INTRODUCTION

The following two facts about modular forms are not at all clear from the definition:

1. Modular forms exist;
2. $\dim M_k(\Gamma)$ is finite.

In fact, a stronger version of (1) is true: interesting modular forms exist. Think of those forms associated with elliptic curves over $\mathbb{Q}$, Galois representations, certain partition functions, and—and this is the topic of this talk—with integral quadratic forms.

The linear pigeon hole principle states that if $n + 1$ pigeons flock together in an $n$-dimensional vector space, then they satisfy a linear relation. In this talk we will find 4 interesting modular forms, elements of the same 3-dimensional vector space.

1. $\theta(z)^8 \in M_4(\Gamma_0(4))$

Define the following function of $q = \exp(2\pi i z)$:

$$\theta(z) := \sum_{d \in \mathbb{Z}} q^{d^2} = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + 2q^{25} + \cdots$$

**Proposition.** $\theta(z/(4z + 1)) = \sqrt{4z + 1} \cdot \theta(z)$

**Lemma** (Poisson summation). $\phi : \mathbb{R} \to \mathbb{C}$ continuous and rapidly decreasing, put

$$\hat{\phi} : \mathbb{R} \to \mathbb{C} : t \mapsto \hat{\phi}(t) := \int_{-\infty}^{\infty} \phi(x) \exp(-2\pi i tx) dt$$

its Fourier transform, then

$$\sum_{n \in \mathbb{Z}} \phi(n) = \sum_{k \in \mathbb{Z}} \hat{\phi}(k).$$

**Proof of the Lemma.** The proof is easy. Define $\psi(x) := \sum_{n \in \mathbb{Z}} \phi(x + n)$. Then $\psi(x)$ is periodic hence has a Fourier series expansion:

$$\psi(x) = \sum_{k \in \mathbb{Z}} c_k \exp(2\pi ikx)$$

where $c_k := \int_0^1 \psi(x) \exp(-2\pi ikx) dx$.

Now we have:

$$\sum_{n \in \mathbb{Z}} \phi(n) = \psi(0) = \sum_{k \in \mathbb{Z}} c_k = \sum_{k \in \mathbb{Z}} \int_0^1 \psi(x) \exp(-2\pi ikx) dx$$

$$= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} \phi(x) \exp(-2\pi ikx) dx$$

$$= \sum_{k \in \mathbb{Z}} \hat{\phi}(k).$$
Proof of the Proposition. Take $\phi(x) = e^{-2\pi i z x^2}$. This is ‘rapidly decreasing’ if the imaginary part of $z$ is positive. Then $\hat{\phi}(t) = (-2iz)^{-1/2}e^{-\pi it^2/2z}$. (Recall that $e^{-\pi x^2}$ is its own Fourier transform.) Poisson summation gives
\[
\sum_n e^{-2\pi i z n^2} = (-2iz)^{-1/2} \sum_k e^{-\pi i k^2/2z}
\]
or in other words:
\[
\theta(z) = (-2iz)^{-1/2}\theta(-1/4z).
\]
This gives the transformation behavior for $z \mapsto -1/4z$. Together with the invariance under $z \mapsto z + 1$ we can calculate the transformation behavior for
\[
\begin{align*}
z \mapsto -\frac{1}{4z} & \mapsto -\frac{1 - 4z}{4z} \mapsto \frac{z}{4z + 1}
\end{align*}
\]
and verify the correctness of the Proposition. □

The transformations $z \mapsto z + 1$ and $z \mapsto \frac{z}{4z + 1}$ generate the group $\Gamma_0(4)$, so it follows that:

**Proposition.** $\theta(z)^8 \in M_4(\Gamma_0(4))$

We have
\[
\theta(z)^8 = 1 + 16q + 112q^2 + 448q^3 + 1136q^4 + \cdots
\]
where the coefficient of $q^n$ is of course the number of vectors of length square root of $n$ in the standard eight dimensional lattice.

2. **Three More Elements of $M_4(\Gamma_0(4))$**

They are:
\[
G_4(z) = \frac{1}{240} + \sum_{n>0} \sigma_3(n)q^n = \frac{1}{240} + q + 9q^2 + 28q^3 + 73q^4 + \cdots
\]
\[
G_4(2z) = \frac{1}{240} + q^2 + 9q^4 + \cdots
\]
\[
G_4(4z) = \frac{1}{240} + q^4 + \cdots
\]

3. dim$_{\mathbb{C}} M_4(\Gamma_0(4)) = 3$

See the picture of the fundamental domain. The quotient $Y(\Gamma)$ of $H$ by $\Gamma_0(4)$ has genus 0 and three cusps. Also, the quotient is a free quotient by $PT\Gamma_0(4) = \Gamma_0(4)/\{\pm 1\}$, i.e. there are no elliptic points. (In fact, the group $PT\Gamma_0(4)$ is torsion-free, for if it would contain some nontrivial torsion element $\pm \gamma$ then the trace of $\gamma$ would be 0 or $\pm 1$, but traces of elements of $\Gamma_0(4)$ are 2 modulo 4.)

(From the freeness it follows that $PT\Gamma_0(4)$ is the fundamental group of the Riemann sphere minus three points: a free group on two generators.)

The freeness of the action is not in contradiction with the fact that $Y(\Gamma_0(4))$ does not carry a universal elliptic curve: elliptic curves with $\Gamma_0(4)$-structure still have the nontrivial automorphism $-1$.

**Proposition.** dim$_{\mathbb{C}} M_4(\Gamma_0(4)) = 3$. 

We could of course just plug in the details into the big formula giving the dimension of a space of modular forms, but that formula is one that I cannot remember. It is the method by which it is deduced rather that I can remember.

**Sketch of proof.** We identify $M_4$ with a subset of the set of meromorphic invariant forms of degree 2 on $H$:

$$M_4(\Gamma_0(4)) = \{ f \in A_4(\Gamma_0(4)) : v_P(f) \geq 0 \text{ and } v_C(f) \geq 0 \}$$

$$= \{ \omega : v_P(f) \geq 0 \text{ and } v_C(f) \geq -2 \}$$

$$= \{ h\omega_0 : \text{div}(h) \geq -2s_1 - 2s_2 - 2s_3 - \text{div}(\omega_0) \}$$

$$= \{ h : \text{div}(h) \geq D \}$$

for some $D$ of degree $-2$. (Note that the degree of the divisor of a differential 1-form on the Riemann sphere is $-2$, hence that the degree of the divisor of any degree 2 form $\omega_0$ is $-4$).

Since on the projective line all divisors of same degree are equivalent, we can take $D$ to be $-2\infty$ and we get the space of polynomials of degree at most 2, which is of dimension 3. \qed

4. **An Elementary Solution of the Eight Squares Problem**

The eight squares problem is the problem of finding the number of vectors of square length $n$ in the lattice $\mathbb{Z}^8$. An elementary solution is one that uses only the five fundamental operations of arithmetic: addition, subtraction, multiplication, division and modular form (Eichler.)

**Theorem.** $\theta(z)^8 = 16G_4(z) - 32G_4(2z) + 256G_4(4z)$.

**Proof.** Apply the linear pigeon hole principle. \qed