3.1 (a) Prove that the set of rational numbers x with height $H(x)$ less than κ contains at most $2\kappa^2 + \kappa$ elements.
(b) * Let $R(\kappa)$ be the set of rational numbers x with height $H(x)$ less than κ. Prove that

$$\lim_{\kappa\to\infty} \frac{\#R(\kappa)}{\kappa^2} = \frac{12}{\pi^2}.$$

3.2 Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be points on the non-singular cubic curve

$$y^2 = x^3 + ax^2 + bx + c,$$

where a, b, c are integers. Let

$$P_3 = (x_3, y_3) = P_1 + P_2 \quad \text{and} \quad P_4 = (x_4, y_4) = P_1 - P_2.$$

(a) Derive formulas for the quantities $x_3 + x_4$ and x_3x_4 in terms of x_1 and x_2. (Note that you should be able to eliminate y_1 and y_2 from these formulas.)
(b) Prove that there is a constant κ, which depends only on a, b, c, such that for all rational points P_1 and P_2,

$$h(P_1 + P_2) + h(P_1 - P_2) \le 2h(P_1) + 2h(P_2) + \kappa.$$

Notice that this greatly strengthens the inequality given in Lemma 2.
(c) Prove that if κ is replaced by a suitably large negative number, then the opposite inequality in (b) will hold. In other words, prove that there is a constant κ, depending only on a, b, c, such that for all rational points P_1 and P_2,

$$-\kappa \le h(P_1 + P_2) + h(P_1 - P_2) - 2h(P_1) - 2h(P_2) \le \kappa.$$

(*Hint.* In (b), replace P_1 and P_2 by $P_1 + P_2$ and $P_1 - P_2$ and use the lower bound $h(2P) \ge 4h(P) - \kappa_0$ provided by Lemma 3.)
(d) Prove that for any integer m there is a constant κ_m, depending on a, b, c, m, such that for all rational points P,

$$-\kappa_m \le h(mP) - m^2h(P) \le \kappa_m.$$

3.3. * Let C be a rational cubic curve given by the usual Weierstrass equation.
(a) Prove that for any rational point $P \in C(\mathbf{Q})$, the limit

$$\hat{h}(P) = \lim_{n\to\infty} \frac{1}{4^n} h(2^n P)$$

exists. The quantity $\hat{h}(P)$ is called the *canonical height* of P. (*Hint.* Try to prove that the sequence is Cauchy.)
(b) Prove that there is a constant κ, depending only on a, b, c, such that for all rational points P we have

$$-\kappa \le \hat{h}(P) - h(P) \le \kappa.$$

(c) Prove that for any integer m and any rational point P,

$$\hat{h}(mP) = m^2\hat{h}(P).$$

(d) Prove that $\hat{h}(P) = 0$ if and only if P is a point of finite order.