Igusa class polynomials

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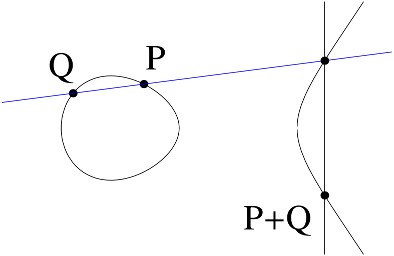
Elliptic curves

- An *elliptic curve* E/k ($\mathrm{char}(k) \neq 2$) is a smooth projective curve
$$y^2 = x^3 + ax^2 + bx + c.$$



- $E(k)$ is an abelian group

Endomorphisms

- $\mathrm{End}(E) =$ (ring of algebraic group morphisms $E \to E$)
  - $(\phi + \psi)(P) = \phi(P) + \psi(P)$
  - $(\phi\psi)(P) = \phi(\psi(P))$
- Examples:
  - For $n \in \mathbf{Z}$, have $n : P \mapsto nP$.
    For "most" E's in characteristic 0, have $\mathrm{End}(E) = \mathbf{Z}$.
  - If $E : y^2 = x^3 + x$ and $i^2 = -1$ in k, then we have
    $$i : (x, y) \mapsto (-x, iy),$$
    and $\mathbf{Z}[i] \subset \mathrm{End}(E)$.
  - If $\#k = q$, we have
    $$\mathrm{Frob} : (x, y) \mapsto (x^q, y^q).$$

The Hilbert class polynomial

The *j-invariant* is

$$j(E) = \frac{6912b^3}{4b^3 + 27c^2} \quad \text{for} \quad E : y^2 = x^3 + bx + c.$$

$$j(E) = j(F) \iff E \cong_{\overline{k}} F$$

Definition

Let K be an imaginary quadratic number field. Its Hilbert class polynomial is

$$H_K = \prod_{\substack{E/\mathbf{C} \\ \mathrm{End}(E) \cong \mathcal{O}_K}} \big(X - j(E)\big) \quad \in \mathbf{Z}[X].$$

Application 1: roots generate the Hilbert class field of K over K.
Application 2: make elliptic curves with prescribed order over $\mathbf{F}_p$.

Curves with prescribed order

- If $p = \pi\overline{\pi}$ in $\mathcal{O}_K$, then $(H_K \bmod p)$ splits into linear factors.
- Let $j_0 \in \mathbf{F}_p$ be a root and let $E_0/\mathbf{F}_p$ have $j(E_0) = j_0$.
- Then a twist E of E_0 has $\mathrm{Frob} = \pi$.
- We get

$$\#E(\mathbf{F}_p) = N(\pi - 1) = p + 1 - \mathrm{tr}(\pi).$$

Computing Hilbert class polynomials (1)

- Any E is complex analytically $\mathbf{C}/\Lambda$ for a lattice Λ
- Endomorphisms induce $\mathbf{C}$-linear maps $\alpha : \mathbf{C} \to \mathbf{C}$ with $\alpha(\Lambda) \subset \Lambda$
- If $\mathrm{End}(E) \cong \mathcal{O}_K$, then $\Lambda = c\mathfrak{a}$ for an ideal $\mathfrak{a} \subset \mathcal{O}_K$ and $c \in \mathbf{C}^*$.
- We get

$$\begin{array}{ccc} \mathrm{Cl}_K & \longleftrightarrow & \dfrac{\{E/\mathbf{C} : \mathrm{End}(E) \cong \mathcal{O}_K\}}{\cong} \\ [\mathfrak{a}] & \longmapsto & \mathbf{C}/\mathfrak{a}. \end{array}$$

Computing Hilbert class polynomials (2)

- Write $\mathfrak{a} = \tau\mathbf{Z} + \mathbf{Z}$ and let $q = \exp(2\pi i\tau)$.
- Then $j(\mathbf{C}/\mathfrak{a}) = j(q) = q^{-1} + 744 + 196884q + \cdots$.
- Compute

$$H_K = \prod_{[\mathfrak{a}]\in\mathcal{CL}_K} (X - j(\mathbf{C}/\mathfrak{a})) \in \mathbf{Z}[X].$$

- Other algorithms:
  - p-adic, [Couveignes-Henocq 2002, Bröker 2006]
  - Chinese remainder theorem. [Chao-Nakamura-Sobataka-Tsujii 1998, Agashe-Lauter-Venkatesan 2004]

Performance

- The Hilbert class polynomial is huge: the degree h_K grows like $|D|^{\frac{1}{2}}$, as do the logarithms of the coefficients.
- Small example: for $K = \mathbf{Q}(\sqrt{-17})$, get

$$\begin{aligned} H_K = x^4 &- 178211040000x^3 \\ &- 75843692160000000x^2 \\ &- 318507038720000000000x \\ &- 2089297506304000000000000 \end{aligned}$$

- Under GRH or heuristics, all three “quasi-linear” $O(|D|^{1+\epsilon})$.
- CRT (the underdog) is now the record holder: constructed a large finite field elliptic curve with $-D > 10^{15}$, $h_K > 10^7$. [Belding-Bröker-Enge-Lauter 2008, Sutherland 2009]

Curves of genus 2

Definition

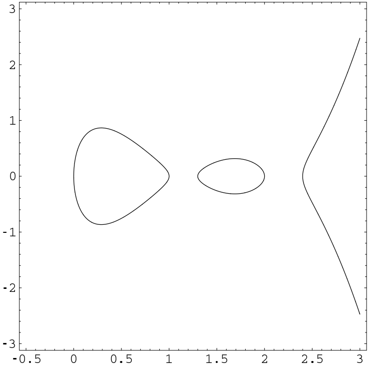
A curve of genus 2 is a smooth geometrically irreducible curve of which the genus is 2.

"Definition" (char.$\neq 2$)

A curve of genus 2 is a smooth projective curve that has an affine model

$$y^2 = f(x), \quad \deg(f) \in \{5, 6\},$$

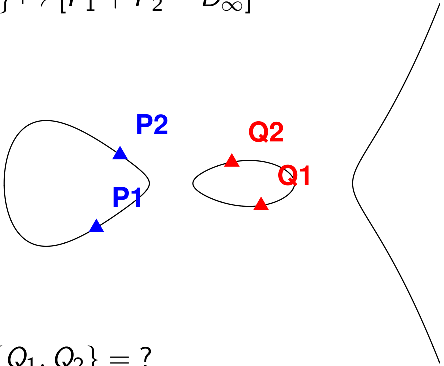
where f has no double roots.



The group law on the Jacobian

The Jacobian: group of equivalence classes of pairs of points.

- More precisely, divisor class group $\mathrm{Pic}^0(C)(k)$
  $\{P_1, P_2\} \mapsto [P_1 + P_2 - D_\infty]$



$\{P_1, P_2\} + \{Q_1, Q_2\} = ?$

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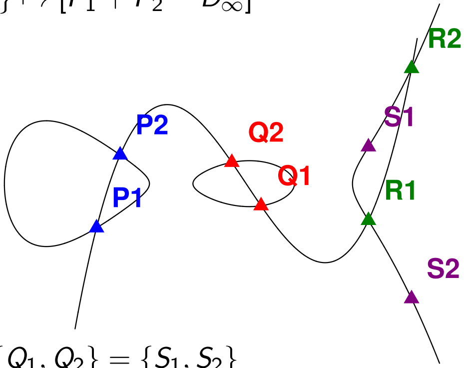


$\{P_1, P_2\} + \{Q_1, Q_2\} = \{S_1, S_2\}$

Igusa class polynomials

- Elliptic curves E have CM if $\text{End}(E)$ is an order in an imaginary quadratic field $K = \mathbf{Q}(\sqrt{r})$ with $r \in \mathbf{Q}$ negative.
- Curves C of genus 2 have CM if $\text{End}(J(C))$ is an order in a *CM field K of degree* 4, i.e. $K = K_0(\sqrt{r})$ with K_0 real quadratic and $r \in K_0$ totally negative.
- Assume K contains no imaginary quadratic field.
- *Igusa's invariants* i_1, i_2, i_3 are the genus-2 analogue of j
- The *Igusa class polynomials* of a quartic CM field K are a set of polynomials of which the roots are the Igusa invariants of curves C of genus 2 with CM by $\mathcal{O}_K$.

Applications

- Roots generate class fields.
  - not of K, but of its "reflex field' (no problem)'
  - not the full Hilbert class field (but we know which field)
  - useful? efficient?
- If $p = \pi\overline{\pi}$ in $\mathcal{O}_K$, construct curve C with

$$\#J(C)(\mathbf{F}_p) = N(\pi - 1) \quad \text{and} \quad \#C(\mathbf{F}_p) = p + 1 - \mathrm{tr}(\pi).$$

Algorithms

1. Complex analytic [Spallek 1994, Van Wamelen 1999]
2. p-adic [Gaudry-Houtmann-Kohel-Ritzenthaler-Weng 2002, Carls-Kohel-Lubicz 2008]
3. Chinese remainder theorem [Eisenträger-Lauter 2005]

None of these had running time bounds:

- denominators
- not known how to bound $|i_n(C)|$.
- algorithms not explicit enough
- no rounding error analysis for alg. 1 (not even for genus 1!!)

Denominators

- CM elliptic curves have "potential good reduction", hence $j(E) \in \overline{\mathbf{Z}}$, hence Hilbert class polynomials are in $\mathbf{Z}[X]$
- CM abelian varieties (such as $J(C)$) also have potential good reduction, but may have

$$(J(C) \bmod \mathfrak{p}) = E_1 \times E_2 \quad \text{and} \quad (C \bmod \mathfrak{p}) = E_1 \cup E_2$$

  for supersingular elliptic curves E_1, E_2.
- In that case, $\exists \iota : \mathcal{O}_K \to \mathrm{End}(E_1 \times E_2)$.
- Can bound denominators by studying the "embedding problem" [Goren-Lauter 2007], [Goren-Lauter (preprint 2010)]

Step 1: Enumerating $\cong$-classes

$$K \otimes \mathbf{R} \cong_{\mathbf{R}-\text{alg.}} \mathbf{C}^2$$

- For Φ an isomorphism and $\mathfrak{a} \subset \mathcal{O}_K$, get a lattice $\Lambda = \Phi(\mathfrak{a}) \subset \mathbf{C}^2$ and $\mathrm{End}(\mathbf{C}^2/\Lambda) = \mathcal{O}_K$.
- Also need a polarization, given by $\xi \in K^*$ with $\xi\mathfrak{a}\overline{\mathfrak{a}}\mathcal{D}_{K/\mathbf{Q}} = \mathcal{O}_K$. Then

$$\frac{\{(\Phi, \mathfrak{a}, \xi)\}}{\sim} \quad \longleftrightarrow \quad \frac{\{C/\mathbf{C} : \mathrm{End}(J(C)) \cong \mathcal{O}_K\}}{\cong}.$$

- symplectic basis gives $\Lambda = \tau\mathbf{Z}^2 + \mathbf{Z}^2$ with $\tau \in \mathrm{Mat}_2(\mathbf{C})$ symmetric with pos. def. imaginary part.

Step 2: Reduction (elliptic case)

For $E = \mathbf{C}/(\tau\mathbf{Z} + \mathbf{Z})$, the number τ is unique up to

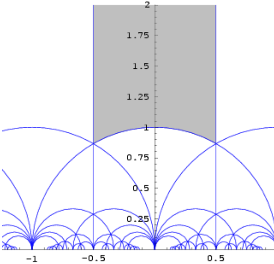
$\mathrm{SL}_2(\mathbf{Z})$

acting via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = (a\tau + b)(c\tau + d)^{-1}.$$

We make τ reduced:

1. $|\mathrm{Re}\,\tau| \leq 1/2$,
2. $|\tau| \geq 1$



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For $E = \mathbf{C}/(\tau\mathbf{Z} + \mathbf{Z})$, the number τ is unique up to

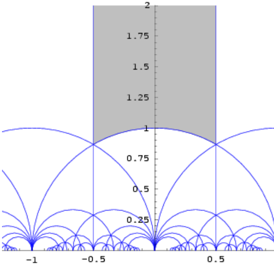
$$\mathrm{SL}_2(\mathbf{Z}) = \{M \in \mathrm{GL}_2(\mathbf{Z}) : M^{\mathrm{t}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\},$$

acting via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = (a\tau + b)(c\tau + d)^{-1}.$$

We make τ reduced:

1. $|\mathrm{Re}\,\tau| \leq 1/2$,
2. $|c\tau + d| \geq 1$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$



Step 2: Reduction

For $J(C) = \mathbf{C}^2/(\tau\mathbf{Z}^2 + \mathbf{Z}^2)$, the matrix τ is unique up to

$$\mathrm{Sp}_4(\mathbf{Z}) = \{M \in \mathrm{GL}_4(\mathbf{Z}) : M^{\mathrm{t}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\},$$

acting via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = (a\tau + b)(c\tau + d)^{-1}.$$

We make τ reduced:

1. entries of $\mathrm{Re}\,\tau$ have absolute value $\leq 1/2$,
2. $|\det(c\tau + d)| \geq 1$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_4(\mathbf{Z})$,
3. $\mathrm{Im}\,\tau = \begin{pmatrix} y_1 & y_3 \\ y_3 & y_2 \end{pmatrix}$ is reduced: $0 \leq 2y_3 \leq y_1 \leq y_2$.

Step 3: Numerical evaluation

- Thomae's formula [1870] gives an equation for C, given τ, in terms of theta constants

$$\theta[c_1, c_2](\tau) = \sum_{v \in \mathbf{Z}^2} \exp(\pi i (v + c_1)\tau(v + c_1)^t + 2\pi i (v + c_1)c_2^t)$$

with $c_1, c_2 \in \{0, \frac{1}{2}\}^2$.

- Write out, get [Bolza 1887, Spallek 1994]

$$i_k(\tau) = \frac{\text{pol. in } \theta\text{'s}}{(\prod \text{all } \theta\text{'s} \neq 0)^4}.$$

- Evaluate Igusa class polynomials numerically.

result

Bounds on Igusa invariants

- For running time bound, need upper bound on

$$|i_k(\tau)| = \frac{|\text{pol. in } \theta\text{'s}|}{(\prod \text{ all } |\theta|\text{'s} \neq 0)^4}.$$

- Have $|\theta(\tau)| < 2$ for reduced τ, so only need lower bound on $|\theta(\tau)|$.
- Got a bound in terms of
  1. upper bound on $\operatorname{Im} \tau_{22}$
  2. lower bound on $|\tau_{12}|$ (allowed to be weak)
- part 2 for free from detailed analysis of Steps 1 and 2.

▸ result

Bounds on $\operatorname{Im} \tau_{22}$

- Bound $\operatorname{Im} \tau_{22}$ by proving existence of alternative τ' and relating it to τ via $A \in \mathrm{Sp}_4(\mathbf{Z})$.
- Elliptic curves: if $\tau = A\tau'$ and $\tau' = x + yi$, then

$$\begin{aligned}\operatorname{Im} \tau = \frac{\operatorname{Im} \tau'}{|c\tau' + d|^2} &= \frac{y}{(cx + d)^2 + (cy)^2} \\ &\leq \begin{cases} y & \text{if } c = 0 \\ y^{-1} & \text{if } c \neq 0 \end{cases} \\ &\leq \max\{y, y^{-1}\}\end{aligned}$$

  Similar results for genus 2.
- To get τ', write $\mathfrak{a} = z\mathfrak{b} + \mathfrak{b}^{-1}$ and maximize $N_{K/\mathbf{Q}}(\mathfrak{b}^2(z - \overline{z})\mathcal{O}_K)$.
  (related to fundamental domains for $\mathrm{SL}_2(\mathcal{O}_{K_0})$).

Result

Theorem

Algorithm computes the Igusa class polynomials of K in time less than

$$\text{cst.}(D_1^{7/2} D_0^{11/2})^{1+\epsilon},$$

where $D_0 = \operatorname{disc} K_0$ and $D_1 D_0^2 = \operatorname{disc} K$. The bit size of the output is between

$$\text{cst.}(D_1^{1/2} D_0^{1/2})^{1-\epsilon} \quad \text{and} \quad \text{cst.}(D_1^2 D_0^3)^{1+\epsilon}.$$

Bottlenecks:

1. quasi-quadratic time theta evaluation (quasi-linear method not proven [Dupont 2006])
2. denominator bounds not optimal (special cases/conjectures [Bruinier-Yang 2006, Yang (to appear)])

What's next?

$g = 1$: In practice, one does not use j, but uses "smaller functions" such as $\sqrt[3]{j}$, Weber functions, and (double) eta quotients.

$g = 2$: Still stuck with Igusa's invariants.

$g = 1$: Useful tool: explicit version of Shimura's reciprocity law, relating Galois action of $\widehat{K}^*$ on values of modular functions to the action of $\mathrm{GL}_2(\widehat{\mathbf{Q}})$ on the modular functions themselves.

$g = 2$: I have been making Shimura's reciprocity law for $g = 2$ more explicit and have some ideas for "smaller functions"

The "embedding problem" (Goren-Lauter)

Given a quartic CM field K (not containing an imag. quadr. field). What are the primes p such that the following exist?

- a maximal order R in the quaternion algebra $B_{p,\infty}/\mathbf{Q}$,
- a fractional right R-ideal $\mathfrak{a}$ with left order R', and
- an embedding of $\mathcal{O}_K$ into the matrix algebra

$$\begin{pmatrix} R & \mathfrak{a}^{-1} \\ \mathfrak{a} & R' \end{pmatrix}$$

such that complex conjugation on $\mathcal{O}_K$ coincides with

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \overline{\alpha} & \overline{\gamma} N(\mathfrak{a})^{-1} \\ \overline{\beta} N(\mathfrak{a}) & \overline{\delta} \end{pmatrix}.$$

Partial answer: we know the splitting behaviour of p in the normal closure of K and we know $p < cD_K$. [GL 2006]