Divisibility Sequences for Elliptic Curves with Complex Multiplication

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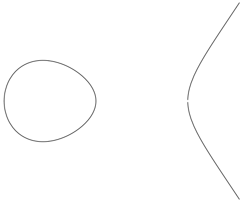
Elliptic Curves

An *elliptic curve* E is a non-singular projective curve E given by

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$



- It has a natural algebraic group structure with neutral element $O = (0 : 1 : 0)$ at infinity.

Elliptic Divisibility Sequences (1)

Let E be an elliptic curve, given by an equation with coefficients in $\mathbb{Z}$.

- Every point $Q \in E(\mathbb{Q})$ can be written uniquely in the form

$$Q = \left(\frac{A}{B^2}, \frac{C}{B^3} \right)$$

with integers A, B, C such that A and C are both coprime to B.

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Let E be an elliptic curve, given by an equation with coefficients in $\mathbb{Z}$. Let $P \in E(\mathbb{Q})$ be a point of infinite order.

- Every point $nP \in E(\mathbb{Q})$ can be written uniquely in the form

$$nP = \left(\frac{A_n}{{B_n}^2}, \frac{C_n}{{B_n}^3} \right)$$

  with integers A_n, B_n, C_n such that A_n and C_n are both coprime to B_n.
- B_n is the largest integer such that nP reduces to $(0:1:0)$ modulo B_n.
- We get a sequence $B_1, B_2, B_3, \ldots$, which we call an *elliptic divisibility sequence*.

Applications

- Source of large primes ...
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  - But what about other rings?
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- Learn about elliptic curves.

Example: $E : y^2 = x^3 - 2x, P = (2, 2)$

$$
\begin{aligned}
B_1 &= 1\\
B_2 &= 2\\
B_3 &= 1\\
B_4 &= 84\\
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- $\log B_m \sim \widehat{h}(P)\ m^2$. (Weil, Siegel)

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- $\log B_m \leq \widehat{h}(P)\ m^2 + C$.

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- $\log B_m \leq \hat{h}(P)\ m^2 + C$.
- $\log B_m = \hat{h}(P)\ m^2 + O\left((\log m)(\log\log m)^3\right)$. (Linear forms in elliptic logarithms, David, 1995)

Example: $E : y^2 = x^3 - 2x, P = (2, 2)$

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\begin{aligned}
B_1 &= 1 \\
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- if $\mathrm{ord}_{\mathfrak{p}}(B_m) >> 0$, then $\mathrm{ord}_{\mathfrak{p}}(B_{mn}) = \mathrm{ord}_{\mathfrak{p}}(B_m) + \mathrm{ord}_{\mathfrak{p}}(n)$. (formal groups / reduction)

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- all but finitely many terms have a new prime factor (Silverman, 1988)

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Lemma

There is a constant $C \neq 0$ in $\mathbb{Z}$ such that

$$\frac{B_n}{D_n} \mid C \prod_{p|n} p B_{n/p}.$$

Proof of Silverman (2)

$$\frac{B_n}{D_n} \leq C \prod_{p|n} p B_{n/p},$$

so

$$\begin{aligned}
\log D_n &\geq \log B_n - \sum_{p|n} (\log B_{n/p} + \log p) - \log C \\
&\geq \widehat{h}(P) n^2 - \sum_{p|n} \widehat{h}(P)(n/p)^2 - o(1) n^2 \\
&= \widehat{h}(P)\ n^2 \left(1 - \sum_{p|n} p^{-2} - o(1) \right) \\
&\geq \widehat{h}(P)\ n^2 (0.547 - o(1)) \to \infty. \quad \square
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- The endomorphisms form a ring $\mathrm{End}(E)$.
- $\mathrm{End}(E)$ is either $\mathbb{Z}$ or $\mathbb{Z}[\omega]$, where $\omega \in \mathbb{C} \setminus \mathbb{R}$ is a zero of a polynomial $X^2 + aX + b$ with $a, b \in \mathbb{Z}$.

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- Example: multiplication by $n \in \mathbb{Z}$
- Example: If $E : y^2 = x^3 + ax$, then $(x, y) \to (-x, iy)$.
- The endomorphisms form a ring $\mathrm{End}(E)$.
- $\mathrm{End}(E)$ is either $\mathbb{Z}$ or $\mathbb{Z}[\omega]$, where $\omega \in \mathbb{C} \setminus \mathbb{R}$ is a zero of a polynomial $X^2 + aX + b$ with $a, b \in \mathbb{Z}$.
- If $\mathrm{End}(E) \neq \mathbb{Z}$, then we say that E has *Complex Multiplication*.

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  - Growth.
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Silverman's proof?

Assume that $\mathcal{O}$ is a principal ring.

- The original proof gives at best

$$\log D_\alpha \geq \widehat{h}(P)\ N(\alpha)\ \left(1 - \sum_{\pi|\alpha} N(\pi)^{-1} - o(1)\right).$$

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- Solution: Inclusion-exclusion.

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- Define $B_{\mathfrak{a}}$ for every ideal $\mathfrak{a}$ by

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- Then generalize the properties.

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Inclusion-exclusion and Mertens' theorem then prove the existence of primitive divisors.

Results (1)

Theorem
For all ideals $\mathfrak{a} \subset \mathcal{O}$ coprime to the conductor of $\mathcal{O}$, except finitely many, the term $B_{\mathfrak{a}}$ has a primitive divisor.

Results (2)

For $\mathbb{Z}$-indexed sequences, the methods give the following:

Theorem

$$\log D_n = \widehat{h}(P)\; n^2 \prod_{p|n}(1-p^{-2}) \;+\; O(n^{\epsilon}),$$

where $\prod_{p|n}(1-p^{-2})$ *is between* $\zeta(2)^{-1} > 0.6079$ *and* 1*.*

(compare to $\log D_n \geq \widehat{h}(P) n^2 (0.547 - o(1))$.)

Results (3)

Suppose that E and P are defined over a number field L and that not all endomorphisms of E are defined over L. Then they are defined over a quadratic extension M/L. Consider the $\mathbb{Z}$-indexed sequence of L-ideals $B_1, B_2, B_3, \ldots$.

Corollary

Define for all $n \in \mathbb{Z}$, the numbers

$$\begin{aligned} r_n &= \#\{p|n \text{ prime} : p \text{ ramifies in } \mathrm{End}(E)\}, \\ s_n &= \#\{p|n \text{ prime} : p \text{ splits in } \mathrm{End}(E)\}. \end{aligned}$$

Then for all but finitely many n, the term B_n has at least $r_n + s_n + 1$ primitive divisors, of which at least s_n split in M/L.

Open problems

- Prove the conjectures of Gunther Cornelissen and Karim Zahidi.
- Give a good definition of divisibility sequence for an abelian variety ...
- indexed by (a subring of) the endomorphism ring.

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