{(K)}(N+\tau)|}{F(N+\tau)}\right)^{\frac{1}{2}} \to \infty \text{ as } N \to \infty.$$

Hence, by (26) and $t = o(N)$,

$$M(N) = (-1)^{K-1}(1+o(1))qtF^{(K)}(N+\tau) + O(\frac{1}{N}) \to 0. \qquad (31)$$

Thus, since $M(N)$ represents an integer, $M(N) = 0$ for $N \geq N_1$. It follows from (31), (29) and (27) that

$$0 = Nt|F^{(K)}(N+\tau)| + O(1) \geq$$

$$\frac{N^2|F^{(K)}(N+\tau)|}{\min_{x\in[N,2N]}(x^2|F^{(K)}(x)|)^{\frac{1}{2}}} + O(1) \geq \left((N+\tau)^2|F^{(K)}(N+\tau)|\right)^{\frac{1}{2}} + O(1)$$

which contradicts (25). □

Corollary 3.7 *Let $\alpha \in \mathbb{R}_{\geq 0}, \beta \in \mathbb{R}, \beta \neq 0, \gamma \in \mathbb{Q}_+$. Suppose $\beta > 0$ whenever $\alpha = 0$. Then $\sum_{N=1}^{\infty} \frac{[\gamma N^\alpha \log^\beta N]}{N!} \notin \mathbb{Q}$.*

Proof. Apply Theorem 3.5 with $a = 1, b = 0, K = [\alpha]$ if $\alpha \notin \mathbb{Z}$ or $\beta > 0$, and $K = \alpha - 1$ otherwise , $f(N) = [\gamma N^\alpha \log^\beta N], F(N) = \gamma N^{\alpha-1}\log^\beta N$. □

Corollary 3.8 *Let $\alpha \in \mathbb{R}_{\geq 0}, 0 < \beta < 1, \gamma \in \mathbb{Q}_+$. Then $\sum_{N=1}^{\infty} \frac{[\gamma N^\alpha \exp(\log^\beta N)]}{N!} \notin \mathbb{Q}$.*

Proof. Apply Theorem 3.5 with $a = 1, b = 0, K = [\alpha], f(N) = [\gamma N^\alpha \exp(\log^\beta N)]$, $F(N) = \gamma N^{\alpha-1}\exp(\log^\beta N)]$. □

4 Linear independence

The method from the previous section enables us to prove the linear independence of the considered sums. Recall that Theorem 3.1 precisely states when $\sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N}(an+b)}$ for $P(x) \in \mathbb{Z}[x]$ is rational.

Theorem 4.1 *Let $a > 0$ and b be integers such that $an + b \neq 0$ for every $n \in \mathbb{N}$. Suppose that $P(x) = \sum_{i=0}^{T} a_i x^i \in \mathbb{Z}[x]$ and that $\sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N}(an+b)}$ is irrational. Let W be a set of functions $F : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy the following properties.*

(i)
$$F(N+x) = \sum_{r=0}^{\infty} \frac{F^{(r)}(N)}{r!} x^r \quad for\ x = o(N)\ \ as\ \ N \to \infty, \tag{32}$$

(ii)
$$F^{(r)}(N) = O(\frac{r!F(N)}{N^r})\ \ uniformly\ for\ r = 0, 1, \ldots\ as\ n \to \infty, \tag{33}$$

(iii) *either there exists a positive integer K such that*

$$F^{(K)}(x) = o(1),\ \frac{F(x)}{x^{K+1}} = o(|F^{(K)}(x)|),\ \lim_{x\to\infty} x^2|F^{(K)}(x)| = \infty \tag{34}$$

or

$$K = 0,\ \lim_{x\to\infty} F(x) = 0,\ \lim_{x\to\infty} xF(x) = \infty, \tag{35}$$

and (iv) *for every pair of functions $F, G \in W$ with corresponding integers $K > L$ one has* $\lim_{x\to\infty} \frac{G^{(k)}(x)}{F^{(k)}(x)} = 0$ *for $k = 0, 1, \ldots, K$; for every pair of functions $F, G \in W$ with $F \neq G$ and corresponding integers $K = L$ one has either* $\lim_{x\to\infty} \frac{G^{(k)}(x)}{F^{(k)}(x)} = 0$ *for $k = 0, 1, \ldots, K$ or* $\lim_{x\to\infty} \frac{F^{(k)}(x)}{G^{(k)}(x)} = 0$ *for $k = 0, 1, \ldots, K$.*
Suppose that for every function $F \in W$ there exists a function $f : \mathbb{N} \to \mathbb{Z}$ such that $\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N}(an+b)}$ is absolutely convergent and $f(N) = (aN + b)F(N) + O(1)$ as $N \to \infty$. Then the numbers $\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N}(an+b)} (f \in W), \sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N}(an+b)}$ and 1 are linearly independent over the rationals.

Proof. Suppose that there exist functions $f_0 := P, f_1, f_2, \ldots, f_M \in W$ such that the numbers $\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N}(an+b)}$ $(i = 0, 1, \ldots, M)$ and the number 1 are linearly dependent over the rationals. Then there exist integers $A_0, A_1, A_2, \ldots, A_M, p$ and $q > 0$, not all zero, such that

$$\frac{p}{q} = \sum_{m=0}^{M} A_m \sum_{N=1}^{\infty} \frac{f_m(N)}{\prod_{n=1}^{N}(an+b)}. \tag{36}$$

It is excluded that $A_1 = \ldots = A_M = 0$. Without loss of generality we may assume that $A_1, \ldots, A_M$ are all nonzero and $M > 0$. Lemma 3.1 and equation

(36) imply

$$\frac{p}{q} = A_0 \sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N}(an+b)} + \sum_{m=1}^{M} A_m \sum_{N=1}^{\infty} \frac{f_m(N)}{\prod_{n=1}^{N}(an+b)} =$$

$$= \frac{p_1}{q_1} + \frac{p_2}{q_2} \sum_{N=1}^{\infty} \frac{1}{\prod_{n=1}^{N}(an+b)} + \sum_{N=1}^{\infty} \frac{\sum_{m=1}^{M} A_m f_m(N)}{\prod_{n=1}^{N}(an+b)}$$

where $p_1, p_2, q_1 > 0$ and $q_2 > 0$ are suitable integers which do not depend on N. Thus

$$A := q_1 q_2 q(\frac{p}{q} - \frac{p_1}{q_1}) = q_1 q \sum_{N=1}^{\infty} \frac{p_2 + q_2 \sum_{m=1}^{M} A_m f_m(N)}{\prod_{n=1}^{N}(an+b)}$$

is an integer. By Lemma 2.1 we obtain that for every positive integer N the number

$$S_N := q_1 q \sum_{s=0}^{\infty} \frac{p_2 + q_2 \sum_{m=1}^{M} A_m f_m(N+s)}{\prod_{n=N}^{N+s}(an+b)}$$

is an integer. Hence

$$S_N = q_1 q_2 q \sum_{s=0}^{\infty} \frac{\sum_{m=1}^{M} A_m F_m(N+s)}{\prod_{n=N}^{N+s-1}(an+b)} + O(\frac{1}{N}). \tag{37}$$

Without loss of generality we may assume by (iv) and (iii) that if K is the integer corresponding to F_M, then

$$\lim_{x\to\infty} \frac{F_m^{(k)}(x)}{F_M^{(k)}(x)} = 0 \quad \text{for } k = 0, 1, \ldots, K \text{ and } m = 0, 1, \ldots, M-1. \tag{38}$$

Let L be a nonnegative integer with $L \leq K$. We obtain from (37) that

$$R_L^*(N) := \sum_{j=0}^{L} (-1)^j \binom{L}{j} S_{N+j} =$$

$$q_1 q_2 q \sum_{j=0}^{L} (-1)^j \binom{L}{j} \sum_{s=0}^{\infty} \left(\frac{\sum_{m=1}^{M} A_m F_m(N+s+j)}{\prod_{n=N}^{N+s+j-1}(an+b)} \right) + O(\frac{1}{N})$$

is an integer too. From Lemma 2.5, (32), and (33) we deduce that

$$R_L^*(N) = (-1)^L q_1 q_2 q \sum_{m=1}^{M} A_m F_m^{(L)}(N) + \sum_{m=1}^{M} O(F_m(N) N^{-L-1}) + O(\frac{1}{N}).$$

Hence, by (38),

$$R_L^*(N) = (-1)^L q_1 q_2 q \sum_{m=1}^{M} A_m F_m^{(L)}(N) + O(F_M(N) N^{-L-1}) + O(\frac{1}{N}). \tag{39}$$

We distinguish between two cases.
(a) Suppose $\limsup_{N\to\infty} N|F_M^{(K)}(N)| = \infty$. Then we put $L = K$ and find by (38) that

$$R_K^*(N) = (-1)^K q_1 q_2 q A_M F_M^{(K)}(N)(1 + o(1)) + O(F_M(N)N^{-K-1}) + O(\frac{1}{N}).$$

Hence we derive a contradiction in the same way as we did in the last lines of the proof of Theorem 3.4.
(b) Suppose $\limsup_{N\to\infty} N|F_M^{(K)}(N)| < \infty$, i.e. $F_M^{(K)}(N) = O(\frac{1}{N})$ as $N \to \infty$. Note that $K \geq 1$ in view of (35). By (27) and (38) we have, for $m = 1, \dots, M$ and $x = o(N)$, that $F_m(N + x) = O(F_m(N)) = O(F_M(N))$. Put $L = K - 1$ in (39). Hence

$$R_{K-1}^*(N) = (-1)^{K-1} q_1 q_2 q \sum_{m=1}^{M} A_m F_m^{(K-1)}(N) + O(F_M(N)N^{-K}) + O(\frac{1}{N}).$$

Define t as in (29). We apply the Mean Value Theorem to the integer $M(N) := R_{K-1}^*(N + t) - R_K^*(N)$. We obtain that

$$M(N) = (-1)^{K-1} q_1 q_2 q t \sum_{m=1}^{M} A_m F_m^{(K)}(N + \tau) + O(F_M(N)N^{-K}) + O(\frac{1}{N})$$

for some τ with $0 < \tau < t$. By (38) we have

$$M(N) = (-1)^{K-1} q_1 q_2 q t(1 + o(1)) A_M F_M^{(K)}(N + \tau) + O\left(\frac{F_M(N)}{N^K} + \frac{1}{N}\right)$$

The further proof proceeds as the proof of Theorem 3.5 from the introduction of $M(N)$ on. $\square$

Remark. It follows from a repeated use of l'Hôpital's rule that condition (iv) can be relaxed. If $\lim_{x\to\infty} \frac{F^{(K)}(x)}{G^{(K)}(x)} = 0$ and $\lim_{x\to\infty} G^{(K-1)}(x) = \infty$, then $\lim_{x\to\infty} \frac{F^{(k)}(x)}{G^{(k)}(x)} = 0$ for $k = 0, 1, \dots, K$. (See [13], Theorem 5.13 with $A = 0$ and $a = \infty$.)

Corollary 4.1 *The numbers* 1, e *and* $\sum_{n=1}^{\infty} \frac{[n^\alpha]}{n!}$ $(\alpha \in \mathbb{R}_+, \alpha \notin \mathbb{Z})$ *are linearly independent over the rationals.*

Corollary 4.2 *Let* $\alpha_1, \dots, \alpha_M$ *be positive real numbers and* $P_1, P_2, \dots, P_M$ *nonzero polynomials with integer coefficients such that the numbers* $\alpha_m \deg P_m$ *are distinct and nonintegral. Then the numbers* $1, e$ *and* $\sum_{N=1}^{\infty} \frac{[N^{\alpha_m} P_m(N)]}{N!}$ $(m = 1, 2, \dots, M)$ *are linearly independent over the rationals.*

Proof. Apply Theorem 4.1 with $P(x) = x$ and $F_m(X) = x^{\alpha_m} P_m(x)$ for $m = 1, \dots, M$. $\square$

Corollary 4.3 *The numbers* 1 *and* $\sum_{n=1}^{\infty} \frac{[n(\log n)^{\alpha}]}{n!}$ $(\alpha \in \mathbb{R})$ *are linearly independent over the rationals.*

Remark Conditions (i), (ii) and (iii) of Theorem 4.1 are satisfied by a large class of functions comprising

$$\gamma x^{\alpha} \quad (\alpha > -1, \alpha \notin \mathbb{Z}, \gamma \in \mathbb{R}_+) \quad \text{with } K = [\alpha] + 1$$

$$\gamma e^{\beta(\log x)^{\alpha}} \quad (0 < \alpha < 1, \beta \in \mathbb{R}_+, \gamma \in \mathbb{R}_+) \quad \text{with } K = 1$$

$$\gamma(\log x)^{\alpha} \quad (\alpha \neq 0, \gamma \in \mathbb{R}_+) \quad \text{with } K = 1 \text{ if } \alpha > 0, K = 0 \text{ if } \alpha < 0$$

$$\gamma(\log \log x)^{\alpha} \quad (\alpha \neq 0, \gamma \in \mathbb{R}_+) \quad \text{with } K = 1 \text{ if } \alpha > 0, K = 0 \text{ if } \alpha < 0.$$

It is therefore possible to apply Theorem 4.1 to sums and products of such functions and polynomials provided that condition (iv) is satisfied.

Example 4.1 *The numbers* 1, $\sum_{n=1}^{\infty} \frac{[(\log n)^{\frac{1}{2}}]}{n!}$ *and* $\sum_{n=1}^{\infty} \frac{[e^{(\log n)^{\frac{1}{2}}}]}{n!}$ *are linearly independent over the rationals.*

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