# SUBSTITUTIONS, ABSTRACT NUMBER SYSTEMS AND THE SPACE FILLING PROPERTY

CLEMENS FUCHS* AND ROBERT TIJDEMAN

Abstract. In this paper we study multi-dimensional words generated by fixed points of substitutions by projecting the integer points on the corresponding broken halfline. We show for a large class of substitutions that the resulting word is the restriction of a linear function mod 1 and that it can be decided whether the resulting word is space filling or not. The proof uses continued lattices and the abstract number system associated with the substitution.

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## 1. Introduction

In 1982 Rauzy [32] introduced the Rauzy fractal as a closure of an infinite sequence of points. He proved that the three parts composing it have disjoint interiors with area 1 and that it forms a tile of $\mathbb{R}^2$ with very nice properties. Since then many researchers have studied the corresponding structures for other substitutions, with quite diverse outcomes (cf. [2, 5, 37, 41, 15, 4, 17, 39, 23, 44]), but a general rule for deciding when a substitution leads to simple tiling of a space is still wanted, especially, because these structures turned out to be useful in the mathematical theory of quasicrystals (for details see [38, 7, 22, 9]). The authors want to reveal some of the underlying principles and thereby to pave the way to a satisfying answer.

The present paper is a generalization of some phenomena which were observed in the special case of the Tribonacci substitution $\sigma : \{0,1,2\} \to \{0,1,2\}$ given by $\sigma(0) = 01, \sigma(1) = 02, \sigma(2) = 0$ by Rosema and Tijdeman [35]. The repeated application of the substitution $\sigma$ with as start value 0 and concatenation as operation yields a sequence of finite words: $u^{(0)} = 0, u^{(1)} = 01, u^{(2)} = 0102, u^{(3)} = 0102010, u^{(4)} = 0102010010201, \ldots$. The limit word $U := 0102010010201010201001020102010010201 01\ldots$ is called the fixed point of $\sigma$. An important role in the analysis is played by the incidence matrix $M_\sigma$ of $\sigma$, the row vectors of which are the

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incidence vectors of $\sigma(0), \sigma(1), \sigma(2)$, respectively, hence in the present case $(1,1,0),(1,0,1)$ and $(1,0,0)$. Its characteristic polynomial equals $x^3 - x^2 - x - 1$. It has one real root $\beta$ with modulus greater than one and two complex roots with modulus smaller than 1. The eigenvectors corresponding to the eigenvalue $\beta$ of $M_\sigma$ equal $\mathbb{R}(\beta^2, \beta, 1)$. From the word $U$ a sequence of points in $\mathbb{Z}^3$ is constructed by putting $P_0 = \vec{0}$ and $P_{m+1} = P_m + \vec{e}_i$ for $m = 0, 1, 2, \dots$ if and only if the letter of $U$ at the position $m$ equals $i$ where $\vec{e}_i$ denotes the $i$-th unit vector. So $P_1 = (1,0,0), P_2 = (1,1,0), P_3 = (2,1,0), P_4 = (2,1,1), P_5 = (3,1,1), \dots$. These points approximate the line $\mathbb{R}(\beta^2, \beta, 1)$ very well. If the points $P_m$ $(m = 0, 1, \dots)$ are projected parallel to this line on a plane, then the famous Rauzy fractal is formed [32]. Instead Rosema and Tijdeman projected the points $P_m$ generated by each word $u^{(n)}$ parallel to the line through the starting point $P_0$ and the end point $P_m$, where $m$ is equal to the length of $u^{(n)}$, on the $(y,z)$-plane. This leads to a sequence of structures which have the Rauzy fractal as a limit. However, by applying a suitable linear transformation depending on $n$ to the $n$-th structure, an increasing sequence of two-dimensional words $(w^{(n)})_{n \geq 0}$ was obtained. It was shown that the derived limit word has the full lattice $\mathbb{Z}^2$ as domain (so it was space-filling) (cf. [35, Theorem 4.7]). Moreover, the normalized value of the word at some lattice point is determined by a linear function modulo 1 (cf. [35, Theorem 4.7]) and the development of the projected words is fully reflected by some two-dimensional representation of the Tribonacci number system (cf. [35, Lemma 4.6]).

The present paper deals with an arbitrary given substitution $\sigma$ mapping $\{0, 1, \dots, k\}$ to finite words in these letters. We assume that $\sigma$ has an invariant word starting with 0. In Section 2 we define the incidence matrix $M_\sigma$, we derive recurrence relations for its entries and show that the growth order of these entries is determined by the eigenvalues with largest modulus. We make the assumption that $M_\sigma$ has a dominant eigenvalue $\beta$ and that all the components of the corresponding eigenvector are positive. Finally, we define the corresponding broken halfline in $\mathbb{R}^{k+1}$. The results in this section are based on the classical Perron-Frobenius theorem.

In Section 3 we define the projection of the integer points on the broken halfline to some hyperplane. We leave the subsequent linear transformation free, but mention some natural choices. One of them, called canonical, leads to simplified formulas. We make the further assumption that the incidence matrix $M_\sigma$ is unimodular. Theorems 2 and 3 describe the structure of the projected word $w^{(n)}$. We define the normalized word $\hat{w}^{(n)}$ and show in Theorem 4 that the limit word $\hat{W}$ has a linear structure mod 1. The method in this section is taken from Berthé and Tijdeman [10].

In Section 4 we define the automaton, the language and the number system associated to $\sigma$. We make a further assumption that $\beta > 1$. We show that the corresponding eigenvector plays a fundamental role in the description of the number system. We present an algorithm to compute the representation of an arbitrary number from $[0,1)$ in this number system. Theorem 4 shows that we have a bijection between the half-open interval $[0,1)$ and the infinite words in the associated language. Theorem 5 gives a certain finiteness criterion, provided that $\beta$ is a Pisot number and states on the other hand that if all elements in $\mathbb{Z}[\beta^{-1}] \cap [0,1)$ have a finite expansion, then $\beta$ is a Pisot number or a Salem number. It is remarkable that this criterion (in the Pisot case) is finite and can be checked effectively. In our analysis of associated number systems we rely on results of Frougny and Solomyak [20] and Akiyama [1].

The results from Sections 2, 3 and 4 are combined in Section 5. Theorem 6 describes the relation between the word $w^{(n)}$ and the number system associated with $\sigma$. It turns out that $w^{(n)}$ consists exactly of the words in the associated language of length at most $n$. Theorem 7 gives a criterion for the limit word $\hat{W}$ to be space filling. As before the criterion is decidable if $\beta$ is a Pisot number. If $\hat{W}$ is space filling, then the dimension of the space filling word $\hat{W}$ equals $\deg\beta - 1$.

In the final Section 6, we give five examples. Example 1 concerns a substitution on three letters leading to a two-dimensional space filling word. Example 2 deals with the flipped Tribonacci substitution which yields a word which is not space filling. Example 3 treats a substitution on three letters with a dominant root of degree 2 and a resulting space filling word of dimension 1. Example 4 shows how a substitution on five letters with a dominant root of degree 3 and two roots of modulus 1 generates a space filling word of dimension 2. Finally, Example 5 is concerned with a substitution on four letters with a dominant Pisot root of degree 2 and two other roots, one outside and one inside the unit circle, that generates a one-dimensional word which is not space filling.

Finally we note that some of the made assumptions are just for convenience, but that this is not the case for the assumptions that $M_\sigma$ is unimodular and has a dominant root $\beta > 1$ and that all components of the corresponding eigenvector are positive.

## 2. Limit word and discretisation of the hyperplane

Let $\Sigma$ be a finite set. By $\Sigma^*$ we denote the set of finite words over the alphabet $\Sigma$ including the empty word $\varepsilon$. A *substitution* is a map $\sigma : \Sigma \to \Sigma^* \backslash \{\varepsilon\}$. This map can be extended to a map $\sigma : \Sigma^* \to \Sigma^*$ by letting $\sigma(\varepsilon) = \varepsilon$ and $\sigma(ws) = \sigma(w)\sigma(s)$ for $w \in \Sigma^*, s \in \Sigma$ where as usual the operation is the concatenation of words. (With this operation the set $\Sigma^*$ is a free monoid with identity $\varepsilon$ generated by $\Sigma$.) We denote by $|w|$ the length of the word $w \in \Sigma^*$, i.e. $|w| = n$ if $w \in \Sigma^n$. Moreover, we set $|w|_s$ for $w \in \Sigma^*, s \in \Sigma$ to be the number of occurrences of letter $s$ in the word $w$. Hence $\sum_{s\in\Sigma} |w|_s = |w|$. The column vector $\vec{w}$ whose components are the quantities $|w|_s$ for

$s \in \Sigma$ is called the *incidence vector* of the word $w$. In the paper the vectors are defined as column vectors and row vectors are presented as transposed vectors. Transposition is denoted by a left superscript $t$.

Let $\mathbb{N}$ denote the non-negative integers and $\Sigma^\omega$ the set of right-infinite words over the alphabet $\Sigma$, i.e. the set of sequences of letters from $\Sigma$ indexed by non-negative integers. We equip $\Sigma^\omega$ with the usual discrete product metric: for $V, W \in \Sigma^\omega$ we set $d(V,W) = 2^{-i}$, where $i$ is the smallest integer such that $V_i \neq W_i$ $(V = (V_n)_{n\in\mathbb{N}}, W = (W_n)_{n\in\mathbb{N}})$ if such an index exists and $i = \infty$ if $V = W$. Convergence of sequences is considered with respect to this metric. Observe that any finite word $w \in \Sigma^*$ can be viewed as an infinite word $w\zeta^\omega \in (\Sigma \cup \{\zeta\})^\omega$ for some $\zeta \notin \Sigma$. So a sequence $(w^{(n)})_{n\geq 0}$ of finite words converges to an infinite word $W$ if and only if for every $l \in \mathbb{N}$ there exists an $N_l \in \mathbb{N}$ such that the first $l$ letters of $W$ and $w^{(n)}$ coincide for every $n \geq N_l$. We denote by $(w)_i$ the letter at position $i$ in the finite or infinite word $w$.

We can extend $\sigma$ to $W = (W_n)_{n\in\mathbb{N}}$ in $\Sigma^\omega$ by putting $\sigma(W) = (\sigma(W_n))_{n\in\mathbb{N}}$ and applying infinite concatenation. Observe that if $(w^{(n)})_{n\geq 0}$ converges to $W \in \Sigma^\omega$ then $(\sigma(w^{(n)}))_{n\geq 0}$ converges to $\sigma(W)$. We say that a finite or infinite word $w$ is a *fixed point* of the substitution $\sigma$ if $\sigma(w) = w$.

In the sequel we will always assume $\Sigma = \{0, 1, \dots, k\}$ and consider the sequence $(u^{(n)})_{n\geq 0}$ given by

$$u^{(n)} = \sigma^n(0) \qquad (n = 0, 1, \dots).$$

Suppose the sequence $u = (u^{(n)})_{n\geq 0}$ converges to a limit word $U = (U_n)_{n\in\mathbb{N}}$ which is a fixed point of the substitution $\sigma$. By relabeling the alphabet $\Sigma$ we may assume that $U_0 = 0$. Since by definition there exists a $N_1$ such that $U_0 = (u^{(n)})_0$ for all $n \geq N_1$, it follows that $(\sigma((u^{(n)})_0))_0 = (u^{(n+1)})_0$. Therefore, we have that

$$\sigma(0) = 0v, \;\; v \in \Sigma^*. \tag{A1}$$

For the sake of simplicity we will always assume that the substitution $\sigma$ satisfies (A1). Therefore $U := \lim_{n\to\infty} u^{(n)}$ is a well-defined infinite word $U_0U_1U_2\cdots$ with $U_0 = 0$.

Let

$$\vec{u}_n = {}^t(|\sigma^n(0)|_0, \dots, |\sigma^n(0)|_k)$$

be the so-called incidence vector of $u^{(n)} = \sigma^n(0)$ and

$$\vec{v}_n = {}^t(|\sigma^n(0)|, \dots, |\sigma^n(k)|).$$

Of special interest is the first coordinate of $\vec{v}_n$, which generates the sequence of lengths $(|u^{(n)}|)_{n\in\mathbb{N}}$ of the sequence $(u^{(n)})_{n\geq 0}$. Put $s^{(n)} = |u_n|$ for $n \in \mathbb{N}$. We will discuss the convergence of the vectors $\vec{u}_n/|u^{(n)}|$. Define the *incidence* *matrix* $M_\sigma$ of the substitution $\sigma$ by

$$M_\sigma = \Big(|\sigma(i)|_j\Big)_{i=0,\dots,k;j=0,\dots,k} \in \mathbb{N}^{(k+1)\times(k+1)}.$$

The incidence matrix contains the global information (the numbers of each letter) of the substitution $\sigma$, but not the local information (the precise order). Let $x^{k+1} - g_k x^k - \cdots - g_0$ be the characteristic equation of $M_\sigma$. Then $M_\sigma^{k+1} = g_k M_\sigma^k + \cdots + g_0 M_\sigma^0$ where $g_k, \dots, g_0$ are rational integers.

It is easy to see that both sequences $\vec{u}_n$ and $\vec{v}_n$ satisfy a linear recurrence relation where $M_\sigma$ is involved. We have

$$(1) \qquad {}^t\vec{u}_n M_\sigma = {}^t\vec{u}_{n+1} \quad \text{and} \quad M_\sigma \vec{v}_n = \vec{v}_{n+1},$$

respectively. Hence

$$\vec{u}_{n+k+1} \;=\; g_k \vec{u}_{n+k} + \cdots + g_0 \vec{u}_n$$

and similarly for $\vec{v}_n$. Therefore the components of the vectors $\vec{u}_n, \vec{v}_n$ satisfy the linear recurrence relation. In particular, $s^{(n+k+1)} = g_k s^{(n+k)} + \cdots + g_0 s^{(n)}$ $(n = 0, 1, \dots)$. Denoting by $\vec{e}_0^{(k+1)}, \dots, \vec{e}_k^{(k+1)}$ the unit column vectors in $\mathbb{R}^{k+1}$, it follows by induction on $n$ that ${}^t M_\sigma^n \vec{e}_0^{(k+1)}$ is the incidence vector of $u^{(n)}$, hence $s^{(n)}$ is the sum of the entries in the row with index 0 of $M_\sigma^n$. More generally we have

**Lemma 1.** *Let $n \geq 0$ and $i \in \{0, 1, \dots, k\}$. Then the $i$-th row vector of $M_\sigma^n$ equals*

$$(|\sigma^n(i)|_0, |\sigma^n(i)|_1, \dots, |\sigma^n(i)|_k)$$

*and the vector of the row sums of $M_\sigma^n$ equals $\vec{v}_n$.*

*Proof.* The assertion holds by definition for $n = 0$. Suppose the statement holds for $n$. Then the $i$-th row vector of $M_\sigma^{n+1}$ equals

$$\begin{aligned}
{}^t\vec{e}_i^{(k+1)} M_\sigma^{n+1} &= {}^t\vec{e}_i^{(k+1)} M_\sigma^n M_\sigma = \sum_{j=0}^{k} |\sigma^n(i)\,|_j \; {}^t\vec{e}_j^{(k+1)} M_\sigma \\
&= \left( \sum_{j=0}^{k} |\sigma^n(i)|_j |\sigma(j)|_0, \dots, \sum_{j=0}^{k} |\sigma^n(i)|_j |\sigma(j)|_k \right) \\
&= (|\sigma^{n+1}(i)|_0, \dots, |\sigma^{n+1}(i)|_k).
\end{aligned}$$

A similar argument proves the second part of the lemma. □

Furthermore, we have the following lemma (where as usual we use the notation $f(x) = o(g(x))$ for $f(x)/g(x) \to 0$ if $x \to \infty$).

**Lemma 2.** *There exist an algebraic integer $\beta \geq 1$, a $p \in \mathbb{N}$ and vectors $\vec{P}_j, \vec{Q}_j$ $(j = 0, 1, \ldots, p-1)$ with entries from $\mathbb{Q}(\beta\xi_p)[x]$ such that*

$$\vec{u}_n = \sum_{j=0}^{p-1} \vec{P}_j(n)\left(\xi_p^j\beta\right)^n + o\left(\beta^n\right), \quad \vec{v}_n = \sum_{j=0}^{p-1} \vec{Q}_j(n)\left(\xi_p^j\beta\right)^n + o\left(\beta^n\right),$$

*where $\xi_p$ denotes a primitive $p$-th root of unity.*

*Proof.* The result follows immediately from the Perron-Frobenius Theorem [8, Theorem 1.4.4] if the incidence matrix $M_\sigma$ is irreducible (cf. [8, p. 2]). If $M_\sigma$ is not irreducible, we can write it (after a relabeling of the alphabet not involving 0) as an upper diagonal block matrix. We denote the square blocks along the diagonal by $M_1, \ldots, M_r$ with $M_1, \ldots, M_r$ irreducible. Since the characteristic polynomial of such a block matrix is given by the product of the characteristic polynomials of the $M_k$, it suffices to consider them. Observe that since all the matrices $M_k$ have non-negative integral entries it follows that the characteristic roots are algebraic integers and therefore at least one of the characteristic roots is $\geq 1$. Now by the Perron-Frobenius Theorem for irreducible non-negative matrices [8, Theorem 1.4.4] it follows that all the characteristic roots of maximum modulus of the matrices $M_k$ are given by $\xi_{p_k}^h\beta_k$ $(h = 0, \ldots, p_k - 1)$ for some $p_k \in \mathbb{N}$ and real algebraic integers $\beta_k \geq 1$. By the theory on recurrences, $\vec{u}_n$ and $\vec{v}_n$ can be expressed as an exponential polynomial with the roots of the characteristic polynomial as base variables. Put $\beta = \max\{\beta_k : k = 1, \ldots, r\}$ and $p = \prod_{|\beta_k|=\beta} p_k$. Then all the terms in the expressions for $\vec{u}_n$ and $\vec{v}_n$ are absorbed in the error term $o(\beta^n)$ except for the base values of the form $\xi_p^h\beta$. The fact that the components of the coefficient-vectors $\vec{P}_j, \vec{Q}_j$ are in the field generated by $\beta\xi_p$ over $\mathbb{Q}$ follows by considering the generating functions of the components, which leads to rational functions with integer coefficients since the vectors $\vec{u}_n$ and $\vec{v}_n$ have integral components, and by using the partial fractal decomposition afterwards. $\square$

Observe that the result recovers part of a result of Lind [24, 25] who characterized for which matrices with entries in $\mathbb{N}$ the spectral radii are Perron numbers. (A Perron number is a real algebraic integer which is larger than all its conjugates.)

From the lemma it follows that if we split the sequence $(u^{(n)})_{n\geq 0}$ according to the arithmetic progressions $n = mp + j$ with $j = 0, \ldots, p-1$, then the incidence matrix $M_\sigma$ for each progression has a dominant root $\beta^p \geq 1$. Moreover, there exists a monic polynomial $P(x) \in \mathbb{Q}(\beta)[x]$ such that the limits

$$\lim_{n\to\infty} \frac{|\sigma^{np+i}(j)|}{P(np+i)\beta^{np+i}} =: a_j(i) \tag{2}$$

exist for all $j = 0, \dots, k; i = 0, \dots, p-1$. Furthermore, we get that

$$\lim_{n\to\infty} \frac{\vec{u}_{np+i}}{|u_{np+i}|} =: \vec{b}(i)$$

exist for all $i = 0, \dots, p-1$. Observe that this need not hold if we replace $p$ with 1, e.g. it does not for a substitution with incidence matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

which has the characteristic equation

$$\beta^5 - \beta^4 - \beta^3 + \beta^2 - \beta + 1 = (\beta - 1)(\beta^2 - \rho^2)(\beta^2 - \rho^{-2}) = 0$$

where $\rho = (1 + \sqrt{5})/2$.

We simplify the formulas by studying the arithmetic progressions with difference $p$ separately. In the sequel we will assume that

$$\text{(A2)} \qquad M_\sigma \text{ has a dominant eigenvalue } \beta,$$

so $p = 1$. We denote $a_j(0), \vec{b}(0)$ by $a_j, \vec{b}$, respectively.

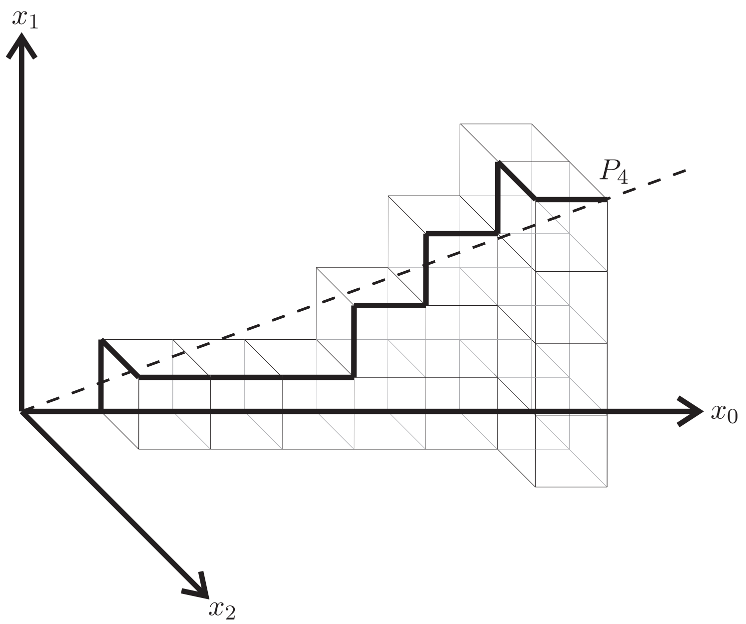



Fig. 1: Broken halfline segment associated to $u^{(4)}$= 0120001010120 for the flipped Tribonacci substitution $\sigma(0) = 01, \sigma(1) = 20, \sigma(2) = 0$.

It is clear that the vector $\vec{a} := {}^t(a_0, \dots, a_k)$ is an eigenvector of $M_\sigma$ and the vector $\vec{b}$ is an eigenvector of ${}^tM_\sigma$, both to the eigenvalue $\beta$. In particular the sequence $u^{(n)}/|u^{(n)}|$ of normed incidence vectors converges to $\vec{b}$.

We remove all letters from $\Sigma$ which do not appear in the limit word $U$, since they do not play a role in the further investigations. (This will lead to a trim automaton associated to $\sigma$ in Section 4.) Moreover, we remove all letters $j$ for which $a_j = 0$. Observe that such letters can appear infinitely often in $U$, as e.g. the letter 2 in $\sigma(0) = 012, \sigma(1) = 111, \sigma(2) = 2$. We assume that

$$a_j > 0 \quad \text{for all} \quad j = 0, \dots, k. \tag{A3}$$

This implies that all sequences $(|\sigma^n(j)|)_{n\geq 0}$ have the same growth order.

The limit word $U$ of the sequence $(u^{(n)})_{n\geq 0} = (\sigma^n(0))_{n\geq 0}$ generates a *broken halfline* in $\mathbb{R}^{k+1}$ in the following way: we start in $\vec{0}$ and for $n = 0, 1, \dots$ go 1 in the direction of the $x_i$-axis when $U_n = i$. (See Fig. 1.) It follows from the convergence of the incidence vectors of $u^{(n)}$ that the broken halfline approximates the halfline $\mathbb{R}_{\geq 0}\vec{b}$.

We will study the projection of the integer points on the broken halfline to some hyperplane not containing $\mathbb{R}\vec{b}$. (Since we ignore linear transformations, we are free in our choice.) The integer points $P_m$ for $m \in \mathbb{N}$ on this line are given by

$$P_0 = \vec{0}, \quad P_{m+1} = P_m + \vec{e}_{U_m}^{(k+1)},$$

We project each integer point parallel to the halfline $\mathbb{R}_{\geq 0}\vec{b}$ to the hyperplane and want to understand the structure of the projections. In general this problem is hard, because the projections form fractals (e.g. the Rauzy-fractal [32] in case we start with the Tribonacci substitution). As in [35] in the Tribonacci case we first investigate the local behaviour by considering for every $n \in \mathbb{N}$ the projections along the broken line segment through $\vec{0}$ and $P_m = \vec{u}_n$ where $m = s^{(n)}$. In the next section we will show that if we use a suitable transformation after applying the projection, then the images lie in the lattice $\mathbb{Z}^k$.

## 3. Continued lattices

We start by defining a linear mapping $\Phi_n$ which projects the integer points on the broken halfline segment associated to $u^{(n)}$ to some hyperplane not containing the incidence vector $\vec{u}_n$. Recall that $\vec{u}_n = {}^tM_\sigma^n \vec{e}_0^{(k+1)}$ by Lemma 1 (see the shaded line in Fig. 1). Hence

$$\Phi_n\big(P_{s^{(n)}}\big) = \Phi_n\left({}^tM_\sigma^n \vec{e}_0^{(k+1)}\right) := \vec{0}. \tag{3}$$

Let $\vec{c}_1, \dots \vec{c}_k$ be such that the vectors ${}^tM_\sigma^k \vec{e}_0^{(k+1)}, \vec{c}_1, \dots, \vec{c}_k$ form a basis of the lattice $\mathbb{Z}^{k+1}$. Then we define, for $n \geq k$,

$$\Phi_n\left({}^tM_\sigma^{n-k} \vec{c}_i\right) := \vec{e}_i^{(k)} \quad (i = 1, \dots, k), \tag{4}$$

where $\vec{e}_i^{(k)}$ denotes the $i$-th $k$-dimensional unit vector.

By assuming that

(A4) $$M_\sigma \text{ is unimodular,}$$

we secure that ${}^tM_\sigma^n\vec{e}_0^{(k+1)}, {}^tM_\sigma^{n-k}\vec{c}_1, \ldots, {}^tM_\sigma^{n-k}\vec{c}_k$ form a basis of $\mathbb{Z}^{k+1}$ for every $n \in \mathbb{N}$. We will hold on this assumption for the rest of the paper.

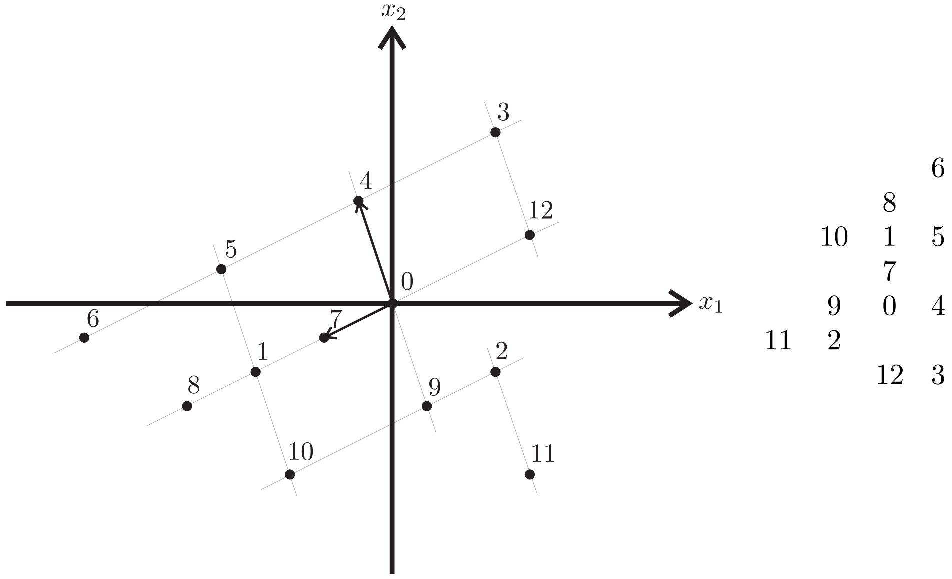



Fig. 2: Result of the projection of the integer points on the broken halfline segment along $u^{(4)}$ to the $(x_1, x_2)$-plane for the flipped Tribonacci substitution $\sigma(0) = 01, \sigma(1) = 20, \sigma(2) = 0$ on the left-hand side and the image in $\mathbb{Z}^2$ after applying $\Phi_4$ with $\vec{c}_1 = \vec{u}_0 = {}^t(1,0,0), \vec{c}_2 = \vec{u}_1 = {}^t(1,1,0)$ on the right-hand side.

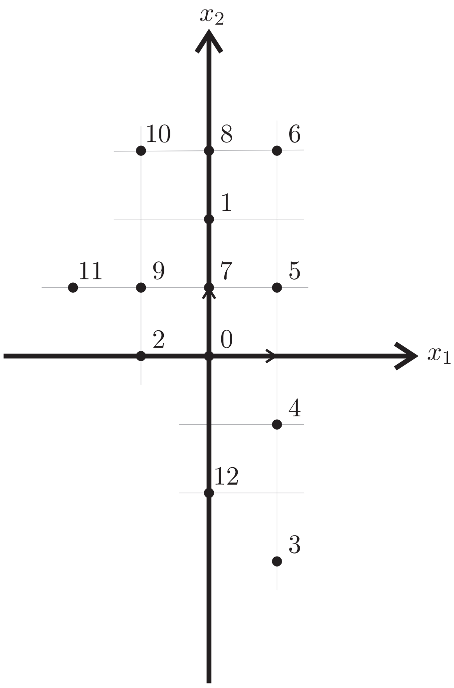
Since we have determined the images of $\Phi_n$ on a basis of the lattice $\mathbb{Z}^{k+1}$ and $\Phi_n$ is linear, it is defined everywhere and the definition for $\Phi_n$ makes sense for all $n \geq 0$. For short, we will call $\Phi_n$ itself a projection.

We may take the vectors $\vec{c}_i$ from the points of the broken halfline corresponding to $u^{(k)}$. In fact we can take $\vec{c}_1 = \vec{e}_0^{(k+1)}$, $\vec{c}_2$ the first point on the broken halfline, which corresponds to the place where the first letter different from 0 in $u^{(k)}$ appears, and so forth. This is clearly valid as all $k+1$ letters have to appear in $u^{(k)}$, since we have removed all letters which do not appear in the limit word from our alphabet. So it is possible to choose the vectors $\vec{c}_i$ such that they are linearly independent over $\mathbb{Z}$. The image of the broken halfline segment associated with $u^{(n)}$ is now a subset of $\mathbb{Z}^k$ (cf. Fig. 2).

Observe that in general it is not possible to take

$$\vec{c}_i = {}^tM_\sigma^{i-1}\vec{e}_0^{(k+1)} = \vec{u}_{i-1}$$

for $i = 1, \dots, k$, even if $M_\sigma$ is unimodular. E.g. the substitution $\sigma(0) = 0112$, $\sigma(1) = 1$, $\sigma(2) = 2$ has a unimodular incidence matrix, but no three consecutive vectors in the sequence $\vec{u}_n$ are a basis of $\mathbb{Z}^3$.




| | | | |
|---|---|---|---|
| | 10 | 8 | 6 |
| | | 1 | |
| 11 | 9 | 7 | 5 |
| | 2 | 0 | |
| | | | 4 |
| | | 12 | |
| | | | 3 |

Fig. 3: Result of the projection of the broken halfline segment associated to $u^{(4)}$ in the canonical case for the flipped Tribonacci substitution $\sigma(0) = 01, \sigma(1) = 20, \sigma(2) = 0$.

Another useful choice is

$$\vec{c}_i = {}^tM_\sigma^k \vec{e}_i^{(k+1)} \quad (i = 0, 1, \dots, k). \tag{5}$$

In this case we can explicitly describe the corresponding projection which we will call the *canonical* projection $\Phi_n^*$. For $\vec{x} \in \mathbb{Z}^{k+1}$ we have

$$\Phi_n^*(\vec{x}) := \Pi({}^tM_\sigma^{-n}\vec{x}), \tag{6}$$

where $\Pi$ is the projection along $\vec{e}_0^{(k+1)}$ and therefore means deletion of the zeroth entry. When we use this special basis, we say that we are in the *canonical case*. We remark that the map $\Phi_n^*$ can be seen as a result of changing the order of projection and transformation: we can as well first apply a linear transformation and then project; as the transformation we choose left multiplication by ${}^tM_\sigma^{-n}$ and we project along the unit vector $\vec{e}_0^{(k+1)}$.

We define

$$\vec{a}_i^{(n)} := \Phi_n\left(\vec{e}_i^{(k+1)}\right) \tag{7}$$

for $i = 0, 1, \dots, k$ and call them *transition vectors* corresponding to $u^{(n)}$. Observe that these vectors tell which step we have to make in $\mathbb{Z}^k$ to get to the next projected point.

In the canonical case we have

$$\vec{a}_i^{(n)} = \Phi_n^* \left( \vec{e}_i^{(k+1)} \right) = \Pi \left( {}^t M_\sigma^{-n} \vec{e}_i^{(k+1)} \right) \tag{8}$$

and therefore the transition vectors in this case are obtained as the row vectors of $M_\sigma^{-n}$ after deletion of their zeroth entries. In general we have the following formula to calculate them.

**Lemma 3.** *We have*

$$\left( \vec{a}_0^{(n+1)}, \dots, \vec{a}_k^{(n+1)} \right) = \left( \vec{a}_0^{(n)}, \dots, \vec{a}_k^{(n)} \right) {}^t M_\sigma^{-1}$$

*for all* $n \geq 0$.

*Proof.* By the substitution $\sigma$ each jump $a_q^{(n)}$ in $w^{(n)}$ is replaced by a series of jumps $a_{(\sigma(q))_i}^{(n+1)}$ according to the word $\sigma(q)$. Therefore we get

$$\vec{a}_q^{(n)} = \sum_{i=0}^{|\sigma(q)|-1} \vec{a}_{(\sigma(q))_i}^{(n+1)} = \vec{a}_0^{(n+1)}(\sigma(q))_0 + \vec{a}_1^{(n+1)}(\sigma(q))_1 + \dots + \vec{a}_k^{(n+1)}(\sigma(q))_k.$$

Thus, the recurrence follows from Lemma 1. □

Let $P_1, \dots, P_m$ with $m = s^{(n)}$ be the integer points on the broken halfline segment. We will define a new $k$-dimensional word using the projections of these points.

A *$k$-dimensional word* is a map from a subset of $\mathbb{Z}^k$ to some alphabet. If we have a sequence $(w^{(n)})_{n \geq 0}$ of $k$-dimensional words, then we say that this sequence converges on $A$ to a $k$-dimensional word $W$ if for every $\vec{x} \in A$ there exists an integer $N$ such that $w^{(n)}(\vec{x}) = W(\vec{x})$ for all $n \geq N$. Observe, that words used up to now are 1-dimensional words defined on $\{0, 1, \dots, n-1\}$ if the word is of length $n$, and on the non-negative integers if it is infinite.

Now, we define $w^{(n)}$ by setting its value at position $\Phi_n(P_i)$ equal to $i$ for $i = 0, \dots, m-1$. Hence $w^{(n)}$ is a $k$-dimensional word with letters from $\mathbb{N}$, i.e.

$$\begin{aligned} w^{(n)} : \Phi_n(A_n) &\longrightarrow \mathbb{N} \\ \Phi_n(P_i) &\longmapsto i \end{aligned}$$

for $i = 0, \dots, s^{(n)} - 1$ where $A_n$ consists of the integer points on the broken line segment from $\vec{0}$ to $\vec{u}_n$. Put $L = \sum_{q \in \Sigma} \{|\sigma(q)| - 1\}$. We have the following

**Theorem 1.** *The sequence of words $w = (w^{(n)})_{n \in \mathbb{N}}$ is well-defined.*
*The domain of $w^{(n+1)}$ is contained in the union of the domain of $w^{(n)}$ and at most $L$ translates of the domain of $w^{(n)}$.*
*Moreover, in the canonical case the projections of $P_m$ in $w^{(n)}$ will be the*

*projections of* ${}^tM_\sigma P_m$ *in* $w^{(n+1)}$ *for* $m = 0, 1, \dots, s^{(n)}-1$ *and* $\sigma^{n+1}(0)-\sigma^n(0)$ *new points are added.*

*Proof.* Suppose $\Phi_n(P_i) = \Phi_n(P_j)$. By the linearity of $\Phi_n$ it follows that $P_i - P_j \in \mathbb{R}\vec{u}_n$. Recall that $\vec{u}_0 = \vec{e}_0^{(k+1)}$ and $M_\sigma$ is unimodular. By induction on $n$ it follows from (1) that the entries of $\vec{u}_n$ are relatively prime. Since $P_i$ and $P_j$ have integral coordinates, we obtain that $P_i - P_j \in \mathbb{Z}\vec{u}_n$. But this implies $i = j$, since $i, j \in \{0, 1, \dots, s^{(n)} - 1\}$, whence $|P_i - P_j| < |\vec{u}_n|$.

If $\vec{x} = (x_1, \dots, x_k) \in \mathbb{Z}^k$ is in the domain of $w^{(n)}$, then, by (4),

$$\vec{x} = \sum_{i=1}^{k} x_i \Phi_n({}^tM^{n-k}\vec{c}_i).$$

When applying $\sigma$ to the word $u^{(n)}$ to obtain $u^{(n+1)}$, we apply ${}^tM_\sigma$ to the broken line segment corresponding to $u^{(n)}$ to obtain the broken line segment corresponding to $u^{(n+1)}$ and therefore $\Phi_{n+1}{}^tM_\sigma$ to get the projections. If $P$ is the integer point on the broken halfline corresponding to $u^{(n)}$ which is projected to $\vec{x}$, then

$$P \in \sum_{i=1}^{k} x_i \ {}^tM_\sigma^{n-k}\vec{c}_i + \mathbb{R}\ {}^tM_\sigma^{n}\vec{e}_0^{(k+1)}.$$

Applying ${}^tM_\sigma$ on the left we obtain

$${}^tM_\sigma P \in \sum_{i=1}^{k} x_i {}^tM_\sigma^{n+1-k}\vec{c}_i + \mathbb{R}\ {}^tM_\sigma^{n+1}\vec{e}_0^{(k+1)}.$$

If we apply $\Phi_{n+1}$, we get that the image of $\vec{x}$ equals

$$\sum_{i=1}^{k} x_i \Phi_{n+1}({}^tM_\sigma^{n+1-k}\vec{c}_i).$$

Thus the vector $\vec{x}$ is also in the domain of $w^{(n+1)}$. Suppose that the broken halfline leaves $\vec{x}$ by a step $\vec{e}_q^{(k+1)}$. Then the subsequent integer points on the broken halfline associated with $u^{(n+1)}$ are obtained by following the letters of the word $\sigma(q)$ and thus by the projections

$$\vec{y}_{s_q} = \vec{x} + \sum_{i=1}^{s_q} \vec{a}^{(n+1)}_{(\sigma(q))_i},$$

where $s_q \in \{1, \dots, |\sigma(q)| - 1\}$. It follows that the projections are contained in the union of the domain of $w^{(n)}$ and $s_q$ translates of it. This argument is valid for $q = 0, 1, \dots, k$. Thus we have proved the second assertion.

Now assume that we are in the canonical case. Since $P_m = {}^t(|u_0 \cdots u_{m-1}|_0, \dots, |u_0 \cdots u_{m-1}|_k)$ and by $\sigma$ each letter $j$ is replaced with

$|\sigma(j)|_0$ zeros, $|\sigma(j)|_1$ ones, and so forth up to $|\sigma(j)|_k$ letters $k$, $P_m$ is mapped to $P_l$ where

$$P_l = \sum_{j=0}^{k} |u_0 \cdots u_{m-1}|_j \, {}^t(|\sigma(j)|_0, \dots, |\sigma(j)|_k).$$

Observe that

$${}^tM_\sigma P_m = \sum_{j=0}^{k} |u_0 \cdots u_{m-1}|_j \, {}^t(|\sigma(j)|_0, \dots, |\sigma(j)|_k) = P_l.$$

Hence $\Phi_n^*(P_m) = \Pi({}^tM_\sigma^{-n} P_m) = \Pi({}^tM_\sigma^{-(n+1)} P_l) = \Phi_{n+1}^*(P_l)$. Consequently, when going from $\sigma^n(0)$ to $\sigma^{n+1}(0)$, the projections of the transformations of $P_0, \dots, P_{s^{(n)}}$ remain in the domain of $w^{(n)}$ and $\sigma^{n+1}(0) - \sigma^n(0)$ new projected points are added. This proves the third statement. $\square$

Observe that the projected points $\vec{0}, \Phi_n(P_1), \dots, \Phi_n(P_{m-1}), \Phi_n(P_m) = \vec{0}$ with $m = s^{(n)}$, which make up the domain of $w^{(n)}$, set up a *roundwalk* (see [10] for a more general setting of this topic), that is a sequence of vectors from $\mathbb{Z}^k$ with starting point equal to endpoint and with the property that $\Phi_n(P_{i+1}) - \Phi_n(P_i) \in \{\vec{a}_0^{(n)}, \dots, \vec{a}_k^{(n)}\}$ for $i = 0, \dots, m-1$. Moreover, the *coding* of this roundwalk, that is the finite word $w_0 \cdots w_m$ over the alphabet $\{0, 1, \dots, k\}$ defined by $w_i = j$ if $\Phi_n(P_{i+1}) - \Phi_n(P_i) = \vec{a}_j^{(n)}$ for $0 \le i \le m-1$, is exactly the word $u^{(n)}$. Clearly, given the transition vectors, the roundwalk is perfectly determined by its coding.

We define, for $i = 1, \dots, k$,

$$b_i^{(n)} := {}^t\vec{c}_i \vec{v}_{n-k} = \sum_{j=0}^{k} c_{i,j} |\sigma^{n-k}(j)| \tag{9}$$

where $\vec{c}_i = {}^t(c_{i,0}, \dots, c_{i,k})$ are the vectors in the definition of $\Phi_n$. Observe that if we divide through by $s^{(n)} = |u^{(n)}|$ and let $n$ go to infinity, then, by (2) and the convention after (A2),

$$\frac{b_i^{(n)}}{s^{(n)}} = \sum_{j=0}^{k} c_{i,j} \frac{\frac{|\sigma^{n-k}(j)|}{P(n-k)\beta^{n-k}}}{\frac{|\sigma^n(0)|}{P(n)\beta^n}} \frac{P(n-k)\beta^{n-k}}{P(n)\beta^n} \longrightarrow \sum_{j=0}^{k} c_{i,j} \frac{a_j}{a_0} \beta^{-k} =: b_i, \tag{10}$$

for $n \to \infty$, which will be important later. We remark that by (5) in the canonical case

$$b_i^{(n)} = {}^t\vec{e}_i^{(k+1)} M_\sigma^k \vec{v}_{n-k} = {}^t\vec{e}_i^{(k+1)} \vec{v}_n = |\sigma^n(i)|$$

and therefore $b_i = \frac{a_i}{a_0}$. Put $b_0 = 1$.

We now use the general theory of roundwalks developed in [40] and [10]. The following result displays some properties of $a_i^{(n)}$ and $b_i^{(n)}$.

**Theorem 2.** *The domain of $w^{(n)}$ is a fundamental domain of the lattice*

$$\Lambda^{(n)} := \mathbb{Z}\left(\vec{a}_1^{(n)} - \vec{a}_0^{(n)}\right) + \ldots + \mathbb{Z}\left(\vec{a}_k^{(n)} - \vec{a}_0^{(n)}\right).$$

*If $d_0\vec{a}_0^{(n)} + d_1\vec{a}_1^{(n)} + \ldots + d_k\vec{a}_k^{(n)} = \vec{0}$ for integers $d_0, d_1, \ldots, d_k \in \mathbb{Z}$, then*

$$d_0 = t \cdot \left|\det\left(\vec{a}_1^{(n)}, \ldots, \vec{a}_k^{(n)}\right)\right| \quad \textit{and}$$
$$d_i = t \cdot \left|\det\left(\vec{a}_1^{(n)}, \ldots, \vec{a}_{i-1}^{(n)}, -\vec{a}_0^{(n)}, \vec{a}_{i+1}^{(n)}, \ldots, \vec{a}_k^{(n)}\right)\right|$$

*for $i = 1, \ldots, k$ and some integer $t$. Moreover, the number $m$ at position $\vec{x} = (x_1, \ldots, x_k) \in \mathbb{Z}^k$ in $w^{(n)}$ is congruent to*

$$\sum_{i=1}^{k} x_i b_i^{(n)} \pmod{s^{(n)}}$$

*and $\vec{x}$ is congruent to $m\vec{a}_0^{(n)}$ $\pmod{\Lambda^{(n)}}$.*

*Proof.* The first and second part of the theorem follow immediately from the much more general theory in [10] (cf. also [35, Lemma 4.2]). For the last part we only have to show that the value of the unit vectors $\vec{e}_i^{(k)} (i = 1, \ldots, k)$ in $w^{(n)}$ are given by $b_i^{(n)}$ for every $n \in \mathbb{N}$. The rest follows immediately from the fact that $g^{(n)}$, which extends $w^{(n)}$ to $\mathbb{Z}^k$ by being constant on cosets of $\Lambda^{(n)}$, is a linear function (see [10, p. 181]).

By (4) the value at $\vec{e}_i^{(k)}$ is equal to $\|{}^tM_\sigma^{n-k}\vec{c}_i\|_1$ where $\|\cdot\|_1$ denotes as usual the sum of the components of the vector. We have, by Lemma 1 and (9),

$$\|{}^tM_\sigma^{n-k}\vec{c}_i\|_1 = \left\|\sum_{j=0}^{k} c_{i,j}\,{}^tM_\sigma^{n-k}\vec{e}_j^{(k)}\right\|_1 = \sum_{j=0}^{k} c_{i,j}\left\|{}^tM_\sigma^{n-k}\vec{e}_j^{(k)}\right\|_1$$
$$= \sum_{j=0}^{k} c_{i,j}|\sigma^{n-k}(j)| = b_i^{(n)}.$$

Observe that the same calculation with ${}^tM_\sigma^n\vec{e}_0^{(k+1)}$ instead of ${}^tM_\sigma^{n-k}\vec{c}_i$ gives the identity $s^{(n)} = |u^{(n)}|$. □

The sequence of lattices $\Lambda^{(n)}$ from the last theorem is called a *continued lattice* (they first appeared in [40] by generalising the continued fraction algorithm). In the canonical case we have some additional properties:

**Theorem 3.** *Assume that we are in the canonical case and let $n \geq 0$. Then*

(i) *$\Lambda^{(n)}$ is a lattice with lattice determinant $s^{(n)}$.*
(ii) *For $i = 1, \ldots, k$ the vector $\vec{e}_i^{(k)}$ is in the same coset of $\Lambda^{(n)}$ as $|\sigma^n(i)|\,\vec{a}_0^{(n)}$.*

(iii) *For* $m = 1, \dots, s^{(n)}$ *we have*

$$\Phi_n^*(P_m) \equiv m\vec{a}_0^{(n)} \pmod{\Lambda^{(n)}}.$$

*Proof.* We start with (i). It follows from (8) that

$$\left(\vec{a}_0^{(n)}, \vec{a}_1^{(n)}, \dots, (i), \dots, \vec{a}_k^{(n)}\right),$$

where $(i)$ indicates that $\vec{a}_i^{(n)}$ is omitted, is the minor of $M_\sigma^{-n}$ corresponding with the entry $|\sigma^n(0)|_i$ of $M_\sigma^n$. Since $\det(M_\sigma) = \pm 1$ by (A4), we obtain from linear algebra and Lemma 1 that, for a fixed choice of the $\pm$-sign,

$$(-1)^i \det\left(\vec{a}_0^{(n)}, \vec{a}_1^{(n)}, \dots, (i), \dots, \vec{a}_k^{(n)}\right) = \pm|\sigma^n(0)|_i$$

for $i = 0, 1, \dots, k$. Hence

$$\begin{aligned} \left|\det\left(\Lambda^{(n)}\right)\right| &= \left|\sum_{i=0}^{k} (-1)^i \det\left(\vec{a}_0^{(n)}, \vec{a}_1^{(n)}, \dots, (i), \dots, \vec{a}_k^{(n)}\right)\right| \\ &= \sum_{i=0}^{k} |\sigma^n(0)|_i = |\sigma^n(0)| = s^{(n)} \end{aligned}$$

and therefore (i) holds.

Next (ii). Let $i \in \{1, \dots, k\}$. The vector ${}^t M_\sigma^n \vec{e}_i^{(k+1)}$ in $\mathbb{Z}^{k+1}$ is transformed to the vector $\vec{e}_i^{(k+1)}$ in $\mathbb{Z}^{k+1}$ and then projected to $\vec{e}_i^{(k)}$ in $\mathbb{Z}^k$. By Lemma 1 we have ${}^t\vec{e}_i^{(k+1)} M_\sigma^n = (|\sigma^n(i)|_0, \dots, |\sigma^n(i)|_k)$. Hence the point $(|\sigma^n(i)|_0, \dots, |\sigma^n(i)|_k)$ is transformed and subsequently projected to $\sum_{j=0}^k |\sigma^n(i)|_j \vec{a}_j^{(n)}$. Thus, by the definition of $\Lambda^{(n)}$,

$$\vec{e}_i^{(k)} = \sum_{j=0}^{k} |\sigma^n(i)|_j \vec{a}_j^{(n)} \equiv \sum_{j=0}^{k} |\sigma^n(i)|_j \vec{a}_0^{(n)} = |\sigma^n(i)| \vec{a}_0^{(n)} \pmod{\Lambda^{(n)}}.$$

Finally (iii). We know that $P_m$ is the sum of $m$ unit vectors, the unit vector $\vec{e}_i^{(k+1)}$ is mapped to $\vec{a}_i^{(n)}$ and $\vec{a}_i^{(n)} \equiv \vec{a}_0^{(n)} \pmod{\Lambda^{(n)}}$ for $i = 0, 1, \dots, k$. Therefore the statement is valid. □

We now define a sequence of new words $\hat{w} = (\hat{w}^{(n)})_{n \geq 0}$, which is closely related to the original sequence. In contrast to $w^{(n)}$ the word $\hat{w}^{(n)}$ will be a $k$-dimensional word defined over the (infinite) alphabet $[0, 1)$. The domain of $\hat{w}^{(n)}$ is the same as the domain of $w^{(n)}$, i.e. equal to $\Phi_n(A_n)$. For $\vec{x}$ in the domain of $w^{(n)}$ we define

$$\hat{w}^{(n)}(\vec{x}) := \left\{ \sum_{i=1}^{k} x_i \frac{b_i^{(n)}}{s^{(n)}} \right\}, \tag{11}$$

where $\{z\}$ denotes the fractional part of $z$, i.e. $\{z\} = z \bmod 1$ (see Fig. 4 and observe that Theorem 2 is the motivation for this definition).

$$
\begin{array}{cccc}
 & & & 6_1 \\
 & & 8_1 & \\
 & 10_1 & 1_1 & 5_0 \\
 & & 7_0 & \\
 & 9_0 & 0_0 & 4_0 \\
11_2 & 2_2 & & \\
 & & 12_0 & 3_0
\end{array}
\qquad\qquad
\begin{array}{cccc}
 & & & \frac{6}{13} \\
 & & \frac{8}{13} & \\
 & \frac{10}{13} & \frac{1}{13} & \frac{5}{13} \\
 & & \frac{7}{13} & \\
 & \frac{9}{13} & 0 & \frac{4}{13} \\
\frac{11}{13} & \frac{2}{13} & & \\
 & & \frac{12}{13} & \frac{3}{13}
\end{array}
$$

$w^{(4)}$ $\qquad$ $\hat{w}^{(4)}$

Fig. 4: The roundwalk $w^{(4)}$ (the subscript indicates the next jump) and its normalization $\hat{w}^{(4)}$ for the flipped Tribonacci substitution $\sigma(0) = 01, \sigma(1) = 20, \sigma(2) = 0$.

The sequence $\hat{w}^{(n)}$ converges to some $k$-dimensional limit word $\hat{W} := \lim_{n\to\infty} \hat{w}^{(n)}$ over the alphabet $[0,1)$ defined on a certain subset of $\mathbb{Z}^k$ in the following way: the domain of $\hat{W}$ is the union of all domains $\Phi_n(A_n)$ of $\hat{w}^{(n)}$, i.e. $\mathrm{dom}\hat{W} := \bigcup_{n=0}^{\infty} \Phi_n(A_n) = \lim_{n\to\infty} \Phi_n(A_n)$, and if $\vec{x} \in \mathrm{dom}\hat{W}$ then

$$\hat{W}(\vec{x}) := \lim_{n\to\infty} \hat{w}^{(n)}(\vec{x}).$$

We have the following result:

**Theorem 4.** *The word $\hat{W}$ is well-defined on dom$\hat{W}$ and*

$$\hat{W}(\vec{x}) = \left\{ \sum_{i=1}^{k} x_i b_i \right\}. \tag{12}$$

*Proof.* The statement is trivially true for $\vec{x} = \vec{0}$. Since $\hat{w}^{(n)}$ is defined by (11) the existence of $\lim_{n\to\infty} \hat{w}^{(n)}(\vec{x})$ follows at once from (10). For the second part we have to show that $\hat{W}(\vec{x}) < 1$; the result then follows from the definition and (10). For $\vec{x} \in \mathrm{dom}\hat{W}$ there exists $n_0 \in \mathbb{N}$ such that $\vec{x} \in \mathrm{dom}\hat{w}^{(n)}$ for all $n \geq n_0$. By Theorem 1 it follows that $1 - \hat{w}^{(n)}(\vec{x})$ can be bounded from below by

$$\frac{|\sigma^k(l)|}{s^{(n_0+k)}} = \frac{\dfrac{|\sigma^k(l)|}{P(k)\beta^k}}{\dfrac{|\sigma^{n_0+k}(0)|}{P(n_0+k)\beta^{n_0+k}}} \frac{P(k)}{P(n_0+k)\beta^{n_0}} \longrightarrow \frac{a_l}{a_0\beta^{n_0}} > 0,$$

for $k = n - n_0 \to \infty$, where $l$ is the jump to get from $w^{(n_0)}(\vec{x})$ to $w^{(n_0)}(\vec{x})+1$ and where we have used that all $a_j > 0$, i.e. assumption (A3). $\square$

Thus the limit word is the restriction of the linear function $\mathbb{Z}^k \longrightarrow [0,1)$ defined by $\vec{x} = {}^t(x_1, \ldots, x_k) \mapsto x_1 b_1 + \ldots + x_k b_k \bmod 1$ to dom$\hat{W}$.

Observe that if $\vec{x} \in \mathrm{dom}\hat{W}$, then there is a minimal $n_0$ such that $\vec{x}$ is in $\mathrm{dom}\hat{w}^{(n_0)}$ and it follows that

$$w^{(n)}(\vec{x}) = \sum_{i=1}^{k} x_i b_i^{(n)} \pmod{s^{(n)}}$$

for all $n \geq n_0$.

It is easy to show that it may happen that the domain of $\hat{W}$ is not $\mathbb{Z}^k$ (see Example 2 in Section 6). Our aim is (and that will be the main result of this paper) to clarify where the limit $\hat{W}$ exists and, especially, when it is defined everywhere on some submodule of $\mathbb{Z}^k$. If the latter is the case, we will say that $\hat{W}$ is *space filling* (has the space filling property).

## 4. Abstract number systems

In this section we describe how to a given substitution $\sigma : \Sigma \to \Sigma^*$, where $\Sigma = \{0, 1, \ldots, k\}$, a number system can be associated. It is well-known (cf. [26]) that a substitution $\sigma$ defines an *automaton* in the following way:

(i) the set of states $Q$ is equal to letters of the alphabet $\Sigma$,
(ii) a transition from state $a$ to state $b$ ($a, b \in \Sigma$) labeled $i$ is added if $b$ occurs in $\sigma(a)$ at position $i$,
(iv) $q_0 = 0$ is the initial state and finally
(v) all states are final states.

So, the associated automaton is given by $\mathcal{M} = (Q, q_0, \Delta, \delta, F)$, where $\Delta = \{0, 1, \ldots, l\}$ with $l := \max\{|\sigma(q)| - 1 : q \in Q\}$ is the set of labels, $\delta : Q \times \Delta \to \Delta$ is the transition function and $F = Q$ denotes the set of final states. As above we denote by $\Delta^*$ the free monoid generated by $\Delta$ for the concatenation product. The neutral element is $\varepsilon$ and the the length of a word $w \in \Delta^*$ is denoted by $|w|$. The function $\delta$ is naturally extended to $Q \times \Delta^*$ by $\delta(q, \varepsilon) = q$ and $\delta(q, dw) = \delta(\delta(q, d), w)$ where $q \in Q, d \in \Delta$ and $w \in \Delta^*$.

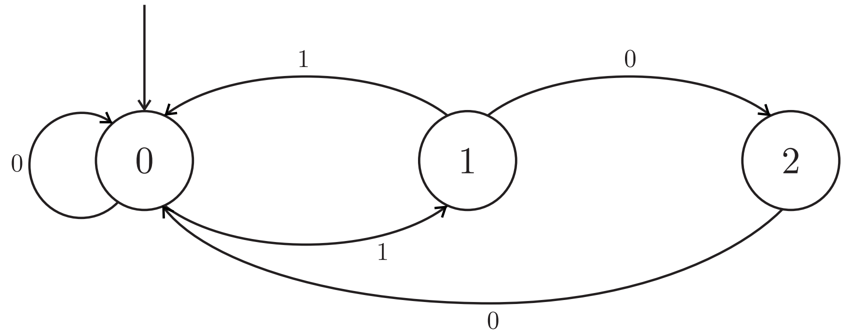



Fig. 5: Automaton associated to the flipped Tribonacci substitution $\sigma(0) = 01, \sigma(1) = 20, \sigma(2) = 0$.

The *language* $\mathcal{L}$ associated to the substitution is the language of the associated automaton $\mathcal{M}$, that is the collection of strings accepted by the automaton (all paths in the automaton from the initial state to a final state), i.e.

$$\mathcal{L} = \{w \in \Delta^* : \delta(q_0, w) \in F\}.$$

Moreover, if $q \in Q$, we denote by $\mathcal{L}_q$ the language accepted in $\mathcal{M}$ in case $q$ is the initial state, i.e.

$$\mathcal{L}_q = \{w \in \Delta^* : \delta(q, w) \in F\}.$$

In particular, $\mathcal{L}_{q_0} = \mathcal{L}_0 = \mathcal{L}$. Observe that from our conditions on the substitution $\sigma$ the automaton is trim, i.e. accessible and coaccessible, finite and deterministic (for more information on automata we refer to [18]) and therefore the languages $\mathcal{L}_q$ are regular languages. For $q \in Q$ and $n \in \mathbb{N}$, we denote by $\mathbf{u}_q(n)$ the number of words of length $n$ accepted from $q$ in $\mathcal{M}$, i.e.

$$\mathbf{u}_q(n) = \#\{w \in \Delta^n : \delta(q, w) \in F\}.$$

The *genealogical ordering* (also known as radix ordering or military ordering) is defined as follows: if $u$ and $v$ are words over $\Delta$, then we define $u < v$, if either $|u| < |v|$ or $|u| = |v|$ and there exist $p, u', v' \in \Delta^*, d, e \in \Delta$ such that $u = pdu', v = pev'$ and $d < e$. This ordering is naturally extended to the set $\Delta^\omega$ of all the infinite words over $\Delta$ according to the lexicographical ordering. The triple $\mathcal{S} = (\mathcal{L}, \Delta, <)$ is called an *abstract number system* (or abstract numeration system), since the words of $\mathcal{L}$ can be enumerated by increasing genealogical ordering leading to a one-to-one correspondence between $\mathbb{N}$ and $\mathcal{L}$ (cf. [27]). It is well-known (and easy to prove) that the $i$-th letter in $\sigma^n(0)$ is the state the automaton $\mathcal{M}$ will be in after it is fed with the $i$-th word of length $n$ of its ordered input language $\mathcal{L}$. Thus we have for the limit word $U = (U_n)_{n\in\mathbb{N}} = \lim_{n\to\infty} \sigma^n(0)$ that $U_i$ is the state the automaton $\mathcal{M}$ will be in after it is fed with the $i$-th word of the language $\mathcal{L}$ if we ignore the words with leading zeroes. For details we refer to [26, Chapter 7].

Next we show that under suitable conditions every real number in the interval $[0, 1)$ can be represented by a unique word from $\mathcal{L}_\infty$ where $\mathcal{L}_\infty$ is the set of infinite words which are limit of a converging sequence of words in $\mathcal{L}$ using the usual infinite product topology on $\Delta^\omega$.

Let $\beta$ be the dominant root of the incidence matrix $M_\sigma$ of $\sigma$ (cf. assumption (A2)) and assume additionally that

$$\text{(A5)} \qquad \text{the dominant root } \beta \text{ is } > 1.$$

Put $\tau = \beta^{-1}$. We remark that under the made conditions $\beta$ has to be irrational: if $\beta \in \mathbb{Q}$, then it would be a rational integer, because $\beta$ is an algebraic integer, and consequently, we would have $\beta = 1$, since $M_\sigma$ is unimodular. Thus $\beta \notin \mathbb{Q}$ by (A5).

Observe that, by remarks made above, $M_\sigma$ is the adjacency matrix of the automaton, i.e. $M_\sigma = (\#\{d \in \Delta : \delta(p,d) = q\})_{p,q\in Q}$. We have the following simple lemma (cf. (2)).

**Lemma 4.** *There is a monic polynomial $P(x) \in \mathbb{Q}(\beta)[x]$ such that the limit*

$$a_q := \lim_{n\to\infty} \frac{\mathbf{u}_q(n)}{P(n)\beta^n}$$

*exists for every $q \in Q$. Moreover, $a_q \in \mathbb{Q}[\tau], a_q \geq 0$.*

*Proof.* We just have to observe that $\mathbf{u}_q(n) = |\sigma^n(q)|$. This follows by induction on $n$. Trivially, $\mathbf{u}_q(1) = |\sigma(q)|$. Clearly, we have that

$$\mathbf{u}_p(n) = \sum_{q\in Q} \#\{d \in \Delta : \delta(p,d) = q\}\ \mathbf{u}_q(n-1) \tag{13}$$

for all $n \geq 1$ and $p \in Q$. Hence the vector $\mathbf{u}(n) = {}^t(\mathbf{u}_{q_0}(n), \ldots, \mathbf{u}_{q_l}(n))$ satisfies the linear recurrence relation $\mathbf{u}(n) = M_\sigma \mathbf{u}(n-1)$ and therefore the same as the vector $\vec{v}_n = {}^t(|\sigma^n(0)|, \ldots, |\sigma^n(k)|)$ considered in Section 2. Since $\beta$ is the dominant root of $M_\sigma$, Lemma 4 follows from Lemma 2 with $p = 1$. Here we use that the minimal polynomial of $\beta$ is monic and has constant $\pm 1$ so that $\beta \in \mathbb{Q}[\tau]$. (For $a_q \in \mathbb{Q}[\tau]$ compare with [34, Lemma 4.1].) $\square$

The monicity of $P$ in Lemma 4 is rather arbitrary. In fact, if we replace $P(n)$ by $\frac{\tau}{a_{q_0}} P(n)$, the limits also exist and lie in $\mathbb{Q}[\tau]$. Thus, dropping the monicity of $P$, we may assume that

$$a_{q_0} = \tau \tag{A6}$$

and we will do so for the rest of the paper. Observe that

$$\beta a_q = \sum_{\substack{d\in\Delta \\ (q,d)\in \mathrm{dom}\delta}} a_{\delta(q,d)}, \tag{14}$$

where $\mathrm{dom}\delta$ denotes the domain of the partial function $\delta$. This means that the vector $\vec{a} = {}^t(a_{q_0}, \ldots, a_{q_k})$ is an eigenvector of $M_\sigma$ to the eigenvalue $\beta$, i.e. the above relation can be rewritten as

$$M_\sigma \vec{a} = \beta \vec{a}. \tag{15}$$

For $q \in Q, t \in \Delta$ we define

$$\alpha_q(t) := \sum_{q'\in Q} \left(a_{q'} \cdot \#\{d < t : \delta(q,d) = q'\}\right).$$

Moreover, we set

$$\alpha_q := \alpha_q(l+1) = \sum_{q'\in Q} \left(a_{q'} \cdot \#\{d \in \Delta : \delta(q,d) = q'\}\right).$$

We remark that $\alpha_q = (M_\sigma \vec{a})_q = \beta a_q$ and in particular $\alpha_0 := \alpha_{q_0} = \beta a_{q_0} = 1$. Set

$$A_q := [0, \alpha_q) = \bigcup_{d=0}^{l} \underbrace{[\alpha_q(d), \alpha_q(d+1))}_{=:A_{q,d}}.$$

We have $A_{q_0} = [0,1)$. We will use the following algorithm:
Let $x \in (0,1)$ and set $w \leftarrow \varepsilon, q \leftarrow q_0$. Then iterate the operations

1. Find $d \in \Delta$ such that $x \in A_{q,d}$
2. $w \leftarrow wd$
3. $x \leftarrow \beta\,(x - \alpha_q(d))$
4. $q \leftarrow (\sigma(q))_d$

The output of the algorithm is the word $w$ which we call the *$\sigma$-representation* of $x$. Conversely we say that $x$ is the *numerical value* of $w$. If $x = 0$ at some stage, then it remains 0 and we say that $x$ has a finite $\sigma$-representation where we stop after the last non-zero value of $q$. The $\sigma$-representation of 0 is by definition 0. Observe that the length of $A_{q,d}$ is $a_{q'}$ for $q' = (\sigma(q))_d$. Hence, the new $x$ is in $A_{q'}$. That is why the algorithm works correctly. The next theorem states that the algorithm induces a bijection between $[0,1)$ and the words in $\mathcal{L}_\infty$.

**Theorem 5.** *Every number $x \in [0,1)$ has a unique $\sigma$-representation $(d_j)_{j=1}^\infty$ in $\mathcal{L}_\infty$ such that*

$$x = \sum_{j=1}^{\infty} \alpha_{\delta(q_0, d_0 \cdots d_{j-1})}(d_j) \tau^{j-1}$$

*with $d_0 = \varepsilon$. Conversely, for every $(d_j)_{j=1}^\infty$ in $\mathcal{L}_\infty$ the above equation gives a unique element $x$ in $[0,1)$.*

*Proof.* This follows immediately from the construction above. ☐

We remark that this result is not new. It can be found in [16, 3.2 Théorème] in the context of substitutions with a primitive incidence matrix $M_\sigma$. Moreover, such number systems were studied extensively for arbitrary regular languages satisfying the conclusion of Lemma 4 in recent years in [28, 29, 21, 34]. To our knowledge our formulation is new, but we tried to be as close as possible to the notation of the mentioned papers.

We consider two important classes of numbers, namely those with periodic and those with finite $\sigma$-representations. We say that an element $x \in \mathbb{Q}(\beta) \cap [0,1)$ has an ultimately periodic $\sigma$-representation if and only if when applying the algorithm we find $(\omega, x, q)$ and $(\omega', x', q')$ such that $x = x'$ and $q = q'$ (compare with [29, Theorem 27]). We call an algebraic integer $\beta$ a *Pisot number* if all conjugates other than itself have modulus less than one and a *Salem number* if the modulus of all the conjugates other than itself is less than or equal to one and at least one is equal to one. Recently, Rigo and Steiner [34] showed that if $\beta > 1$ is a Pisot number, then the set of real numbers in $[0,1)$ with finite or ultimately periodic $\sigma$-representation equals

$\mathbb{Q}(\beta) \cap [0,1)$. This generalized the well-known result on Rényi's classical $\beta$-expansion [33, 31] by Bertrand [14] and Schmidt [42] and a first attempt in [29] for abstract number systems. Moreover, they showed that if $\beta$ is neither a Pisot nor a Salem number, then there exists at least one point in $\mathbb{Q}(\beta) \cap [0,1)$ which has an infinite $\sigma$-representation which is not ultimately periodic.

We denote by $\mathrm{Fin}(\beta)$ the set of all numbers $x \in [0,1)$ whose $\sigma$-representation is finite. We claim that

$$\mathrm{Fin}(\beta) \subseteq \mathbb{Z}a_{q_0} + \mathbb{Z}a_{q_1} + \ldots + \mathbb{Z}a_{q_k}. \tag{16}$$

Indeed, by the theorem every element in $\mathrm{Fin}(\beta)$ is a finite linear combination of terms of the form $a_q\tau^{n-1}$. Since this is one of the coordinates of the vector $\tau^{n-1}\vec{a} = M_\sigma^{-k}\vec{a}$ and $M_\sigma^{-1}$ has integral entries by (A4), it follows that $a_q\tau^{n-1} \in \mathbb{Z}a_{q_0} + \mathbb{Z}a_{q_1} + \ldots + \mathbb{Z}a_{q_k}$. This proves (16). By $a_{q_0} = \tau$, we especially have $\mathbb{Z}[\tau] \subseteq \mathbb{Z}a_{q_0} + \mathbb{Z}a_{q_1} + \ldots + \mathbb{Z}a_{q_k} =: \Omega$. The following lemma gives information about $\Omega$.

**Lemma 5.** *Let $\Omega = \mathbb{Z}a_{q_0} + \mathbb{Z}a_{q_1} + \ldots + \mathbb{Z}a_{q_k}$. Then*

(i) *$\Omega$ is a $\mathbb{Z}$-module that is not necessarily free.*

(ii) *If $\Omega$ is free, then $\dim\Omega = \deg\beta$ and therefore*

$$\mathbb{Z}[\tau] \subseteq \Omega \cong \mathbb{Z}[\tau] \cong \mathbb{Z}^{\deg\beta}$$

*as $\mathbb{Z}$-modules. In this case we can choose a basis from $a_{q_0}, \ldots, a_{q_k}$.*

(iii) *If $\vec{u}_0, \ldots, \vec{u}_{k-1}$ are linearly independent over $\mathbb{Z}$, then $\Omega = \mathbb{Z}[\tau]$.*

*Proof.* Clearly, $\Omega$ is a $\mathbb{Z}$-module. For a substitution with incidence matrix

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 5 \\ 0 & 1 & 3 \end{pmatrix}$$

we have

$$(a_0, a_1, a_2) = \left(\tau, -\frac{19}{47}\tau + \frac{13}{47}, \frac{5}{47}\tau + \frac{4}{47}\right),$$

where $\tau^2 - 5\tau + 1 = 0$. Hence $\Omega$ is not necessarily free.

Next (ii). Since $\mathbb{Z}[\tau] \subseteq \Omega$, the dimension is at least $\deg\beta$. On the other hand, from $a_q \in \mathbb{Q}[\tau]$ it follows that $a_i, 1, \tau, \ldots, \tau^{\deg\beta - 1}$ are linearly dependent over $\mathbb{Z}$ for $i = 0, 1, \ldots, k$. If $\Omega$ is free and therefore has a basis, its dimension is equal to $\deg\beta$. Obviously, the basis can be chosen from the set of generators.

Finally (iii). We already observed above that $\tau^n \in \Omega$ for every $n \in \mathbb{N}$, because of the zeroth coordinate in the equation $\tau\vec{a} = M_\sigma^{-1}\vec{a}$. Hence $\Omega = \mathbb{Z}[\tau]$ if the zeroth row vectors of $M_\sigma, M_\sigma^0, M_\sigma^{-1}, \ldots, M_\sigma^{2-k}$ are linearly independent over $\mathbb{Z}$. Since $M_\sigma$ is unimodular, this is equivalent with the zeroth row vectors of $M_\sigma^{k-1}, M_\sigma^{k-2}, \ldots, M_\sigma, M_\sigma^0$ being linearly independent over $\mathbb{Z}$, which is the same as $\vec{u}_0, \ldots, \vec{u}_{k-1}$ being linearly independent over

$\mathbb{Z}$. □

Observe that it is easy to decide whether $\Omega$ is free or not, since we only have to pick a basis from $a_{q_0}, \ldots, a_{q_k}$ (so that all the other elements are linear combinations of them with integer coefficients).

As in the classical case of $\beta$-expansions (e.g. in [1, 3, 20]) we consider the condition

$$\mathrm{Fin}(\beta) = (\mathbb{Z}a_{q_0} + \mathbb{Z}a_{q_1} + \ldots + \mathbb{Z}a_{q_k}) \cap [0,1). \tag{F}$$

This condition says that all possible candidates indeed have a finite $\sigma$-representation. We denote the conjugates of $x, a_q, \beta$ in $\mathbb{Q}(\beta)$ by $x^{(j)}, a_q^{(j)}, \beta^{(j)}$ for $j = 0, 1, \ldots, \deg\beta - 1, q \in Q$ with the convention that $x^{(0)} = x, a_q^{(0)} = a_q, \beta^{(0)} = \beta$. By using the methods from [1, 3, 20] we prove

**Theorem 6.**

(a) *Let $\beta$ be a Pisot number. Then $\beta$ has the property* (F) *if and only if every element of the finite set*

$$\left\{x \in \Omega \,:\, |x^{(j)}| \leq M_j \ (j \geq 0)\right\},$$

*where $M_0 = \max\{\alpha_q \,:\, q \in Q\}$ and, for $j = 1, \ldots, \deg\beta - 1$,*

$$M_j = \frac{\max\left\{|a_q^{(j)}| \,:\, q \in Q\right\} \cdot \#\Delta \cdot \#\Sigma}{1 - |\beta^{(j)}|}$$

*has a finite $\sigma$-representation in every language $\mathcal{L}_q$ (with $q \in Q$). Moreover, this can be checked effectively.*

(b) *If $\mathbb{Z}[\tau] \cap [0,1) \subseteq \mathrm{Fin}(\beta)$, then $\beta$ is a Pisot number or a Salem number.*

*Proof.* We first prove (a). We show that the validity of (F) can be decided by checking only the finitely many elements $x$ in the displayed set. First let $x \in \Omega \cap [0,1)$ with expansion

$$x = \sum_{j=0}^{\infty} \sum_{q \in Q} a_q \varepsilon_{q,j} \tau^j,$$

where $\varepsilon_{q,j} = \#\{s < d_j \,:\, \delta(\delta(q_0, d_0 \cdots d_{j-1}), s) = q\}$ are integers and $d_0 d_1 \cdots$ is the $\sigma$-representation of $x$, and $a_q \in \mathbb{Q}[\tau]$. Let $\epsilon > 0$ and $m$ so large that

$$\left|x^{(j)}\right| \left(\left|\beta^{(j)}\right|\right)^m < \epsilon$$

for $j = 1, \ldots, \deg\beta - 1$. We consider

$$y := x\beta^m - \sum_{j=0}^{m-1} \sum_{q \in Q} a_q \varepsilon_{q,j} \tau^{j-m} = \sum_{j=0}^{\infty} \sum_{q \in Q} a_q \varepsilon_{q,m+j} \tau^j. \tag{17}$$

Obviously, we have $y \in \Omega$, $0 \leq y < \alpha_q$ for some $q \in Q$ and $|y^{(j)}| < \epsilon + M_j$ for $j > 0$. Therefore, since $\epsilon > 0$ is arbitrary, it suffices to check whether or not all expressions in the set

$$\left\{y \in \Omega \,:\, |y^{(j)}| \le M_j \text{ for } j = 0,1,\ldots,\deg\beta - 1\right\} \tag{18}$$

have a finite $\sigma$-representation in all languages $\mathcal{L}_q$ with $q \in Q$. Equivalently, we may consider all the elements contained in

$$\left\{x \in \mathbb{Z}[\tau] \,:\, |x^{(j)}| \le b_j\right\},$$

where $b_j = M_j \cdot \min\{n \in \mathbb{N} \,:\, na_{q_i} \in \mathbb{Z}[\tau] \text{ for } i = 1,\ldots,k\}$ ($j = 0,1,\ldots,\deg\beta - 1$). It is now easy to compute these numbers effectively, since we can solve the equations $B\vec{x} = \vec{z}$ where

$$B = \begin{pmatrix} 1 & \tau & \cdots & \tau^{\deg\beta-1} \\ 1 & \tau^{(1)} & \cdots & (\tau^{(1)})^{\deg\beta-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \tau^{(\deg\beta-1)} & \cdots & (\tau^{(\deg\beta-1)})^{\deg\beta-1} \end{pmatrix},$$

$\vec{x} \in \mathbb{Z}^{\deg\beta}$ are the coordinates of $x$ in the basis $1,\ldots,\tau^{\deg\beta-1}$, and with $\|\vec{z}\|_\infty \le \max\{b_j : \ j = 0,\ldots,\deg\beta - 1\}$ and where $\|\ \|_\infty$ denotes the usual maximum norm. Since $1,\tau,\ldots,\tau^{\deg\beta-1}$ is a basis of $\mathbb{Q}(\beta)$, the determinant of $B$, which is equal to the discriminant of the number field, is different from zero. Therefore, we get

$$\begin{aligned}\|\vec{x}\|_\infty = \|B^{-1}\vec{z}\|_\infty \le \|B^{-1}\|_\infty \cdot \|\vec{z}\|_\infty \\ \le \max_{j=0,\ldots,\deg\beta-1}\left\{\sum_{i=0}^{\deg\beta-1}\left|(B^{-1})_{i,j}\right|\right\} \cdot \max_{j=0,\ldots,\deg\beta-1}\{b_j\},\end{aligned}$$

where $(B^{-1})_{i,j}$ is the element at position $(i,j)$ in $B^{-1}$. Thus we have an effective bound for $\|\vec{x}\|_\infty$ and it suffices to show that all elements in (18) obtained by such $\vec{x}$ have a finite $\sigma$-representation.

To conclude the first part, we only have to point out that when applying the expansion algorithm to one of the elements in the set in (18), we remain in the same set, since this just means to increase $m$ by 1 in (17), and therefore the possible period (observe that by the result in [34] all elements have an ultimately periodic $\sigma$-representation) is bounded by the cardinality of this set.

Next (b). Suppose $\mathbb{Z}[\tau] \cap [0,1) \subseteq \mathrm{Fin}(\beta)$. Assume that $\beta$ has a conjugate $\gamma = \beta^{(j)}$ with $\gamma \neq \beta, |\gamma| > 1$. Let $\eta = \max\{\beta^{-1}, |\gamma|^{-1}\}$. Take $x := [\beta^m] - \beta^m + 1$, where $[z]$ is the largest integer less than or equal to $z$. It is plain that $0 < x < 1$. By assumption $x$ has a finite expansion

$$x = \sum_{j=0}^{k}\sum_{q\in Q} a_q \varepsilon_{q,j}\tau^j.$$

Taking conjugates we get

$$[\beta^m] - \gamma^m + 1 = \sum_{j=0}^{k} \sum_{q\in Q} a_q^{(j)} \varepsilon_{q,j} \gamma^{-j}.$$

Subtracting the two expansions we end up with

$$\gamma^m - \beta^m = \sum_{j=0}^{k} \sum_{q\in Q} \varepsilon_{q,j} (a_q \tau^j - a_q^{(j)} \gamma^{-j})$$

and by observing that $0 \le \varepsilon_{q,j} \le l$ we see that the right-hand side is bounded by

$$2 \max \left\{ \left| a_q^{(j)} \right| \,:\, q \in Q, j = 0, \ldots, \deg\beta - 1 \right\} \cdot \#\Delta \cdot \#\Sigma \cdot (1-\eta)^{-1},$$

whereas the left-hand side of the equation is unbounded. This contradiction shows that $\beta$ is a Pisot number or a Salem number. $\square$

An alternative proof that the set (17) is finite follows from the fact that all these elements are contained in the set of solutions of the norm form inequality $N_{\mathbb{Q}(\beta)/\mathbb{Q}}(z) \le b_0 b_2 \cdots b_{\deg\beta-1}$, which has finitely many solutions by Schmidt's famous result on norm form equations [43, Satz 2, p. 5] (see also [19]). However, this method is ineffective.

The second statement of Theorem 6 implies that from (F) we can conclude that $\beta$ is a Pisot number or a Salem number. Moreover, we remark that the statement holds not only in the case of number systems associated to the underlying substitution, which is considered throughout this paper, but in the context of abstract number systems as well.

It follows from our proof that if an element of $\Omega \cap [0,1)$ does not have a finite expansion, then its expansion is ultimately periodic (this follows also from a result of Rigo and Steiner [34]). If so, from the proof of Theorem 6(a) we can deduce an upper bound for the preperiod depending on the input $x$ (using the $\epsilon$-condition) and an upper bound for the period (viz. the cardinality of the set (18)). In particular, only finitely many distinct periods can occur.

Finally, we point out that the condition in the first part of the theorem can be simplified if the automaton has the property that

$$\delta(q_0, w) = q_0 \text{ infinitely often as } |w| \to \infty. \tag{19}$$

This is the case e.g. for the Tribonacci expansion considered in [35] and more generally for all so-called $\beta$-substitutions, that are substitutions of the form $\sigma(0) = 0^{n_0}1, \sigma(1) = 0^{n_1}2, \ldots, \sigma(k-1) = 0^{n_{k-1}}k, \sigma(k) = 0^{n_k}$, where $n_0, n_k > 0$ and $n_i \ge 0$ $(i = 1, \ldots, k-1)$. Then the abstract number system associated to it generates the usual $\beta$-expansion and the incidence matrix has the form

$$\begin{pmatrix} n_0 & 1 & 0 & \cdots & 0 \\ n_1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ n_{k-1} & 0 & 0 & \cdots & 1 \\ n_k & 0 & 0 & \cdots & 0 \end{pmatrix}$$

(cf. [29]).

If we have the additional property (19), it suffices to check whether or not all elements in

$$\Big\{x \in \Omega \cap [0,1) \,:\, |x^{(j)}| \le M_j \ (j \ge 1)\Big\}$$

have a finite expansion in the language $\mathcal{L}$. (It is not necessary to check all languages $\mathcal{L}_q$ as above.) This follows from the proof of Theorem 6(a), since we can choose some large $m$ such that $\delta(q_0, d_0 \cdots d_{m-1}) = q_0$.

## 5. The space filling property

In this section we will link the number system associated to the substitution introduced in the previous section with the words $\hat{w}^{(n)}$ and show that the question whether this word converges to a word that is space filling or not can be decided by means of the first part of Theorem 6. From now on we write $a_0, a_1, \ldots, a_k$ instead of $a_{q_0}, a_{q_1}, \ldots, a_{q_k}$.

Write $C$ for the matrix with column vectors ${}^tM_\sigma^k \vec{e}_0^{(k+1)}, \vec{c}_1, \ldots, \vec{c}_k$ introduced in Section 3. We mention that in the canonical case $C = {}^tM_\sigma^k$. Put $\vec{a} = {}^t(a_0, \ldots, a_k)$ and $\vec{b} = {}^t(b_0, \ldots, b_k)$, where the $b_i$ are defined in (10). We derive the following important relation between $\vec{b}$ and $\vec{a}$.

**Lemma 6.** *For all $n \in \mathbb{N}$ we have*

$$\vec{b} = \tau^{n-1}\ {}^tCM_\sigma^{n-k}\vec{a}. \tag{20}$$

*Proof.* By (15), (A6) and (10) we have

$$\tau^{n-1}\ {}^tCM_\sigma^{n-k}\vec{a} = \beta^{1-n}\ {}^tC\beta^{n-k}\vec{a} = \beta^{1-k}\ {}^tC\vec{a} = \vec{b}.$$

Thus we have proved the assertion. □

In case that $\vec{c}_i$ equals the incidence vector $\vec{u}_i = {}^tM_\sigma^i \vec{e}_0^{(k+1)}$ of $u^{(i)}$ for $i = 1, \ldots, k$ we get

$$b_i = \tau^{k-1}\ {}^t\vec{c}_i\vec{a} = \tau^{k-1}\ {}^t\vec{e}_0^{(k+1)} M_\sigma^i \vec{a} = \tau^{k-1}\ {}^t\vec{e}_0^{(k+1)} \beta^i \vec{a} = \tau^{k-i-1} a_0 = \tau^{k-i}$$

for $i = 1, \ldots, k$.

By (20) with $n = 1$ we have

$$\mathbb{Z} + \mathbb{Z}b_1 + \ldots + \mathbb{Z}b_k = \mathbb{Z}a_0 + \ldots + \mathbb{Z}a_k = \Omega.$$

In case $\vec{c}_i = \vec{u}_i$ for $i = 1, \ldots, k$ this implies $\Omega = \mathbb{Z}[\tau]$ by Lemma 5(iii).

In Section 3 we also introduced the projections $\Pi : \mathbb{Z}^{k+1} \to \mathbb{Z}^k$ and $\Phi_n : \mathbb{Z}^{k+1} \to \mathbb{Z}^k$ and the transition vectors $\vec{a}_i^{(n)}$ $(i = 0, 1, \dots, k)$. The identity (20) has the following consequence.

**Lemma 7.** *For all* $n \in \mathbb{N}$ *and* $i = 0, 1, \dots, k$ *we have*

$$(b_1, \dots, b_k) \cdot \vec{a}_i^{(n)} \equiv a_i \tau^{n-1} \pmod 1. \tag{21}$$

*Proof.* Let the inverse matrix of $M_\sigma^{(n-k)} C$ have respective column vectors $\vec{x}_0^{(n)}, \dots, \vec{x}_k^{(n)}$, hence $M_\sigma^{(n-k)} C \vec{x}_i^{(n)} = \vec{e}_i^{(k+1)}$ for $i = 0, 1, \dots, k$. Then, for $i = 1, \dots, k$, by (7) and the definition and linearity of $\Phi_n$,

$$\begin{aligned} \vec{a}_i^{(n)} &= \Phi_n(\vec{e}_i^{(k+1)}) = \Phi_n(M_\sigma^{(n-k)} C \vec{x}_i^{(n)}) \\ &= x_{i0}^{(n)} \Phi_n({}^t M_\sigma^n \vec{e}_0^{(k+1)}) + \sum_{j=1}^{k} x_{ij}^{(n)} \Phi_n({}^t M_\sigma^{(n-k)} \vec{c}_j) = \sum_{j=1}^{k} x_{ij}^{(n)} \vec{e}_j^{(k)} = \Pi \vec{x}_i^{(n)}, \end{aligned}$$

where $\vec{x}_i^{(n)} = {}^t(x_{i0}^{(n)}, \dots, x_{ik}^{(n)}) \in \mathbb{Z}^{k+1}$ $(i = 0, \dots, k)$ by the unimodularity of $M_\sigma$ and $C$. We obtain, by (20) and (15),

$$\begin{aligned} (b_1, \dots, b_k) \cdot \vec{a}_i^{(n)} &= \tau^{n-1} \Pi({}^t\vec{a}\ {}^t M_\sigma^{(n-k)} C)\ \Pi(\vec{x}_i^{(n)}) \\ &= \tau^{n-1}\ {}^t\vec{a}\ {}^t M_\sigma^{(n-k)} C \vec{x}_i^{(n)} - \tau^{n-1}\ {}^t\vec{a}\ {}^t M_\sigma^{(n)} \vec{e}_0^{(k+1)} x_{i0}^{(n)} \\ &= \tau^{n-1}\ {}^t\vec{a} \vec{e}_i^{(k+1)} - \tau^{n-1}\ x_{i0}^{(n)} \beta^n a_0 = \tau^{n-1} a_i - x_{i0}^{(n)} \equiv \tau^{n-1} a_i \pmod 1, \end{aligned}$$

since $x_{i0}^{(n)} \in \mathbb{Z}$. □

From the proof of Lemma 7 it follows that

$$\Phi_n(\vec{x}) = \Pi\left(C^{-1}\ {}^t M_\sigma^{k-n} \vec{x}\right)$$

for all $\vec{x} \in \mathbb{Z}^{k+1}$ (compare with (6) for $\Phi_n^*$). Especially, this means that the transition vectors are equal to the columns of $C^{-1}$ after deletion of the zeroth entries.

We now define a function $f$ from $\mathbb{Z}^k$ to $(\mathbb{Z}a_0 + \ldots + \mathbb{Z}a_k) \cap [0, 1)$ by

$$f(\vec{x}) := f(x_1, \dots, x_k) := (b_1, \dots, b_k) \cdot \vec{x} \pmod 1 = \left\{ \sum_{i=1}^{k} x_i b_i \right\}$$

where $\vec{x} = {}^t(x_1, \dots, x_k)$. This function is linear modulo 1. Further, by (21),

$$f(\vec{a}_i^{(n)}) \equiv a_i \tau^{n-1} \pmod 1. \tag{22}$$

Moreover, if $a_0, \dots, a_k$ is a basis of the $\mathbb{Z}$-module $\Omega = \mathbb{Z}a_0 + \ldots + \mathbb{Z}a_k$, then it is bijective and by (21) we have, provided that $a_i \tau^{n-1} < 1$,

$$\vec{a}_i^{(n)} = f^{-1}\left(a_i \tau^{n-1}\right).$$

Generally, $f$ is an epimorphism of $\mathbb{Z}$-modules. By (12) the limit word $\hat{W}$ is the restriction of $f$ on its domain, i.e. for all $\vec{x} \in \mathrm{dom}\hat{W}$ we have $f(\vec{x}) = \hat{W}(\vec{x})$. Therefore, studying $f$ results in getting information on $\hat{W}$.

Using these facts together with Theorem 1 we will characterize the domain of the words $\hat{w}^{(n)}$.

**Theorem 7.** *If $\vec{x} \in \mathbb{Z}^k$ is in the domain of $\hat{w}^{(n)}$, then $f(\vec{x})$ has a finite $\sigma$-representation $d_1 \dots d_n$ of length at most $n$, i.e.*

$$f(\vec{x}) = \sum_{j=1}^{n} \alpha_{\delta(q_0, d_0 \cdots d_{j-1})}(d_j)\tau^{j-1}$$

*with $d_0 = \varepsilon$. Conversely, for all elements $\vec{x}$ in the domain of $\hat{w}^{(n)}$ the $\sigma$-representations of $f(\vec{x})$ are different and all words in $\mathcal{L}$ of length at most $n$ appear.*

*Moreover, if the $\sigma$-representation of $f(\vec{x})$ for $\vec{x}$ in the domain of $\hat{w}^{(n-1)}$ is $d_0 \cdots d_{n-1}$, $q = \delta(q_0, d_0 \cdots d_{n-1})$ and $w^{(n)}(x) = m$, then*

$$w^{(n)}\left(\vec{x} + \sum_{i=1}^{s} \vec{a}^{(n)}_{(\sigma(q))_i}\right) = m + s$$

*and the $(m+s)$-th letter in $u^{(n)}$ is $(\sigma(q))_s$ for $s = 1, \dots, |\sigma(q)|$, whereas*

$$\vec{x} + \sum_{i=1}^{|\sigma(q)|} \vec{a}^{(n)}_{(\sigma(q))_i} = \vec{x} + \vec{a}^{(n-1)}_q$$

*is in the domain of $w^{(n-1)}$.*

*Proof.* We prove the assertion by induction on $n$.

For $n = 0, 1$ the statement is trivially true. Observe that $\delta(q_0, d_0) = 0$, which gives the second part of the statement, namely the jumps we have to make with the transition vectors when going from the origin (the only point in the domain of $\hat{w}^{(0)}$) to the points in $\mathrm{dom}\hat{w}^{(1)}$ obtained from $u^{(1)} = \sigma(0)$.

Suppose $\vec{x} \in \mathbb{Z}^k$ is in the domain of $\hat{w}^{(n-1)}$ and has the $\sigma$-representation $d_0 d_1 \cdots d_{n-1}$. Then, by the induction hypothesis,

$$f\left(\vec{x}\right) = \sum_{j=1}^{n-1} \alpha_{\delta(q_0, d_0 \cdots d_{j-1})}(d_j)\tau^{j-1}.$$

Let $\delta(q_0, d_0 d_1 \cdots d_{n-1}) = q$. Then $\vec{x}$ is also in the domain of $\hat{w}^{(n)}$ and the new elements in $\hat{w}^{(n)}$ which originate from $\vec{x}$ are given by

$$\vec{y}_s := \vec{x} + \sum_{i=1}^{s} \vec{a}^{(n)}_{(\sigma(q))_i} \ (s = 1, \dots, |\sigma(q)| - 1).$$

For $s = |\sigma(q)|$ we get the successor of $\vec{x}$ in $w^{(n-1)}$ according to (14). We have, by (22),

$$\begin{aligned} f\left(\vec{x}+\sum_{i=1}^{s}\vec{a}^{(n)}_{(\sigma(q))_i}\right) &\equiv f\left(\vec{x}\right)+\sum_{i=1}^{s} f\left(\vec{a}^{(n)}_{(\sigma(q))_i}\right) \\ &\equiv \sum_{j=1}^{n-1}\alpha_{\delta(q_0,d_0\cdots d_{j-1})}(d_j)\tau^{j-1}+\sum_{i=1}^{s} a_{(\sigma(q))_i}\tau^{n-1} \\ &= \sum_{j=1}^{n}\alpha_{\delta(q_0,d_0\cdots d_{j-1})}(d_j)\tau^{j-1} \pmod 1, \end{aligned}$$

where $d_n = s$, because, by the previous section,

$$\begin{aligned} \alpha_{\delta(q_0,d_0\cdots d_{n-1})}(d_n) &= \alpha_q(s) \\ &= \sum_{q'\in Q}\left(a_{q'}\cdot\#\{t<s\,:\,\delta(q,t)=q'\}\right)=\sum_{i=1}^{s} a_{(\sigma(q))_i}. \end{aligned} \tag{23}$$

By the second part of Theorem 5, $\sum_{j=1}^{n}\alpha_{\delta(q_0,d_0\cdots d_{j-1})}(d_j)\tau^{j-1}$ is the $\sigma$-representation of some number in $[0,1)$. It follows that the congruence is in fact an equality. Moreover, since the value of $w^{(n)}$ is uniquely determined by the word $d_0d_1\cdots d_n$, we conclude (by Theorem 5) that this is really the representation of $f(\vec{y}_s)$ which proves the assertion. Obviously, all representations are different and all words in $\mathcal{L}$ of length at most $n$ appear.

The second part of the theorem follows at once from the consideration above together with Theorem 1. □

We need still another lemma (which is implicitly contained in the last theorem) to characterize the space filling property of the word $\hat{W}$.

**Lemma 8.** *If $\vec{x}\in\mathbb{Z}^k$ is in the domain of $\hat{w}^{(n)}$, then*

$$(\vec{x}+\ker f)\cap \mathrm{dom}\hat{w}^{(n)}=\{\vec{x}\},$$

*where* $\ker f$ *denotes the kernel of the map* $f$.

*Proof.* We prove the statement by induction on $n$. For $n=0$ the result is trivial. Assume now that $\vec{x},\vec{y}\in\mathrm{dom}\hat{w}^{(n)}$ and $\vec{x}-\vec{y}\in\ker f$, i.e. $f(\vec{x})=f(\vec{y})$. It follows from the proof of Theorem 1 that we can write

$$\vec{x}=\vec{x}_0+\sum_{i=1}^{s}\vec{a}^{(n)}_{(\sigma(q))_i},\quad \vec{y}=\vec{y}_0+\sum_{i=1}^{s'}\vec{a}^{(n)}_{(\sigma(q'))_i},$$

for certain $\vec{x}_0,\vec{y}_0\in\mathrm{dom}\hat{w}^{(n-1)}$ and $s,s',q,q'\in\Sigma$. Therefore,

$$f(\vec{x}_0)+\sum_{i=1}^{s}a_{(\sigma(q))_i}\tau^{i-1}\equiv f(\vec{x})=f(\vec{y})\equiv f(\vec{y}_0)+\sum_{i=1}^{s'}a_{(\sigma(q'))_i}\tau^{i-1}$$

modulo 1 by (22), and we have equality by the same argument as at the end of the proof of the theorem above. Since the representation of a number in $\Omega \cap [0,1)$ is unique, it follows that $s = s', q = q'$ and $f(\vec{x}_0) = f(\vec{y}_0)$, but the latter equality contradicts to the induction hypothesis, unless $\vec{x}_0 = \vec{y}_0$ and therefore $\vec{x} = \vec{y}$. □

The next result shows that all elements in the domain are completely determined by their $\sigma$-representations and the transition vectors.

**Corollary 1.** *If $\vec{x} \in \mathrm{dom}\hat{W}$ with $\sigma$-representation $d_1 \cdots d_n$ of length $n$, then*

$$\vec{x} = \sum_{j=1}^{n} \sum_{i=1}^{d_j} \vec{a}^{(j)}_{(\sigma(\delta(q_0, d_1 d_1 \cdots d_{j-1})))_i}.$$

*Proof.* By Theorem 7, (23) and (21) we have

$$\begin{aligned} f(\vec{x}) &= \sum_{j=1}^{n} \alpha_{\delta(q_0, d_0 \cdots d_{j-1})}(d_j) \tau^{j-1} = \sum_{j=1}^{n} \sum_{i=1}^{d_j} a_{(\sigma(\delta(q_0, d_0 \cdots d_{j-1})))_i} \tau^{j-1} \\ &\equiv (b_1, \ldots, b_k) \cdot \left( \sum_{j=1}^{n} \sum_{i=1}^{d_j} \vec{a}^{(j)}_{(\sigma(\delta(q_0, d_1 d_1 \cdots d_{j-1})))_i} \right) \pmod 1. \end{aligned}$$

The vector in brackets is in the domain of $\hat{W}$ by Theorem 1. Furthermore we recall $f(\vec{x}) = (b_1, \ldots, b_k) \cdot \vec{x} \bmod 1$. The assertion now follows from Lemma 8. □

We remark that it follows from Theorem 7 and its corollary that the *Rauzy color* of $w^{(n)}$ at position $\vec{x} \in \mathrm{dom} w^{(n)}$, that is the index of the transition vector to make the jump from $w^{(n)}(\vec{x})$ to $w^{(n)}(\vec{x}) + 1$, is equal to the state $\delta(q_0, d_0 \cdots d_n)$ the automaton $\mathcal{M}$ is in after it is fed with the representation $d_0 \cdots d_n$. (If $\vec{x} \in \mathrm{dom} w^{(h)}$ for some $h < n$ we read $d_0 \cdots d_n$ as $d_0 \cdots d_h 0 \cdots 0$.)

| | | | | | | |
|---|---|---|---|---|---|---|
| 00011 | | 1101 | 1001 | 0111 | 0100001 | 0001111001 |
| 100001 | 011001 | | 0001 | 1111001 | 110011 | 100011 |
| 000001 | 111 | 11 | 1 | 011 | 0011001 | 000011 |
| | 010001 | 001 | $\varepsilon$ | | | |
| 111101 | | | | 01 | | 111111 |
| | 001101 | | 1111 | 11001 | 10001 | 01101 |
| 1101111 | 1001111 | 0111111 | | 0011 | 00001 | |

Fig. 6: All finite $\sigma$-representations of $f(\vec{x})$, i.e. all elements of $\hat{W}(\vec{x})$, where $\vec{x} = (x_1, x_2)$ has $\max\{|x_1|, |x_2|\} \le 3$ in the canonical case for the flipped Tribonacci substitution $\sigma(0) = 01, \sigma(1) = 20, \sigma(2) = 0$ (compare with Fig. 3).

From Lemma 8 we see that if we factor out the kernel of $f$, we get a well defined isomorphism of $\mathbb{Z}$-modules given by

$$\begin{aligned} f^*: \mathbb{Z}^k/\ker f &\longrightarrow (\mathbb{Z}a_0+\ldots+\mathbb{Z}a_k)\cap[0,1)=\Omega\cap[0,1)\\ \vec{x}+\ker f &\longmapsto f(\vec{x}). \end{aligned}$$

Observe that by Lemma 5(i) the $\mathbb{Z}$-module $\mathbb{Z}^k/(\ker f)$ is not always isomorphic to some sublattice of $\mathbb{Z}^k$. If $\Omega$ is free, then we can map $\hat{w}^{(n)}$ to some sublattice of $\mathbb{Z}^k$ of dimension $\deg\beta-1$ by a projection $\Psi$ along the kernel of $f$ as follows. Let $1, b_{i_1}, \ldots, b_{i_{\deg\beta-1}}$ be a basis of $\Omega$. Then each $b_i$ with $i\notin\{i_1,\ldots,i_{\deg\beta-1}\}$ can be expressed as a linear combination in this basis. We denote the corresponding vector by $\vec{x}_i\in\mathbb{Z}^k$, where we have removed the zeroth coordinate, hence $f(\vec{x}_i)=0$ and $\vec{x}_i$ has an entry 1 in position $i$ and 0 at every position $j\notin\{i_1,\ldots,i_{\deg\beta-1},i\}$. Now the map $\Psi$ can be described by a linear combination of such vectors $\vec{x}_i$ for any given point in the domain of $\hat{w}^{(n)}$ such that all the coordinates at positions $i$ with $i\notin\{i_1,\ldots,i_{\deg\beta-1}\}$ become 0. Thus

$$\Psi\left(\mathbb{Z}^k/\ker f\right)=\bigoplus_{i\in\{i_1,\ldots,i_{\deg\beta-1}\}}\mathbb{Z}\vec{e}_i^{(k)}\cong\mathbb{Z}^{\deg\beta-1}$$

and especially we have

$$\mathrm{dom}\Psi(\hat{W})=\Psi\left(\mathrm{dom}\hat{W}\right)=\bigoplus_{i\in\{i_1,\ldots,i_{\deg\beta-1}\}}\mathbb{Z}\vec{e}_i^{(k)}.$$

In other words, if $\vec{x}$ is in the domain of $\hat{w}^{(n)}$, then there is a representative in the same coset mod ker $f$ which is in this sublattice of $\mathbb{Z}^k$.

If $\Omega$ is free, then $\Psi$ maps $\hat{w}^{(n)}$ onto the above sublattice of $\mathbb{Z}^k$. By combining Theorems 6 and 7 we get the following result concerning the space filling property of the the limit word $\hat{W}=\lim_{n\to\infty}\hat{w}^{(n)}$. This answers the question whether the limit word fills the submodule $\mathbb{Z}^k/(\ker f)$, that is the corresponding sublattice of $\mathbb{Z}^k$ if $\Omega$ is a free $\mathbb{Z}$-module.

**Theorem 8.** *The limit word $\hat{W}$ is space filling if and only if the associated number system has the finiteness property* (F)*. If $\beta$ is a Pisot number, then the space filling property is decidable.*

*Proof.* By Theorem 7 the domain of the limit word $\hat{W}$ is in bijection with $\mathrm{Fin}(\beta)$ by sending $\vec{x}\in\mathrm{dom}\hat{W}$ to $f(\vec{x})$. Therefore Lemma 8 shows that the domain of $\hat{W}$ consists exactly of all elements $\vec{x}\in\mathbb{Z}^k$ for which $f(\vec{x})$ is in $\mathrm{Fin}(\beta)$. The first statement now follows from the observation that the domain of $f$ equals $(\mathbb{Z}a_0+\ldots+\mathbb{Z}a_k)\cap[0,1)$. Since by part (a) of Theorem 6 property (F) is decidable if $\beta$ is Pisot, the space filling property is decidable in this case. This completes the proof. $\square$

Observe that in the Pisot case the space filling property of $\hat{W}$ can therefore be decided effectively by checking if all elements from the finite set given in Theorem 6 admit only finite representations in all languages $\mathcal{L}_q$ $(q\in Q)$.

The above mentioned space filling properties carry over without any troubles to the BV-words $v$ (see [11, 12, 13]) studied in [10] and [45].

## 6. Examples

In this section we will give five examples illustrating the theory developed in the previous sections. The first example is a substitution on three letters which gives a limit word that is space filling everywhere in $\mathbb{Z}^2$. The second example is the flipped Tribonacci substitution that was already mentioned at some previous occasions in this paper and which leads to a word that is not space filling. The next two examples give words that are space filling in a 1- and 2-dimensional submodule that can be projected to a line and a plane, respectively. The last example gives a word on four letters that is in a 1-dimensional submodule which projects to a line but is not space filling. We hope that these examples will give a better insight to the reader.

Example 1.

As a first example we consider the substitution

$$\sigma(0) = 0001, \sigma(1) = 02, \sigma(2) = 0 \tag{E1}$$

with incidence matrix and its inverse given by

$$M_\sigma = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$M_\sigma^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & -1 \end{pmatrix}.$$

The characteristic polynomial of $M_\sigma$ is $x^3 - 3x^2 - x - 1$ and therefore $M_\sigma$ has a dominant eigenvalue $\beta \approx 3.383$ and $\tau = \beta^{-1} \approx 0.296$. By calculating the normalized eigenvector of $M_\sigma$ to the eigenvalue $\beta$ we get

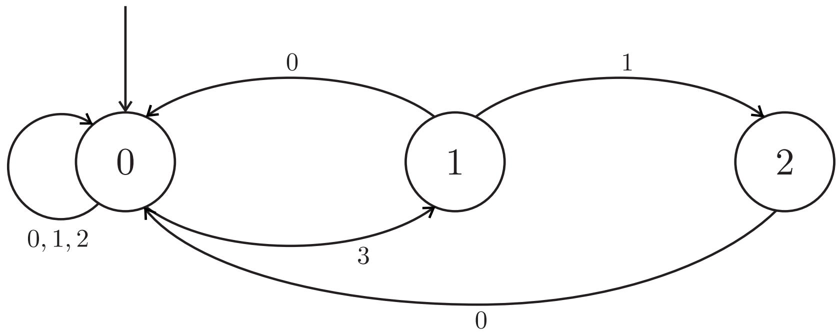
$$\vec{a} = (a_0, a_1, a_2) = (\tau, \tau^2 + \tau^3, \tau^2).$$

Hence, the $\mathbb{Z}$-module $\Omega$ equals $\mathbb{Z}[\tau]$.

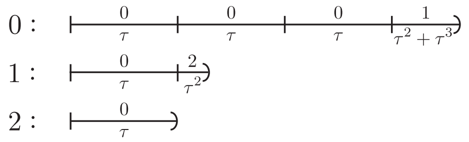
By using $M_\sigma^{-1}$ it is easy to compute the transition vectors for the canonical projection $\Phi_n^*$. We have

| $n$ | $\vec{a}_0^{(n)}$ | $\vec{a}_1^{(n)}$ | $\vec{a}_2^{(n)}$ |
|---|---|---|---|
| 1 | ${}^t(0,1)$ | ${}^t(0,-3)$ | ${}^t(1,-1)$ |
| 2 | ${}^t(1,-1)$ | ${}^t(-3,4)$ | ${}^t(-1,-2)$ |
| 3 | ${}^t(-1,-2)$ | ${}^t(4,5)$ | ${}^t(-2,6)$ |

The automaton associated to $\sigma$ is given by:




The associated number system scheme as described in the algorithm in Section 4 is given by




Therefore, we get the following projections of $u^{(1)} = 0001$, $u^{(2)} = 0001000$ $1000102$, $u^{(3)} = 000100010001020001000100010200010001000102000 10$ for $n = 1, 2, 3$ in the canonical case:

$w^{(1)}$

| |
|---|
| 3 |
| 2 |
| 1 |
| 0 |

$w^{(2)}$

| | | | |
|---|---|---|---|
| 12 | | | |
| 8 | 13 | | |
| 4 | 9 | | |
| 0 | 5 | 10 | |
| | 1 | 6 | 11 |
| | | 2 | 7 |
| | | | 3 |

$w^{(3)}$

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | 42 | | | |
| | | | 28 | 46 | | |
| | | 43 | 14 | 32 | | |
| | | 29 | 0 | 18 | 36 | |
| | 44 | 15 | 33 | 4 | 22 | 40 |
| | 30 | 1 | 19 | 37 | 8 | 26 |
| 45 | 16 | 34 | 5 | 23 | 41 | 12 |
| 31 | 2 | 20 | 38 | 9 | 27 | |
| 17 | 35 | 6 | 24 | | 13 | |
| 3 | 21 | 39 | 10 | | | |
| | 7 | 25 | | | | |
| | | 11 | | | | |

From the three images above we would expect that the limit word $\hat{W}$ will be space filling. Indeed that is the case since the associated number system has property (F). This follows either by applying Theorem 6, where we have to check whether all elements of the form $x_0 + x_1\tau + x_2\tau^2$ with $\|(x_0, x_1, x_2)\|_\infty \leq 162$ have a finite expansion in the language $\mathcal{L}$ (since the remark after Theorem 6 applies), or by applying the criterion in [1]. The latter is allowed since $\beta$ is a cubic Pisot unit and the number system is the usual $\beta$-expansion of a number (cf. the end of Section 4).

Since $\Omega$ is free with full dimension and $(b_0, b_1, b_2) = (1, \tau + \tau^2, \tau)$, we have $f(x_1, x_2) = x_1(\tau + \tau^2) + x_2\tau \bmod 1$. This follows from the fact that we are in the canonical case and thus the $b_i$'s are obtained from the $a_i$'s by dividing them by $\tau$. The table below gives the expansions of all $f(x_1, x_2)$ with $-3 \le x_1 \le 3, -8 \le x_2 \le 3$:

| | | | | | | |
|---|---|---|---|---|---|---|
| 2121 | 0111 | 1211 | **3** | 0301 | 20211 | 00111 |
| 1121 | 2221 | 0211 | **2** | **31** | 10211 | 21211 |
| 0121 | 1221 | **301** | **1** | **21** | 00211 | 11211 |
| 2231 | 0221 | **201** | $\varepsilon$ | **11** | **22** | 01211 |
| 1231 | **302** | **101** | **211** | **01** | **12** | **23** |
| 02231 | **202** | **001** | **111** | **221** | **02** | **13** |
| **303** | **102** | **212** | **011** | **121** | **231** | **03** |
| **203** | **002** | **112** | **222** | **021** | **131** | 30301 |
| **103** | **213** | **012** | **122** | 300211 | **031** | 20301 |
| **003** | **113** | **223** | **022** | 200211 | 310211 | 10301 |
| 220201 | **013** | **123** | 301211 | 100211 | 210211 | 00301 |
| 120201 | 230201 | **023** | 201211 | 000211 | 110211 | 220211 |

Observe that the expansions of length at most 3, which are the bold entries in the table above, are at the positions given by the domain of $w^{(3)}$ above. Moreover, the values of these entries are precisely the $\sigma$-representations of the corresponding values of $w^{(3)}$ with zeros added on the right to complete to three digits and, conversely, when we order the bold face numbers in the table lexicographically, their rank numbers are presented in $w^{(3)}$.

Example 2.

As second example we consider the flipped Tribonacci substitution

$$\sigma(0) = 01, \sigma(1) = 20, \sigma(2) = 0. \tag{E2}$$

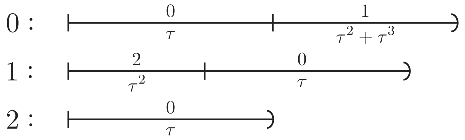
The incidence matrix and its inverse are given by

$$M_\sigma = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad M_\sigma^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

The characteristic polynomial is $x^3 - x^2 - x - 1$ and therefore $M_\sigma$ has the dominant eigenvalue $\beta \approx 1.839$ and $\tau = \beta^{-1} \approx 0.544$. The normalized eigenvector of $\beta$ is again given by $\vec{a} = (a_0, a_1, a_2) = (\tau, \tau^2 + \tau^3, \tau^2)$, and therefore we have $\Omega = \mathbb{Z}[\tau]$. By using $M_\sigma^{-1}$ it is easy to produce the following list of transition vectors when we take $\vec{c}_1 = {}^t(1,0,0), \vec{c}_2 = {}^t(1,1,0)$ (compare with Fig. 2)

| $n$ | $\vec{a}_0^{(n)}$ | $\vec{a}_1^{(n)}$ | $\vec{a}_2^{(n)}$ |
|---|---|---|---|
| 1 | ${}^t(1,0)$ | ${}^t(-1,1)$ | ${}^t(-1,-1)$ |
| 2 | ${}^t(-1,-1)$ | ${}^t(2,1)$ | ${}^t(0,2)$ |
| 3 | ${}^t(0,2)$ | ${}^t(-1,-3)$ | ${}^t(2,-1)$ |
| 4 | ${}^t(2,-1)$ | ${}^t(-2,3)$ | ${}^t(-3,-2)$ |
| 5 | ${}^t(-3,-2)$ | ${}^t(5,1)$ | ${}^t(1,5)$ |

The associated automaton is already given in Fig. 5 and the number system scheme is given by




We have already seen some projected words in Fig. 4, so we discuss whether the projection of the fixed point is space filling. This time we cannot argue with the criterion from [1] since the number system is not the usual $\beta$-expansion, which would be the usual Tribonacci expansion. According to Theorem 6 we have to check if all elements $x_0 + x_1\tau + x_2\tau^2$ with $\|(x_0, x_1, x_2)\|_\infty \le 53$, where we have

$$\|B^{-1}\|_\infty \le 1.111 \quad \text{and} \quad M_1 = M_2 \le \frac{2.094 \cdot 2 \cdot 3}{0.262} \le 47.954,$$

have a finite expansion in $\mathcal{L}$ (observe that the remark after Theorem 6 applies and therefore we have only to check the expansions in the language $\mathcal{L}$). But already when we consider $1 - \tau$ (cf. table below) and apply the algorithm we get the expansion $(011)^\omega$:

$$1-\tau = \tau^2+\tau^3 = \tau(\tau+\tau^2) = \tau^2+(\tau^2+(\tau^2+\tau^4))\tau^2 = \tau\cdot\tau+\tau^2\cdot\tau^2+(\tau^2+\tau^3)\cdot\tau^3$$

and we are back in the starting position. Hence the projected fixed point will not be space filling (compare with Fig. 6). Below we give a table of all expansions of $f(x_1, x_2) = x_1(\tau + \tau^2) + x_2\tau \bmod 1$ for $-3 \le x_1 \le 3, -3 \le x_2 \le 3$ (in the upper part of the table we have the values for $-3 \le x_1 \le 1$ and the in the lower part the values for $2 \le x_1 \le 3$):

| | | | | | |
|---|---|---|---|---|---|
| 00011 | $1111(011)^\omega$ | 1101 | 1001 | 0111 | |
| 100001 | 011001 | $0(011)^\omega$ | 0001 | 1111001 | |
| 000001 | 111 | 11 | 1 | 011 | |
| $011100(011)^\omega$ | 010001 | 001 | $\varepsilon$ | $11(011)^\omega$ | $\rightarrow$ |
| 111101 | $110(011)^\omega$ | $100(011)^\omega$ | $(011)^\omega$ | 01 | |
| $011000(011)^\omega$ | 001101 | $000(011)^\omega$ | 1111 | 11001 | |
| 1101111 | 1001111 | 0111111 | $010(011)^\omega$ | 0011 | |

| | | |
|---|---|---|
| | 0100001 | 0001111001 |
| | 110011 | 100011 |
| | 0011001 | 000011 |
| $\rightarrow$ | $10(011)^\omega$ | $01111(011)^\omega$ |
| | $00(011)^\omega$ | 111111 |
| | 10001 | 01101 |
| | 00001 | $11100(011)^\omega$ |

We remark that it is clear from the main result in [34] that only ultimately periodic expansions appear (since the dominant root of $M_\sigma$ is a Pisot number and $\Omega \subseteq \mathbb{Q}[\tau]$). It can be shown that $(011)^\omega$ is the only period which appears in this case.

Example 3.

Let the substitution $\sigma$ be given by

$$\sigma(0) = 01, \sigma(1) = 0002, \sigma(2) = 0. \tag{E3}$$

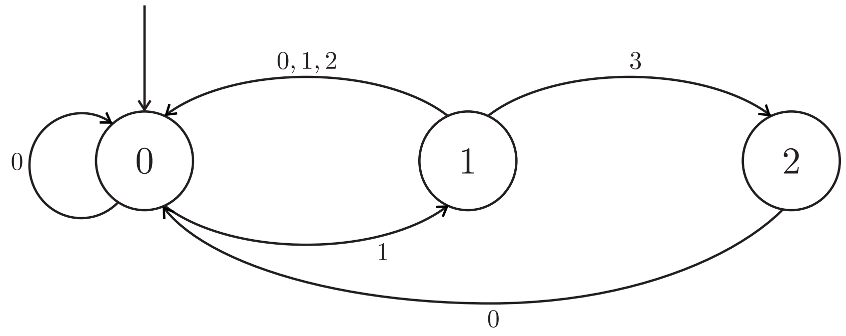
The incidence matrix and its inverse are

$$M_\sigma = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad M_\sigma^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -3 \end{pmatrix}.$$

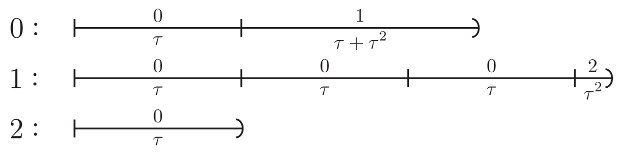
The characteristic polynomial is $x^3 - x^2 - 3x - 1 = (x^2 - 2x - 1)(x + 1)$ and therefore $M_\sigma$ has a dominant eigenvalue $\beta = 1 + \sqrt{2}$. Thus $\tau = \beta^{-1} = \frac{1}{5}(1 - \sqrt{2})$ and $\tau^2 + 2\tau = 1$. The eigenvector of $M_\sigma$ to $\beta$ is given by $\vec{a} = (a_0, a_1, a_2) = (\tau, \tau + \tau^2, \tau)$. Thus $\Omega = \mathbb{Z}[\tau]$, but $\deg\beta = 1 < k = 2$. Using $M_\sigma^{-1}$ we get the following list of transition vectors in the canonical case:

| $n$ | $\vec{a}_0^{(n)}$ | $\vec{a}_1^{(n)}$ | $\vec{a}_2^{(n)}$ |
|---|---|---|---|
| 1 | ${}^t(0,1)$ | ${}^t(0,-1)$ | ${}^t(1,-3)$ |
| 2 | ${}^t(1,-3)$ | ${}^t(-1,4)$ | ${}^t(-3,8)$ |
| 3 | ${}^t(-3,8)$ | ${}^t(4,-11)$ | ${}^t(8,-20)$ |
| 4 | ${}^t(8,-20)$ | ${}^t(-11,28)$ | ${}^t(-20,49)$ |

The associated automaton is given by




and the associated number system scheme reads



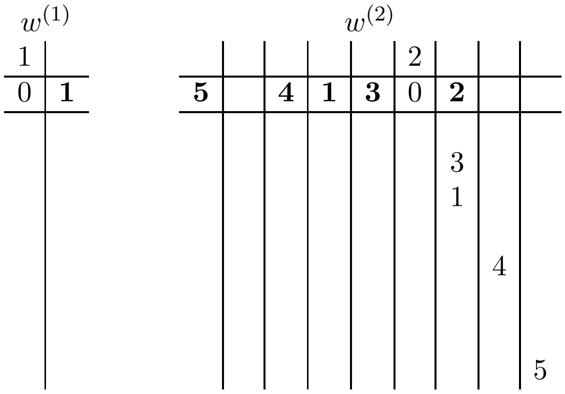

Again the number system has the property (F), since it is the usual $\beta$-expansion and Akiyama's criterion [1] applies. Thus $\sigma$ is space filling. In the canonical case we have $(b_0, b_1, b_2) = (1, 1 + \tau, \tau)$. Consequently $f(x_1, x_2) = x_1(1+\tau) + x_2\tau \bmod 1$. We take $b_0 = 1, b_1 = 1 + \tau$ as a basis for $\Omega$. The kernel of $f$ is therefore generated by $\vec{x}_2 = {}^t(x_1, x_2) = {}^t(-1, 1)$, since $b_2 = b_1 - b_0$.

Therefore, the projected fixed point of $\sigma$ completely fills $\Psi\left(\mathbb{Z}^2/\ker f\right) = \mathbb{Z}\vec{e}_1^{(2)}$ which is therefore the domain of $\Psi(\hat{W})$.

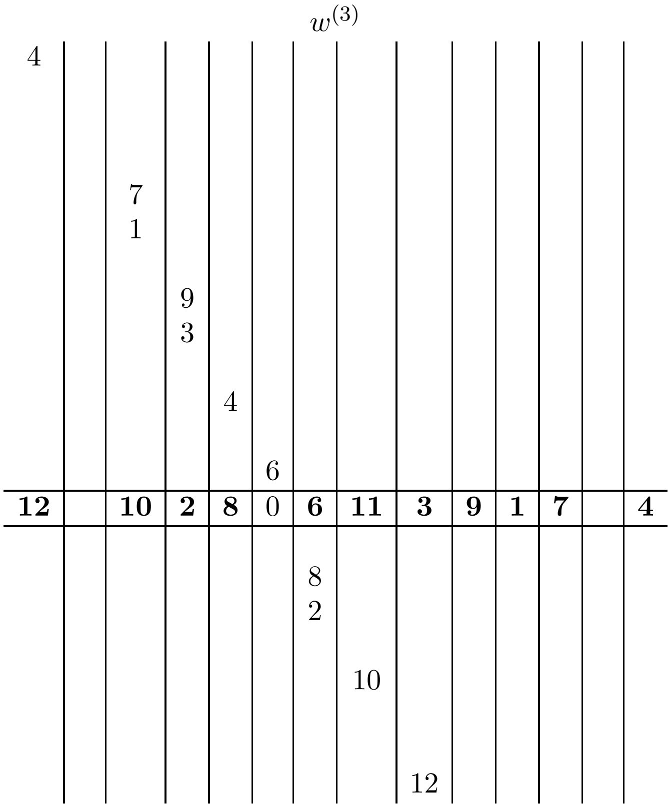
We give a list of the expansions of $f(x_1, 0) = x_1(1+\tau) \bmod 1$ for $-5 \leq x_1 \leq 8$, i.e. the $\sigma$-representations of the corresponding numbers above:

**13** | 0111 | **12** | **01** | **11** | $\varepsilon$ | **1** | **121** | **011** | **111** | **001** | **101** | 12121 | **012**

For the canonical projections of the words $u^{(1)} = 01, u^{(2)} = 010002$ in thin letter type and their $\Psi$-images in bold letter type we get




and for $u^{(3)} = 0100020101010$ we get



Observe that the expansions of length at most 3 (the bold entries in the table above) are exactly those that appear in the domain of $\Psi(w^{(3)})$.

Example 4.
We consider the substitution defined by

$$\sigma(0) = 01, \sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 4, \sigma(4) = 0. \tag{E4}$$

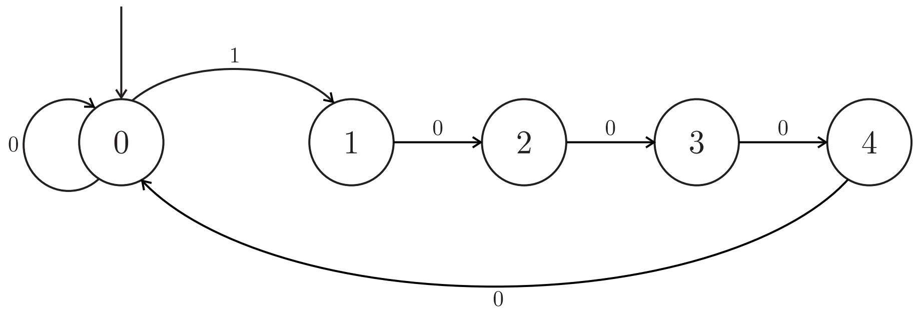
We have taken this example from [9, p. 21]. The incidence matrix is

$$M_\sigma = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

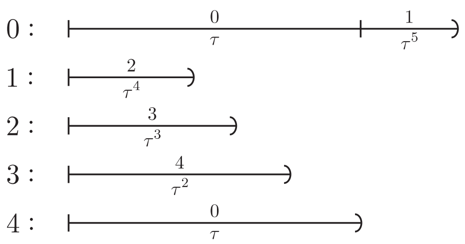
The characteristic polynomial of $M_\sigma$ is equal to $x^5 - x^4 - 1 = (x^3 - x - 1)(x^2 - x + 1)$. Thus, there is a dominant root $\beta \approx 1.325$ which is a cubic Pisot unit, and a quadratic factor the roots of which have absolute value equal to 1. We have $\tau = \beta^{-1} \approx 0.755$ and $\tau^3 + \tau^2 = 1$. The eigenvalue of $M_\sigma$ to $\beta$ is

$$\vec{a} = (a_0, a_1, a_2, a_3, a_4) = (\tau, \tau^5, \tau^4, \tau^3, \tau^2).$$

Therefore, $\Omega = \mathbb{Z}[\tau]$. The associated automaton is




and the associated number system scheme



By using

$$M_\sigma^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

it is easy to get the transition vectors. E.g. we have for $n = 11$ that

$$u^{(11)} = 012340010120123012340123400123400 1$$

and

$$\left(\vec{a}_0^{(11)}, \vec{a}_1^{(11)}, \vec{a}_2^{(11)}, \vec{a}_3^{(11)}, \vec{a}_4^{(11)}\right) = \begin{pmatrix} 0 & 0 & 1 & -2 & 1 \\ 1 & -1 & 0 & 1 & -2 \\ -2 & 3 & -1 & 0 & 1 \\ 1 & -3 & 3 & -1 & 0 \end{pmatrix}.$$

In the canonical case $(b_0, b_1, b_2, b_3, b_4) = (1, \tau^4, \tau^3, \tau^2, \tau)$. We pick $b_0 = 1, b_3 = \tau^2, b_4 = \tau$ as a basis for $\Omega$. Since $b_1 = -b_0 + b_3 + b_4$ and $b_2 = b_0 - b_3$, the projection $\Psi$ is given by the vectors $\vec{x}_1 = {}^t(1, 0, -1, -1), \vec{x}_2 = {}^t(0, 1, 1, 0)$. In fact

$$\mathrm{dom}\Psi\big(\hat{W}\big) = \Psi\left(\mathbb{Z}^4/\ker f\right) = \mathbb{Z}\vec{e}_3^{(4)} + \mathbb{Z}\vec{e}_4^{(4)},$$

since the associated number system has the finiteness property. This follows again by the criterion in [1], since the number system gives the usual $\beta$-expansion. We get

$$\Psi\left(w^{(11)}\right)$$

$$\begin{array}{cccccccc} 33 & & & & & 29 & 14 & \\ 7 & 27 & 12 & 32 & 18 & 3 & 23 & \\ 16 & 1 & 21 & 6 & 26 & 11 & 31 & \\ & 9 & 30 & 15 & 0 & 20 & 5 & 25 \\ & 19 & 4 & 24 & 8 & 28 & 13 & \\ & & & & 17 & 2 & 22 & \\ & & & & & 10 & & \end{array}$$

This roundwalk can be obtained by using the images of the transition vectors in the $(x_3, x_4)$-plane, which are given by

$$\Psi\left(\vec{a}_0^{(11)}\right) = \begin{pmatrix} 0 \\ 0 \\ -3 \\ 1 \end{pmatrix}, \Psi\left(\vec{a}_1^{(11)}\right) = \begin{pmatrix} 0 \\ 0 \\ 4 \\ -3 \end{pmatrix}, \Psi\left(\vec{a}_2^{(11)}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \end{pmatrix},$$

$$\Psi\left(\vec{a}_3^{(11)}\right) = \begin{pmatrix} 0 \\ 0 \\ -3 \\ -3 \end{pmatrix}, \Psi\left(\vec{a}_4^{(11)}\right) = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 1 \end{pmatrix}.$$

It can clearly be seen how the plane is filled up. Since $(b_0, b_1, b_2, b_3, b_4) = (1, \tau^4, \tau^3, \tau^2, \tau)$, the corresponding list of expansions of $f(0, 0, x_3, x_4) =$

$x_3\tau^2 + x_4\tau \bmod 1$ is given by the following table, where all finite $\sigma$-representations of $f(\vec{x}) = x_3\tau^2 + x_4\tau \bmod 1$ with $\vec{x} = (0, 0, x_3, x_4)$ and $-4 \leq x_3 \leq 4, -3 \leq x_4 \leq 3$ are shown (the upper part of the table contains the expansions for $-4 \leq x_3 \leq -2$, the middle part the expansion for $-1 \leq x_3 \leq 1$ and the lower part for $2 \leq x_3 \leq 4$):

| | | | | |
|---|---|---|---|---|
| | **10000100001** | 00100001000001 | 00000001000001 | |
| | **00000100001** | **10000000001** | **00010000001** | |
| | **00100000001** | **00000000001** | **10000001** | |
| | 0100001000000001 | **00001000001** | **10000001** | $\rightarrow$ |
| | 1000010000001 | **00100001** | **00000001** | |
| | 0000010000001 | 1000000000001 | 0001000000001 | |
| | 0010000000001 | 0000000000001 | 0100000000001 | |
| | 01000001000001 | 00001000000001 | **100000001** | |
| | **100001** | **0010000001** | **000000001** | |
| | **000001** | **1** | **0001** | |
| $\rightarrow$ | **001** | $\varepsilon$ | **01** | $\Rightarrow$ |
| | **01000001** | **00001** | **1000000001** | |
| | 1000001000001 | **0010000001** | **1000000001** | |
| | 0000001000001 | 0100001000001 | **0000100001** | |
| | **000100001** | 100001000001 | 00100001000000001 | |
| | **010000001** | 000001000001 | 100000000001 | |
| | **1000001** | 001000000001 | 000000000001 | |
| $\Rightarrow$ | **0000001** | **0100001** | 000010000001 | |
| | **0001000001** | 100001000000001 | 001000001000001 | |
| | **0100000001** | 000001000000001 | 100000000000001 | |
| | 100000010000001 | 001000000000001 | 000000000000001 | |

It can be seen that the expansions of length at most 11, which are again the bold entries, are exactly at the places of $\mathrm{dom}\Psi(w^{(11)})$.

EXAMPLE 5.

Finally let the substitution be defined by

$$\sigma(0) = 02111111, \sigma(1) = 30, \sigma(2) = 1000000, \sigma(3) = 0. \tag{E5}$$

The incidence matrix and its inverse are given by

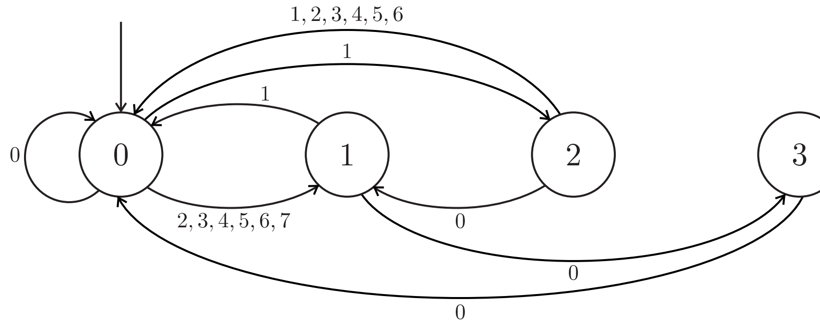
$$M_\sigma = \begin{pmatrix} 1 & 6 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 6 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad M_\sigma^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -6 \\ 1 & 0 & -6 & 35 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

The characteristic polynomial is equal to $x^4 - x^3 - 12x^2 - 7x - 1 = (x^2 - 4x - 1)(x^2 + 3x + 1)$ and therefore it has a dominant Pisot root given by $\beta = 2 + \sqrt{5}$ and $\tau = \sqrt{5} - 2$. The two roots of the other polynomial are $-\frac{1}{2}(3 \mp \sqrt{5})$, which lie on both sides of the unit circle. The eigenvalue of $M_\sigma$

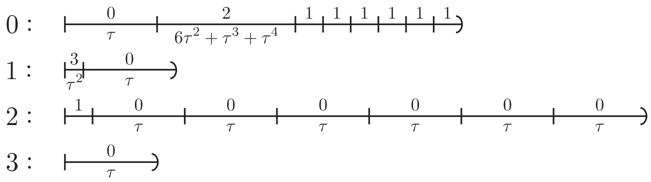
to $\beta$ is

$$(a_0, a_1, a_2, a_3) = (\tau, \tau^2 + \tau^3, 6\tau^2 + \tau^3 + \tau^4, \tau^2).$$

The associated automaton is




and the number system scheme reads




We have $(b_0, b_1, b_2, b_3) = (1, \tau+\tau^2, 6\tau+\tau^2+\tau^3, \tau)$ and by choosing $b_0, b_3$ as a basis we get $\vec{x}_1 = (1, 0, 3), \vec{x}_2 = (0, 1, -19)$ for the description of the map $\Psi$. E.g. the word $u^{(2)} = 02111111100000303030303030$ and the transition vectors in the canoncial case are given by

$$\left(\vec{a}_0^{(2)}, \vec{a}_1^{(2)}, \vec{a}_2^{(2)}, \vec{a}_3^{(2)}\right) = \begin{pmatrix} 1 & -6 & 35 & -1 \\ 0 & -6 & 36 & 1 \\ -1 & 41 & -244 & -5 \end{pmatrix}$$

and their projections are

$$\left(\Psi\left(\vec{a}_0^{(2)}\right), \Psi\left(\vec{a}_1^{(2)}\right), \Psi\left(\vec{a}_2^{(2)}\right), \Psi\left(\vec{a}_3^{(2)}\right)\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & -55 & 335 & 17 \end{pmatrix}.$$

We give a list of $\sigma$-expansions of $f(0, x_2) = \tau x_2 \bmod 1$ for $10 \le x_2 \le 10$ (in the upper part the values $-10 \le x_2 \le -3$, in the middle part $-2 \le x_2 \le 5$ and in the lower part the remaining values are shown):

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $2(13)^\omega$ | 61 | 01471 | $123(13)^\omega$ | 1671 | 511 | 01 | $1(13)^\omega$ | $\rightarrow$ | |
| $\rightarrow$ | 1571 | 411471 | $\varepsilon$ | 1 | 1471 | $3(13)^\omega$ | 71 | 0411471 | $\Rightarrow$ |
| $\Rightarrow$ | 1371 | 2171 | 611 | 01571 | 1271 | | | | |

Indeed the expansions of length at most 2 are at the positions where the above roundwalk passes. Note that $-3\tau$ corresponds to $1(13)^\omega$, since

$$\begin{aligned} 1 - 3\tau &= \tau \cdot 1 + (\tau^2 + \tau^3)\tau + (3\tau)\tau^2 \\ &= \tau \cdot 1 + (\tau^2 + \tau^3)\tau + (\tau + 7\tau^2 + 2\tau^3 + \tau^4)\tau^2 + \tau^2\tau^3 + (3\tau)\tau^4 \end{aligned}$$

and we are back in state 0. Since there are values which have an infinite expansion, the limit word $\hat{W}$ is not space filling.

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