The two-dimensional Prouhet-Tarry-Escott Problem

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Abstract

In this paper we generalize the Prouhet-Tarry-Escott problem (PTE) to any dimension. The one-dimensional PTE problem is the classical PTE problem. We concentrate on the two-dimensional version which asks, given parameters $n, k \in \mathbb{N}$, for two different multi-sets $\{(x_1, y_1), \ldots, (x_n, y_n)\}$, $\{(x'_1, y'_1), \ldots, (x'_n, y'_n)\}$ of points from $\mathbb{Z}^2$ such that $\sum_{i=1}^n x_i^j y_i^{d-j} = \sum_{i=1}^n x_i'^j y_i'^{d-j}$ for all $d, j \in \{0, \ldots, k\}$ with $j \leq d$. We present parametric solutions for $n \in \{2, 3, 4, 6\}$ with optimal size, i.e., with $k = n - 1$. We show that these solutions come from convex $2n$-gons with all vertices in $\mathbb{Z}^2$ such that every line parallel to a side contains an even number of vertices and prove that such convex $2n$-gons do not exist for other values of n. Furthermore we show that solutions to the two-dimensional PTE problem yield solutions to the one-dimensional PTE problem. Finally we address the PTE problem over the Gaussian integers.

1 Introduction

We introduce the following problem, which we call *the general Prouhet-Tarry-Escott Problem*:

Problem 1 ***(PTE$_r$)*** *Given $k, n, r \in \mathbb{N}$ find two different multi-sets $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of points from $\mathbb{Z}^r$ where $\mathbf{a}_i = (a_{i1}, \ldots, a_{ir}), \mathbf{b}_i = (b_{i1}, \ldots, b_{ir})$ for $i = 1, \ldots, n$ such that*

$$\sum_{i=1}^{n} a_{i1}^{j_1} a_{i2}^{j_2} \cdots a_{ir}^{j_r} = \sum_{i=1}^{n} b_{i1}^{j_1} b_{i2}^{j_2} \cdots b_{ir}^{j_r}$$

for all nonnegative integers $j_1, \ldots, j_r$ with $j_1 + j_2 + \ldots j_r \leq k$.

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In the sequel k is called the *degree*, n the *size*, and r the *dimension* of the solution. Solutions satisfying $n = k+1$ are called *ideal solutions*, since nontrivial solutions with $n \leq k$ do not exist. We indicate Problem PTE_r with $n = k+1$ as $\text{PTE}_r(k)$.

An equivalent formulation of PTE_r is as follows.

Problem 2 *Given $k,n,r \in \mathbb{N}$ find two different multi-sets $\{\mathbf{a_1},\ldots,\mathbf{a_n}\},\{\mathbf{b_1},\ldots,\mathbf{b_n}\}$ of points from $\mathbb{Z}^r$ such that*

$$\sum_{i=1}^{n} P(\mathbf{a}_i) = \sum_{i=1}^{n} P(\mathbf{b}_i)$$

for every polynomial $P \in \mathbb{Z}[\mathbf{x}]$ where $\mathbf{x} \in \mathbb{Z}^r$ is of total degree $\leq k$.

The main question is for which k, n, r proper solutions to PTE_r exist, and in particular for which k, r proper solutions to $\text{PTE}_r(k)$ exist.

The classical Prouhet-Tarry-Escott Problem (PTE_1) is an old problem tracing back to works of Euler and Goldbach ([3]). It is an open question ([6]) whether solutions to $\text{PTE}_1(k)$ exist for every $k \in \mathbb{N}$. At present, ideal solutions are only known for $k \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11\}$ (see [2],[5]). No values of k are known for which no ideal solution exist.

In the present paper we study the case $r = 2$. We give parametric solutions to $\text{PTE}_2(k)$ for $k \in \{1, 2, 3, 5\}$. Our approach is geometric. Additional geometric examples for $\text{PTE}_1(k)$ solutions can be found in [1]. First we give some definitions.

A *lattice direction* $\text{lin}\{(p,q)\}$ is a (non-degenerated) linear subspace which is generated by a vector $(p,q) \in \mathbb{Z}^2$. We call two sets F_1, F_2 *tomographically equivalent* with respect to direction $\text{lin}\{(p,q)\}$ if

$$|F_1 \cap (\mathbf{x} + \text{lin}\{(p,q)\})| = |F_2 \cap (\mathbf{x} + \text{lin}\{(p,q)\})| \text{ for all } \mathbf{x} \in \mathbb{Z}^2.$$

Note that opposite directions are identified.

Problem 3 ***(GP$_2$)*** *Given $k, n \in \mathbb{N}$ find a set of $k+1$ distinct directions and two sets of n points from $\mathbb{Z}^2$ such that the sets are tomographically equivalent with respect to all $k+1$ directions.*

Again we call a solution *ideal* if $n = k+1$ and indicate the problem in this case as $\text{GP}_2(k)$.

The following problem turns out to be equivalent with $\text{GP}_2(k)$.

Problem 4 *Given k find a convex $(2k+2)$-gon with all vertices in $\mathbb{Z}^2$ such that every line parallel to a side contains an even number of vertices.*

Our results in Sections 2 and 3 show that this problem is solvable only for $k = 1, 2, 3$ and 5. In Section 4 we prove that every solution to $\mathrm{GP}_2(k)$ yields a solution to $\mathrm{PTE}_2(k)$ and we remark that solutions to $\mathrm{PTE}_2(k)$ lead to classes of solutions to $\mathrm{PTE}_1(k)$. Section 5 contains a result on the general PTE problem. The final section deals with the PTE problem for Gaussian integers.

2 Solutions to the Geometric Problems

We construct solutions to $\mathrm{GP}_2(k)$ for $k = 1, 2, 3$ and 5. See Figure 1 for an illustration of a $\mathrm{GP}_2(5)$ solution.

It is clear from the geometry that the property of two sets being tomographically equivalent remains invariant under affine transformation ($f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ with $A \in \mathbb{Z}^{2\times 2}$ non-singular and $\mathbf{b} \in \mathbb{Z}^2$). For PTE this is known as Frolov's Theorem ([5]).

The following results all have in common that the directions of the points

$$(p_0, q_0), \ldots, (p_k, q_k), (-p_0, -q_0), \ldots, (-p_k, -q_k)$$

viewed from the origin are listed in the order of increasing angle with the positive $x-$axis. By consecutive addition of the direction vectors $(p_0, q_0), \ldots, (p_k, q_k)$, $-(p_0, q_0), \ldots, -(p_k, q_k)$ we obtain alternately the points of F_1 and F_2. Such sets are called lattice $\mathcal{L} - gons$.

2.1 Ideal solutions of degree $k = 1$

We choose parameters $a, b, c, d \in \mathbb{Z}$ such that $\mathrm{lin}\{(a, b)\}$ and $\mathrm{lin}\{(c, d)\}$ are different directions. Then

$$F_1 = \{(0,0), (a+c, b+d)\}, \quad F_2 = \{(a,b), (c,d))\}$$

are tomographically equivalent with respect to both directions and yield solutions to $\mathrm{PTE}_2(1)$. Every solution yields a parallelogram with $(\frac{a+c}{2}, \frac{b+d}{2})$ as point of symmetry. It is easy to verify that the solutions are not all equivalent under affine transformations.

2.2 Ideal solutions of degree $k = 2$

We choose parameters $a, b, c \in \mathbb{Z}$ such that $\mathrm{lin}\{(a, 0)\}$, $\mathrm{lin}\{(b, c)\}$ and $\mathrm{lin}\{(b-a, c)\}$ are different directions. It is clear, that

$$F_1 = \{(0,0), (a+b, c), (2b-a, 2c)\}, \quad F_2 = \{(a,0), (2b, 2c), (b-a, c)\}$$

are tomographically equivalent with respect to both directions and yield solutions to $\mathrm{PTE}_2(2)$. Note that (b, c) is a point of symmetry. A convex solution is for example obtained by taking $a = b = c = 1$. It is easy to verify that the solutions are not all equivalent under affine transformations.

2.3 Ideal solutions of degree $k = 3$

We choose parameters $a, b, c \in \mathbb{N}$, where $b|ac$, and such that $\mathrm{lin}\{(a, 0)\}$, $\mathrm{lin}\{(b, c)\}$, $\mathrm{lin}\{(0, ac/b)\}$ and $\mathrm{lin}\{-b, c\}$ are different directions. It is clear, that

$$\begin{aligned} F_1 &= \{(0,0), (a+b, c), (a, 2c+ac/b), (-b, c+ac/b)\}, \\ F_2 &= \{(a,0), (a+b, c+ac/b), (0, 2c+ac/b), (-b, c)\} \end{aligned}$$

are tomographically equivalent with respect to both directions and yield solutions to $\mathrm{PTE}_2(3)$. Note that $(\frac{a}{2}, c + \frac{ac}{2b})$ is a point of symmetry. A convex solution is for example obtained by taking $a = b = c = 1$. It is easy to verify that the solutions are not all equivalent under affine transformations.

2.4 Ideal solutions of degree $k = 5$

We choose parameters $a, b \in \mathbb{N}$ such that $\mathrm{lin}\{(2a, 0)\}$, $\mathrm{lin}\{(b, b)\}$, $\mathrm{lin}\{(a, 3a)\}$, $\mathrm{lin}\{(0, 2b)\}$, $\mathrm{lin}\{(-a, 3a)\}$ and $\mathrm{lin}\{-b, b\}$ are different directions. It can be easily verified that

$$\begin{aligned} F_1 &= \{(0,0), (2a+b, b), (3a+b, 3a+3b), (2a, 6a+4b), (-b, 6a+3b), (-a-b, 3a+b)\}, \\ F_2 &= \{(2a,0), (3a+b, 3a+b), (2a+b, 6a+3b), (0, 6a+4b), (-a-b, 3a+3b), (-b, b)\} \end{aligned}$$

are tomographically equivalent with respect to both directions and yield solutions to $\mathrm{PTE}_2(5)$. Note that $(a, 3a + 2b)$ is a point of symmetry. A convex solution is for example obtained by taking $a = b = 1$. It is easy to verify that the solutions are not all equivalent under affine transformations.

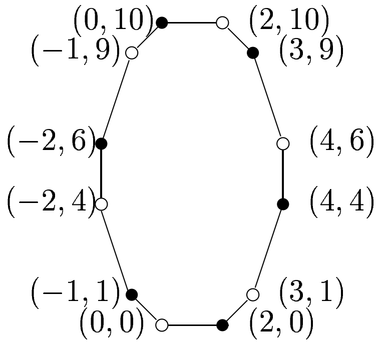

Fig. 1. Solution of $GP_2(5)$ with $a = b = 1$. White points indicate the points of F_1; black points indicate the points of F_2.

3 Nonexistence of Solutions to $\mathbf{GP}_2(k)$

In this section we prove that Problem $GP_2(k)$ and Problem 1.4 are equivalent. Subsequently we prove that these problems do not admit solutions for $k \notin \{1, 2, 3, 5\}$. Because of a result of Gardner and Gritzmann only the case $k = 4$ has to be treated here.

For the next lemma it is irrelevant that the vertices are in $\mathbb{Z}^2$.

Lemma 5 *Let $k \in \mathbb{N}$. The solutions to $GP_2(k)$ are precisely the solutions to Problem 1.4.*

PROOF. $\Rightarrow$. Suppose there are a set of $k + 1$ distinct directions and two sets of $k + 1$ points such that the sets are tomographically equivalent with respect to all the directions. Consider the convex hull of the $2(k + 1)$ points. In each of the $k + 1$ directions there are two lines such that each line contains a point from each set and there are no points outside the strip between the lines. Hence the $2(k + 1)$ points form a convex $2k + 2$-gon where each point is a proper vertex and two opposite sides are parallel. Moreover, because of the tomographic property, every line parallel to a side contains as many vertices from one set as from the other, hence an even number of vertices in total.

$\Leftarrow$. Let be given a convex $(2k + 2)$-gon such that every line parallel to a side contains an even number of vertices. Go around the polygon and put the vertices alternately in set F_1 and in set F_2. So we get two sets of $k + 1$ points each such that every side of the polygon contains a point from F_1 and one from F_2. The sides of the polygon define $k+1$ distinct directions. Start with any two adjacent vertices. They belong to different sets and are connected by a line in one of the $k + 1$ directions. Move the line keeping its direction towards the other vertices. If the next vertex is met by the shifting line, another vertex is met simultaneously by the condition of Problem 1.4. Because of the convexity there are no more than two vertices on the line. By an induction argument one vertex is adjacent to a point in F_1 and therefore in F_2 and the other is adjacent to a point in F_2 and therefore in F_1. We conclude that F_1 and F_2 are tomographically equivalent with respect to the $k+1$ directions determined by the sides of the polygon. □

By a result of Gardner and Gritzmann, [4] Theorem 4.5, there do not exist lattice $\mathcal{L}$-gons for more than six directions, i.e., the solution for $k = 5$ with 6 points is the solution to $\mathrm{GP}_2(k)$ with maximal k. After Section 2 the only remaining value to be considered is $k = 4$. In view of Lemma 3.1 it suffices to establish the following result to show that there are no solutions in that case.

Theorem 6 *There does not exist a convex 10-gon with all vertices in $\mathbb{Z}^2$ such that every line parallel to a side contains an even number of vertices.*

PROOF. Suppose there exists a convex 10-gon with all vertices in $\mathbb{Z}^2$ such that every line parallel to a side contains 0 or 2 vertices. Arrange the five directions of the sides in such a way that when going around counterclockwise the directions are $(p_0, q_0), \ldots, (p_4, q_4)$, $-(p_0, q_0), \ldots, -(p_4, q_4)$. We apply a rational transformation such that the first direction becomes $\mathrm{lin}\{(1,0)\}$ and the fourth $\mathrm{lin}\{(0,1)\}$. The lattice need no longer consist of integer points, but there exists a positive integer D such that the coordinates of the images of all the lattice points are multiples of D^{-1}. Thus, by replacing all coordinates by their D-multiple, we obtain a convex 10-gon with all vertices in $\mathbb{Z}^2$ such that every line parallel to a side contains 0 or 2 vertices, and the first direction is the direction of the positive x-axis and the fourth direction that of the positive y-axis.

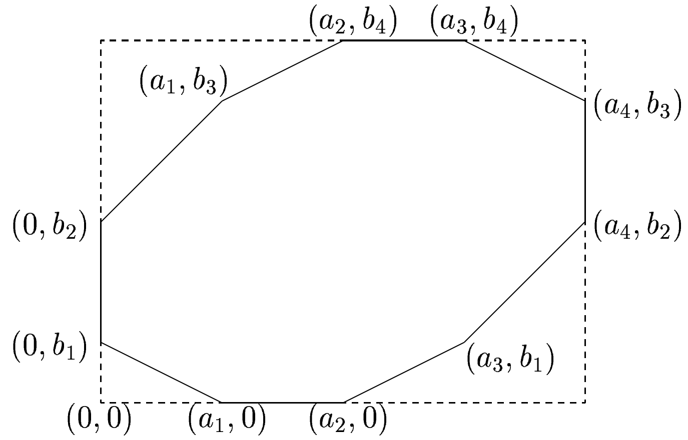

Fig. 2. Location of the 10-gon after the affine transformation. There are horizontal and vertical sides.

Without loss of generality we may assume that the ten vertices of the 10-gon in counterclockwise order are given by

$$(a_1, 0), (a_2, 0), (a_3, b_1), (a_4, b_2), (a_4, b_3), (a_3, b_4), (a_2, b_4), (a_1, b_3), (0, b_2), (0, b_1).$$

(See Figure 1.) Here we have already used that on every horizontal and every vertical line there are 0 or 2 vertices.

The tomography condition in the fifth and the second direction imply that

$$\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3 - b_1}{a_3 - a_1} = \frac{b_4 - b_2}{a_4 - a_2} = \frac{b_4 - b_3}{a_4 - a_3}$$

and

$$\frac{b_2}{a_4 - a_1} = \frac{b_1}{a_3 - a_2} = \frac{b_3 - b_1}{a_4} = \frac{b_4 - b_2}{a_3} = \frac{b_4 - b_3}{a_2 - a_1},$$

respectively. Hence

$$\frac{a_4 - a_1}{a_2} = \frac{a_4}{a_3 - a_1} = \frac{a_3}{a_4 - a_2} = \frac{a_2 - a_1}{a_4 - a_3} = \frac{a_3 - a_2}{a_1}.$$

Put $a := a_1, b := a_2 - a_1, c := a_3 - a_2, d := a_4 - a_3$. Then we obtain

$$\frac{b + c + d}{a + b} = \frac{a + b + c + d}{b + c} = \frac{a + b + c}{c + d} = \frac{b}{d} = \frac{c}{a}.$$

Successively we find $ab = cd, ad + cd = bc, db + d^2 = b^2$. However, the latter equation is not solvable in nonzero integers. □

4 The Relation between $\mathbf{GP}_2$ and $\mathbf{PTE}_2$

In this section we show that every solution of GP_2 leads to a solution of PTE_2. However, it is not clear whether the converse is true. For notational convenience, we write

$$\{(x_1, y_1), \ldots, (x_n, y_n)\} \stackrel{k}{=} \{(x'_1, y'_1), \ldots, (x'_n, y'_n)\}$$

if the sets $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ and $\{(x'_1, y'_1), \ldots, (x'_n, y'_n)\}$ are solutions to PTE_2 of degree k.

Lemma 7 *Given $k + 1$ different directions* $\operatorname{lin}\{(p_i, q_i)\}$ $(i = 0, \ldots, k)$ *the polynomials*

$$(q_0 x - p_0 y)^k, \ldots, (q_k x - p_k y)^k \in \mathbb{R}[x, y]$$

form a basis of the vector space V_k which is generated by the monomials $y^k, x^1 y^{k-1}, \ldots, x^{k-1} y^1, x^k$.

PROOF. Every polynomial $(q_i x - p_i y)^k$ can be expressed in the monomial basis $\mathcal{B} = (y^k, x^1 y^{k-1}, \dots, x^{k-1} y^1, x^k)$ as $\left(\binom{k}{0}(q_i)^0(-p_i)^k, \dots, \binom{k}{k}(q_i)^k(-p_i)^0\right)$. Thus we only have to show that these $k+1$ vectors are linearly independent, i.e. we have to show that the matrix

$$C = \left(\binom{k}{j}(q_i)^j(-p_i)^{k-j}\right)_{i,j=0,1,\dots k} \in \mathbb{R}^{(k+1)\times(k+1)}$$

is non-singular. Suppose first, that $p_0 \cdots p_k \neq 0$. By setting $t_i = -q_i/p_i$, and by defining the Vandermonde matrix $C' = (t_i^j)_{i,j}$, we obtain

$$\det(C) = \det(C') \cdot \prod_{i=0}^{k} \binom{k}{i}(-p_i)^k = \prod_{i \neq j}(t_i - t_j) \cdot \prod_{i=0}^{k} \binom{k}{i}(-p_i)^k \neq 0,$$

since $t_{i_0} = t_{j_0}$, that is $\frac{q_{i_0}}{p_{i_0}} = \frac{q_{j_0}}{p_{j_0}}$, contradicts that the $k+1$ directions are different. If $p_i = 0$ (and thus $p_j \neq 0$ for $i \neq j$), then a similar proof applies by taking into account that in this case the $i-$th row of C contains a multiple of the $i-$th unit vector. □

Theorem 8 *Let be given two different sets* $F_1 = \{(x_1, y_1), \dots, (x_n, y_n)\}, F_2 = \{(x'_1, y'_1), \dots, (x'_n, y'_n)\}$ *of points from* $\mathbb{Z}^2$ *which are tomographically equivalent with respect to* $k+1$ *different directions. Then* F_1 *and* F_2 *are solutions to* PTE_2 *of degree* k*, i.e.,*

$$\sum_{i=1}^{n} x_i^j y_i^d = \sum_{i=1}^{n} x_i'^j y_i'^d, \qquad \textit{for all integers } d, j \geq 0 \textit{ with } d + j \leq k.$$

PROOF. Let us denote the directions by $\operatorname{lin}\{(p_i, q_i)\}$, $i = 0, 1, \dots, k$. These directions are lattice directions since $F_1, F_2 \subset \mathbb{Z}^2$.

For every $d, j \in \mathbb{N}$ with $d + j \leq k$, we know by Lemma 7 that there are $\alpha_0, \dots, \alpha_k \in \mathbb{R}$ such that

$$x^j y^d = \sum_{i=0}^{k} \alpha_i (q_i x - p_i y)^{j+d}. \tag{1}$$

For $i = 1, \dots, k$ on every line g in direction $\operatorname{lin}\{(p_i, q_i)\}$, there is a one-to-one correspondence between points of $F_1 \cap g$ and $F_2 \cap g$, thus

$$\{(q_i x_1 - p_i y_1), \dots, (q_i x_n - p_i y_n)\} = \{(q_i x'_1 - p_i y'_1), \dots, (q_i x'_n - p_i y'_n)\}.$$

So, if we evaluate (1) at the points of F_1 and F_2 we obtain:

$$\sum_{l=1}^{n} x_l^j y_l^d - \sum_{l=1}^{n} x_l'^j y_l'^d = \sum_{l=1}^{n}\sum_{i=0}^{k} \alpha_i (q_i x_l - p_i y_l)^{d+j} - \sum_{l=1}^{n}\sum_{i=0}^{k} \alpha_i (q_i x'_l - p_i y'_l)^{d+j} = 0$$

for all $d, j \in \mathbb{N}$ with $1 \leq d + j \leq k$. Because of $|F_1| = |F_2| = n$ we obtain that (4) also holds if $d + j = 0$. This means that we obtain a solution to PTE$_2$:

$$\{(x_1, y_1), \ldots, (x_n, y_n)\} \stackrel{k}{=} \{(x'_1, y'_1), \ldots, (x'_n, y'_n)\}.$$

□

Corollary 9 *The parametric solutions given in Section 2 provide solutions to $PTE_2(k)$ for $k \in \{1, 2, 3, 5\}$.*

Remark 10 *Solutions of PTE_2 lead to solutions of PTE_1, e.g., by taking only the $x-$components of the solution. It may happen that $\{x_1, \ldots, x_n\} = \{x'_1, \ldots, x'_n\}$, hence they provide trivial solutions to PTE_1. But it is always possible to rotate the two tomographically equivalent sets in the solution to $PTE_2(k)$ in such a way that they are not tomographically equivalent with respect to the direction* $\text{lin}\{(1, 0)\}$. *Since $\{x_1, \ldots, x_n\} \neq \{x'_1, \ldots, x'_n\}$, they lead to a non-trivial $PTE_1(k)$ solution. Consequently, the stated $PTE_2(k)$ solutions all lead to parametric $PTE_1(k)$ solutions by rotating (or by affine transforming) the sets F_1 and F_2.*

Remark 11 *A solution to $PTE_2(k)$ may, by rotating in different ways and projecting it afterwards on the x-axis, lead to different solutions to $PTE_1(k)$ which are not affinely-equivalent. For example, by applying the affine transformation $x \mapsto Ax$ with $A = \begin{pmatrix} -2 & 8 \\ -1 & -2 \end{pmatrix}$ to the sets*

$$F_1 = \{(0, 0), (3, 1), (4, 6), (2, 10), (-1, 9), (-2, 4)\},$$
$$F_2 = \{(2, 0), (4, 4), (3, 9), (0, 10), (-2, 6), (-1, 1)\}$$

of Section 2.4 (i.e. $a = b = 1$), we obtain the following degree 5 solutions to $PTE_2(5)$ (see Figure 3):

$$F'_1 = \{(0, 0), (2, -5), (40, -16), (76, -22), (74, -17), (36, -6)\},$$
$$F'_2 = \{(-4, -2), (24, -12), (66, -21), (80, -20), (52, -10), (10, -1)\}.$$

It can be easily checked that the projection on the $x-$axis, i.e.

$$\{0, 2, 40, 76, 74, 36\} \stackrel{5}{=} \{-4, 24, 66, 80, 52, 10\}$$

and the projection on the y-axis

$$\{0, -5, -16, -22, -17, -6\} \stackrel{5}{=} \{-2, -12, -21, -20, -10, -1\}$$

are not affinely equivalent.

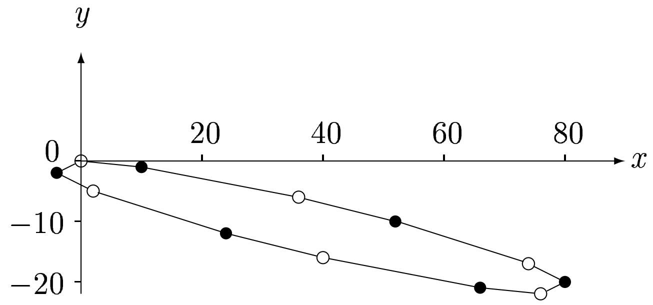

Fig. 3. Affine image of the sets from Figure 1 representing two non-equivalent $PTE_1(5)$ solutions (see Remark 11). The white points represent the points of F_1'; the black points represent the points of F_2'.

5 A Generalization of a Theorem of Prouhet

If one cannot prove the existence of ideal solutions to PTE_2 for given k, one may wonder for which k and n PTE_2 is solvable. We prove a theorem about the existence of large solutions to PTE_2, which is in the spirit of the theorem proved by Prouhet (see [7], [8]).

Theorem 12 *For every degree $k \in \mathbb{N}$ there exist solutions*

$$\{(x_1, y_1), \ldots, (x_n, y_n)\} \stackrel{k}{=} \{(x_1', y_1'), \ldots, (x_n', y_n')\}$$

to PTE_2, where $n = 2^k$.

PROOF. Given $k+1$ lattice directions $\operatorname{lin}\{(p_0, q_0)\}, \ldots, \operatorname{lin}\{(p_k, q_k)\}$, we have to construct sets of cardinality 2^k being tomographically equivalent with respect to these directions, thus leading to PTE_2 solutions by Theorem 8. They can be obtained by taking $U_2 = \{(0,0), (p_1{+}p_2, q_1{+}q_2)\}$, $V_2 = \{(p_1, q_1), (p_2, q_2)\}$, and by recursively defining

$$U_{i+1} = V_i \cup (\theta_{i+1}(p_{i+1}, q_{i+1}) + U_i), \quad V_{i+1} = U_i \cup (\theta_{i+1}(p_{i+1}, q_{i+1}) + V_i),$$

for $i = 2, \ldots, k{+}1$. The $\theta_i \in \mathbb{Z}$ have to be chosen such that $V_i \cap (\theta_{i+1}(p_{i+1}, q_{i+1}) + U_i) = \emptyset$ and $U_i \cap (\theta_{i+1}(p_{i+1}, q_{i+1}) + V_i) = \emptyset$. It is clear that if the θ_i are chosen sufficiently large, then this can be achieved. The sets $F_1 = U_{k+1}$ and $F_2 = V_{k+1}$ have the desired properties by definition. □

6 The Prouhet-Tarry-Escott Problem over the Gaussian Integers

To our knowledge, PTE_1 over the Gaussian integers has not been investigated yet. Thus, we ask for two sets $\{\xi_1, \ldots, \xi_n\}$, $\{\xi'_1, \ldots, \xi'_n\}$ of Gaussian integers such that $\sum_{i=1}^{n} \xi_i^d = \sum_{i=1}^{n} {\xi'_i}^d$ for $d = 0, \ldots, k$. Of course, we obtain solutions of points lying on a straight line from integer solutions of PTE_1. But proper Gaussian solutions can be obtained from proper solutions of PTE_2, setting $\xi_i = x_i + y_i i$, $\xi'_i = x'_i + y'_i i$.

Besides, there are proper Gaussian solutions to PTE which do not arise as solutions of the mentioned form. This can been seen, for example, by considering

$$\{0, 2i, 2+i\} \stackrel{2}{=} \{i, 1+i, 1+i\}.$$

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