AN INTRODUCTION TO MODULI SPACES OF CURVES

MAARTEN HOEVE

ABSTRACT. Notes for a talk in the seminar on modular forms and moduli spaces in Leiden on October 24, 2007.

CONTENTS

1. INTRODUCTION

Up to now in the seminar we have only considered moduli spaces of elliptic curves, i.e. curves of genus 1 with a marked rational point. The goal of this talk is to give an introduction to moduli spaces of curves of genus greater than 1.

The theory of such moduli spaces is quite old. It started with Riemann's remark that a general Riemann surface with g holes is determined by $3g-3$

parameters. Nowadays the theory is quite well developed. We can construct moduli spaces of curves and we know their basic properties.

Still the theory for genus greater than 1 is much less explicit than the elliptic curves case, where we can represent the moduli space as the quotient of the upper half plane by $\mathrm{SL}_2(\mathbb{Z})$. There are still many open questions about the moduli spaces of curves and it is a very active area of research.

1.1. **References.** Unfortunately, there is no good textbook on moduli of curves. Still we try to give some references.

- This talk is mostly based on David Mumford's lectures on *Curves and their Jacobians*, [Mum]. These lecture notes are an excellent introduction. They contain many important ideas, but almost no proofs.
- There is also the book *Moduli of Curves* by Harris and Morrison, [HM]. This book is more detailed than Mumford's notes, but is still quite informal and lacks many proofs.
- For the construction of the coarse moduli space of curves using geometric invariant theory, see the book [GIT] by Mumford. The appendix of this book also contains many references to articles on moduli.

2. Curves of low genus

2.1. **Curves.** With a *smooth curve* over an algebraically closed field k we mean an integral scheme of dimension 1 which is smooth and proper over k. If $k = \mathbb{C}$, then a smooth curve 'is' a compact Riemann surface. All smooth curves are automatically projective.

A *family of curves* over a base scheme S is a smooth and proper morphism $f : C \to S$ for which every geometric fibre is a smooth curve.

The most important invariant of a curve C is its genus

$$g(C) := \dim_k H^1(C, \mathcal{O}_C).$$

By Serre-duality the genus is equal to the dimension of $H^0(C, \Omega_C)$. Over $\mathbb{C}$ the genus is the number of holes in the Riemann surface. This number completely determines the topology of the Riemann surface.

2.2. **Classification of curves.** Let us try to classify all curves of small genus g.

- $g = 0$: Over an algebraically closed field there is only one curve of genus 0, the projective line. (If the field is not algebraically closed there may be more).
- $g = 1$: This is basically the case of elliptic curves, which we have seen in the previous talks.
- $g = 2$: Suppose C has genus 2. One checks easily that in this case the canonical line bundle Ω_C gives a $2 : 1$ morphism to $\mathbb{P}^1$. This means that C is *hyperelliptic*. By Riemann-Hurwitz the morphism $C \to \mathbb{P}^1$ has 6 branch points. We can assume that one of these points lies over ∞. Then C is given by an equation $y^2 = f(x)$ where f has degree 5 (if $\mathrm{char}(k) > 5$).
- $g = 3$: Suppose that C has genus 3. Then there are two cases:
  (1) C is hyperelliptic and is given by an equation $y^2 = f(x)$ where f has degree 7 (Riemann-Hurwitz).
  (2) The canonical line bundle Ω_C gives an embedding $C \hookrightarrow \mathbb{P}^{g-1} = \mathbb{P}^2$. Since every $\omega \in H^0(C, \Omega_C)$ has $2g - 2 = 4$ zeroes, every line in $\mathbb{P}^2$ intersects C in 4 points. So C is given by an equation of degree 4, a quartic. Conversely, every smooth quartic defines a curve of genus 3.
- $g = 4$: Again there are two cases:
  (1) C is hyperelliptic and is given by an equation $y^2 = f(x)$ where f has degree 9.
  (2) The canonical line bundle gives an embedding $C \hookrightarrow \mathbb{P}^{g-1} = \mathbb{P}^3$. The image is the zero set of a polynomial of degree 2 (*quadric*) and degree 3 (*cubic*). The quadric may be singular. Conversely, any such intersection is the image of a canonical embedding of a genus 4 curve.

We can continue this way, but it gets harder and harder to write down a list of all curves of genus g. Still we want to know something about curves of every genus.

In these examples one sees that the curves of genus g all occur in algebraic families. For instance, almost all curves of genus 3 occur in the family of quartics in $\mathbb{P}^2$ (the hyperelliptic curves can just be seen as degenerate quartics). So one can ask the question: can we find an algebraic family of curves in which every curve of genus g occurs exactly once?

A parameter space for such a family is called a moduli space. Such moduli spaces exist for smooth curves. For instance, $\mathbb{A}^1$ is a moduli space for elliptic curves via the j-invariant. However, the above question is too naive, and we will need to reformulate it to get the right definition of such moduli spaces.

3. The moduli problem

The idea of the moduli space of curves is to find a variety that classifies all smooth curves. The goal of this section is to say exactly which properties we want this variety to have.

3.1. The moduli functor.

3.1. The moduli functor. Roughly speaking we want a moduli space for smooth curves to be a scheme which parametrizes all smooth curves.

A naive definition might be that a moduli space for smooth curves is a scheme $\mathcal{M}$ such that over an algebraically closed field k one has a bijection

$$\mathcal{M}(k) \stackrel{\sim}{\leftrightarrow} \{\text{smooth curves}\}/\cong .$$

The problem with this definition is that it says nothing about the scheme structure of $\mathcal{M}$. We may just take $\mathcal{M}$ to be a disjoint union of points $\mathrm{Spec}(k)$, one for each isomorphism class. This is definitely not something we want.

To determine the scheme structure of $\mathcal{M}$, we need to look at families of curves. Given a family $f : C \to S$, we get a map $S(k) \to \mathcal{M}(k)$. If $\mathcal{M}$ is a moduli space, we want this map to come from a morphism of schemes $S \to \mathcal{M}$. This leads to the following definition.

Definition 3.1. A *fine moduli space for smooth curves* is a scheme $\mathcal{M}$ with a family $\pi : \mathcal{C} \to \mathcal{M}$ of curves such that given any family of curves $f : C \to S$ there is a unique morphism $h : S \to \mathcal{M}$ such that $C \cong h^*\mathcal{C} = S \times_{\mathcal{M}} \mathcal{C}$ (as S-schemes). The family $\pi : \mathcal{C} \to \mathcal{M}$ is called the *universal curve*.

If such a moduli space exists, it is unique up to unique isomorphism. Indeed given another moduli space $\mathcal{M}'$ and universal curve $\pi' : \mathcal{C}' \to \mathcal{M}'$, we get morphisms $\mathcal{M} \to \mathcal{M}'$ and $\mathcal{M}' \to \mathcal{M}$ by the universal property, which must be each others inverse.

We can rephrase the definition in terms of representable functors. Define a functor

$$\begin{aligned}\mathcal{F} : (\text{Schemes})^\circ &\to (\text{Sets})\\ S &\mapsto \{\text{families of curve over } S\}/\cong,\\ (S \to S') &\mapsto (C/S' \mapsto S \times_{S'} C).\end{aligned}$$

Then a fine moduli space for smooth curves is a scheme which represents this functor. Such a scheme is unique by Yoneda's lemma. The above functor is called the *moduli functor*.

If S is a connected scheme, then the genus in any family of curves over S is constant. So if a fine moduli space exists, it can be written as a disjoint union $\coprod_{i=0}^{\infty} \mathcal{M}_g$, where $\mathcal{M}_g$ is a fine moduli space for smooth curves of genus g. So from now on we will usually fix a genus g and only look at curves with that genus. We write $\mathcal{C}_g$ for the universal curve over $\mathcal{M}_g$.

3.2. **Why the moduli functor is not representable.** In the previous section we have given the definition of a fine moduli space. This is what a moduli space should ideally be. However, *fine moduli spaces do not exist if the objects you want to classify have automorphisms*.

The problem is that given a curve C_0 which has non-trivial automorphisms, we can make a family $C \to S$ for which every fibre is isomorphic to C_0 but C is not isomorphic to $C_0 \times S$. If a fine moduli space $\mathcal{M}$ would exist, there is a morphism $h : S \to \mathcal{M}$ such that $C \cong h^*\mathcal{C}$. Since each fibre is isomorphic to C_0, h factors through the closed point corresponding to the isomorphism class of C_0. This would mean that $C \cong h^*\mathcal{C} = C_0 \times S$, a contradiction.

Example 3.2. Let us give a construction of such a family of curves over $\mathbb{C}$. We start with a curve C_0 over $\mathbb{C}$ and a non-trivial automorphism σ on C_0. Define an action of $\mathbb{Z}$ on $\mathbb{C}$ by $k * z = z + 2\pi i k$ for $k \in \mathbb{Z}$and $z \in \mathbb{C}$ and an action of $\mathbb{Z}$ on $\mathbb{C} \times C_0$ by

$$k * (z, P) = (z + 2\pi i k, \sigma^k(P))$$

for $P \in C_0$. These actions commute with the projection $\mathbb{C} \times C_0 \to \mathbb{C}$. The quotient $\mathbb{C}/2\pi i\mathbb{Z}$ is isomorphic to $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ by the exponential map. The quotient $\mathbb{C} \times C_0/\mathbb{Z}$ gives a family of curves over $\mathbb{C}^*$ for which every fibre is isomorphic to C_0. However, this family is not isomorphic to $\mathbb{C}^* \times C_0$, since otherwise the action of $\mathbb{Z}$ on C_0would be trivial.

Another example is the following.

Example 3.3. Suppose that we want to make a moduli space for vector spaces over some (algebraically closed) field k. A family of vector spaces is a vector bundle. Hence we define a moduli functor $\mathcal{M}$ by

$$\mathcal{M}_r(S) = \{\text{vector bundles of rank } r \text{ on } S\}/\cong .$$

We claim that this functor is not representable.

Suppose that it is represented by a scheme X. The vector space k^r over $\mathrm{Spec}(k)$ gives a closed k-rational point $x \in X$. Pick a curve C and any non-trivial vector bundle $\mathcal{V}$ on C. Then $\mathcal{V}$ determines a morphism $C \to X$. There is an open cover $\{U_i\}$ such that $\mathcal{V}|_{U_i}$ is trivial. Hence the restriction of $C \to X$ to U_i is just the constant map $U_i \to \{x\}$. But then the morphism $C \to X$ must be the constant map with image x. This would imply that $\mathcal{V}$ is trivial, a contradication.

The point is that allthough on each U_i the vector bundle is trivial, we can glue these trivial bundles in a non-trivial way, because each vector space has automorphisms.

For every genus we can find a curve with a non-trivial automorphism group. So a fine moduli space for smooth curves of genus g does not exist. There are several solutions to this problem.

3.3. **Solution 1: coarse moduli spaces.** In the first solution we weaken the demands for a moduli space. Instead of looking at fine moduli spaces, we look at coarse moduli spaces.

Definition 3.4. Suppose $\mathcal{F}$ is moduli functor. A *coarse moduli space* for $\mathcal{M}$ is a scheme M and a morphism $\phi : \mathcal{M} \to h_M$ such that

(1) For all algebraically closed fields k the map
$$\phi(k) : \mathcal{M}(k) \to M(k)$$
is a bijection.
(2) Given any scheme S and morphism $\psi : \mathcal{M} \to h_S$, there is a unique morphism $\chi : M \to S$ such that $\psi = h_\chi \circ \phi$.

Here h_S is the functor $T \mapsto \mathrm{Mor}(T, S)$ and $h_\chi : h_M \to h_S$ is $\chi \circ -$. Also we write $\mathcal{M}(k)$ for $\mathcal{M}(\mathrm{Spec}(k))$. If a coarse moduli space exists, it is unique by the second property.

Coarse moduli spaces for smooth curves exist. We denote them with M_g, where g is the genus. The big problem with these spaces is that they map be singular. Also, there is no universal curve over M_g.

Example 3.5. The projective line is a coarse moduli space for elliptic curves (by the j-invariant).

3.4. **Solution 2: rigidifying the problem.** The second solution is to change our moduli problem. We look at curves with some extra structure, such that curves with this extra structure have no automorphisms. This is called *rigidifying* the problem.

For elliptic curves this is done by looking at $\Gamma(n)$ and $\Gamma_1(n)$ structures, see Peter's talk. For example, you can only look at $X(n)$ for $n \geq 3$.

For curves of higher genus you can for instance look at pointed curves. An n-pointed curve is a tuple $(C, P_1, \ldots, P_n)$, where C is a curve of genus g and $P_1, \ldots, P_n$ are distinct (rational) points on C. The moduli space of n-pointed curves denoted with $\mathcal{M}_{g,n}$. The idea is that if an automorphism fixes enough points, it must be the identity.

3.5. **Solution 3: stacks.** The final solution is extending the category of schemes. The moduli problem may not be representable by a scheme, but maybe it is representable by something else. This something else is called stacks.

This solution is probably most satisfying. The moduli stack of curves classifies exactly what you want, is smooth and keeps track of all automorphisms. It is also the most technically involved solution, so we will not say more about it here.

4. Construction of the moduli space

In this section we sketch three possible constructions of the moduli space of curves.

4.1. **Approach 1: Teichmüller space.** Like in the case of $g = 1$ we can give an analytic construction of the moduli space over $\mathbb{C}$. Let Π_g be the group generated by elements $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ with the relation $\prod_{i=1}^{g} [\alpha_i, \beta_i] = 1$. Then fundamental group of a compact Riemann-surface is isomorphic to Π_g.

Define as a set the Teichmüller space

$$T_g = \{(C, \phi) \mid C \text{ a complex Riemann-surface and } \phi : \Pi_g \xrightarrow{\sim} \pi_1(C)\}/\sim,$$

where $(C,\phi) \sim (C',\phi')$ if there is an isomorphism $f : C \to C'$ such that $(f_*) \circ \phi$ differs from ϕ' by an inner automorphism (i.e. conjugation by an element). Also, we demand that ϕ is orientation preserving, i.e. $\phi(\alpha_i) \cdot \phi(\beta_i) = 1$, where $\cdot$ is the intersection pairing.

Example 4.1. Consider the case $g = 1$. Then $\Pi_1 = \mathbb{Z}^2$ and $\pi_1(C) = H_1(C,\mathbb{Z}) = \mathbb{Z}^2$. One now easily checks that T_1 is the upper half plane.

Using analytic methods it is not hard to put a complex structure on T_g. It is a deep theorem that T_g is a bounded domain in $\mathbb{C}^{3g-3}$.

Let Γ_g be the group of automorphisms of Π_g which preserve the orientation modulo the group of inner automorphisms. Then Γ_g acts on T_g by $(C,\phi) * \gamma = (C,\phi \circ \gamma)$. The quotient T_g/Γ_g is a coarse moduli space for smooth curves of genus g, since any two isomorphisms $\Pi_g \to \pi_1(C)$ differ by an element of Γ_g.

Since T_g is contractible, the group Γ_g is the fundamental group of $\mathcal{M}_g$ (in the sense of stacks, since the action is not free). So Γ_g is an important topological invariant of the moduli space.

Remark 4.2. Allthough we know that T_g is a bounded complex domain, there is no explicit description of this space. So in general it is not possible to write down 'modular forms' on T_g for the group Γ_g.

4.2. **Approach 2: Geometric invariant theory.** The construction of the coarse moduli space using Teichmüller space is an analytic construction. There is also a purely algebraic construction of the coarse moduli space using geometric invariant theory. This approach was invented by David Mumford, see . The idea is a follows.

It is 'easy' to show the existence of fine moduli schemes for subspaces of $\mathbb{P}^n$ with a fixed Hilbert polynomial, because then you can use equations. These moduli spaces are called *Hilbert schemes* and were constructed by Grothendieck.

But every curve is canonically embedded in a projective space: the line bundle $\Omega_{C/k}^{\otimes 3}$ defines a closed immersion $C \hookrightarrow \mathbb{P}(H^0(C,\Omega_{C/k}^{\otimes 3}))$. This projective space is isomorphic to $\mathbb{P}^N$ where $N = 5g-6$. Unfortunately, the isomorphism $\mathbb{P}(H^0(C,\Omega_{C/k}^{\otimes 3})) \cong \mathbb{P}^N$ is not canonical, it depends on the choice of a basis of $H^0(C,\Omega_{C/k}^{\otimes 3})$ up to multiplication to a constant.

To remedy this, define a new moduli functor $\mathcal{H}_g$ by

$$\mathcal{H}_g(S) = \left\{ \begin{array}{c} \text{families of genus } g \text{ curves } f : C \to S \\ \text{with an isomorphism } \mathbb{P}(f_*(\Omega_{C/S}^{\otimes 3})) \cong \mathbb{P}^N \times S \end{array} \right\}.$$

Then one can show that $\mathcal{H}_g$ is representable by a closed subscheme of the Hilbert scheme of $\mathbb{P}^N$.

The coarse moduli space M_g is the quotient of $\mathcal{H}_g$ by the natural action of the automorphism group PGL_N of $\mathbb{P}^N$. Unfortunately this group is not a finite group, so one must be careful in taking this quotient. This is where geometric invariant theory comes in. This is a theory about taking quotients of algebraic varieties by algebraic groups like PGL_n. The end result is that the coarse moduli space M_g exists and that it is a quasi-projective variety.

4.3. **Approach 3: The Torelli map.** A final approach is to use the Torelli map, which sends a curve to its Jacobian. In this way one can realize the moduli space of curves as a closed subset of the moduli space of (principally polarized) abelian varieties. This last moduli space is easier to construct. In particular, in the complex case one can write down modular forms on this space. By restriction one can get 'modular forms' on $\mathcal{M}_g$.

Remark 4.3. The problem of determining $\mathcal{M}_g$ as a subset of the moduli space of abelian varieties is known as the *Schottky problem*. Put another way, the Schottky problem is to determine which abelian varieties are Jacobians. Not all abelian varieties can be Jacobians, since M_g has dimension $3g-3$ while the moduli space of abelian varieties has dimension $\frac{1}{2}g(g+1)$.

Theoretically this problem is solved: we know that the theta functions of Jacobian satisfy a certain differential equation which other theta functions do not. Practically this criterion is almost impossible to check and people are still looking at better ways to distinguish Jacobians.

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