1 Metrized line bundles

K a numberfield, M_K set of all places. M_K^∞ infinite and M_K^0 finite places.

Definition 1.1. An **Arakelov Surface** (A.S.) is a pair $(X, (\mu_\sigma)_{\sigma \in M_K^\infty})$ with X a regular integral surface, projective over $\mathcal{O}_K$ s.t. X_K is geometrically connected. And the μ_σ are $1-1$ forms on $X_\sigma(\mathbb{C})$ s.t. $\int_{X_\sigma(\mathbb{C})} \mu_\sigma = 1$. We call $g(X) := g(X_K)$ the genus of X.

Example 1. $X = \mathbb{P}^1_{\mathbb{Z}}$, z the standard coordinate on $\mathbb{C} = \mathbb{P}^1_C \setminus \{\infty\}$ and

$$\mu = \frac{i}{2\pi} \frac{\mathrm{d}\, z \wedge \mathrm{d}\, \overline{z}}{(1 + z\overline{z})^2}$$

Example 2. X the elliptic curve given by $y^2 + y = x^3 - x^2$, $X_(\mathbb{C}) \cong \mathbb{C}/\tau\mathbb{Z} + \mathbb{Z}$ with $\tau \simeq 0.5000 + i0.22988$ and

$$\mu = \frac{\Im(\tau)}{2i} \mathrm{d}\, z \wedge \mathrm{d}\, \overline{z}$$

If $g(X) \geq 1$ then $\dim \Omega^1(X_\sigma(\mathbb{C})) = g \geq 1$ and the space $\Omega^1(X_\sigma(\mathbb{C}))$ has an hermitian inner product $\langle \_, \_ \rangle_\sigma$ given by

$$\langle a, b \rangle = \frac{i}{2} \int_{X_\sigma(\mathbb{C})} a \wedge \overline{b}.$$

Definition 1.2. The canonical $1-1$ form on $X_\sigma(\mathbb{C})$ is the form

$$\mu^{can}_{X_\sigma} := \frac{i}{2g} \sum_{j=1}^{g} \omega_j \wedge \overline{\omega_j}$$

where $\omega_1, \ldots, \omega_g$ are an orthonormal basis for $\langle \_, \_ \rangle_\sigma$

Definition 1.3. Let $\mathcal{L}$ be a line bundle on an A.S. X. An **admissible metric** on $\mathcal{L}_\sigma$ is a smooth Hermitian metric $|\_|_\sigma$ on $\mathcal{L}_\sigma$ such that

$$\frac{1}{\pi i} \partial\overline{\partial} \log |s|_\sigma = \deg \mathcal{L}_\sigma \mu_\sigma$$

for some (hence any) locally generating section on $X_\sigma(\mathbb{C})$. An **admissible bundle** on X is a linebundle with an amdissible metric on all $\mathcal{L}_\sigma$.

Example 3. $\mathcal{L} = \mathcal{O}_X$ and $|\_|_\sigma = |\_|$ the standard hermitian innerproduct on $\mathbb{C}$. Since for holomorphic f one has

$$\frac{1}{\pi i} \partial\overline{\partial} \log |f(z)| = \frac{1}{\pi i} \partial\overline{\partial} \frac{1}{2} \log(f(z)\overline{f(z)}) = \frac{2\pi i}{\partial} \frac{1}{2f(z)\overline{f(z)}} f(z)\, \mathrm{d}\, \overline{f(z)} = 0$$

This example also shows the (hence any) part in the definition since

$$|f(z)s|_\sigma = \log(|f(z)| \cdot |s|_\sigma) = \log|f(z)| + \log|s|_\sigma$$

Remark: if $|\_|_\sigma$ is admissible and $c \in \mathbb{R}_{>0}$ then $c|\_|_\sigma$ is admissible.

Proposition 1.4. *Let X be an A.S. and $P \in X_\sigma(\mathbb{C})$. Then there is a unique smooth function $G(P,z) : X_\sigma(\mathbb{C}) \to \mathbb{R}_\geq 0$ s.t.*

- *P is the unique zero of G, and it is of order 1*
- *$\frac{1}{\pi i}\partial\overline{\partial}\log G(P,z) = \mu$ for $z \neq P$*
- *$\int_{X_\sigma(\mathbb{C})} \log G(P,z)\mu = 0$*

For this G one as $G(P,z) = G(z,P)$.

Definition 1.5. $g(P,Q) = \log G(P,Q)$ is called the Green's function (associated to $X_\sigma(\mathbb{C})$).

Corollary 1.6. *Let D be a divisor on X then $\mathcal{O}_X(D)$ can be made admissible in a canonical way.*

Proof. For each σ we write $D_\sigma = \sum_{x \in X_\sigma(\mathbb{C})} n_x x$ and define $|\_|_\sigma$ pointwise by

$$|1|_\sigma(y) = \prod G(x,y)^{n_x}$$

for $y \notin \operatorname{Supp} D_\sigma$. And extend continously. □

Definition 1.7. The compactified picard group of X denoted $\operatorname{Pic}_C(X)$ is the group of isomorphism classes admissible line bundles on X. (isomorphisms between admissible line bundles should respect the metrics)

Theorem 1.8. *The sequence:*

$$0 \to \mathbb{R}^{r_1+r_2}/\log(\mathcal{O}_K*) \to \operatorname{Pic}_C(X) \to \operatorname{Pic}(X) \to 0$$

is exact.

Definition 1.9. An compactified divisor on X is a formal sum

$$D = D_{fin} + \sum_{\sigma \in M_K^\infty} a_\sigma X_\sigma$$

With D_{fin} a cartier divisor and $a_\sigma \in \mathbb{R}$. The principal divisor asociated to $f \in K(X)^*$ is

$$\operatorname{div} f := \operatorname{div}_{fin}(f) + \sum_{\sigma \in M_K^\infty} a_\sigma(f) X_\sigma$$

where $a_\sigma(f) := -\int_{X_\sigma(\mathbb{C})} \log|f|_\sigma \mu_\sigma$. The compactified class group $\operatorname{Cl}_C X$ is the group of compactified divisors modulo the principal divisors.

If D is a compactified divisor then we can associate to it an admissible line bundle $\mathcal{O}_X(D)$. This is just $\mathcal{O}_X(D_{fin})$ with the canonical metrics scaled by $e^{-a_\sigma/\varepsilon_\sigma}$.

Theorem 1.10.

$$\mathcal{O}_X \colon \operatorname{Cl}_C X \to \operatorname{Pic}_C X$$

is an isomorphism.

2 intersection theory on arakelov surfaces

The irreducible divisors of X are the irreducible cartier divisors together with X_σ for $\sigma \in M_K^\infty$. An irreducible divisor is called vertical if it is an irreducible component of X_σ for some $\sigma \in M_K$ (not nessesarilly infinite!). All the other irreducible divisors are horizontal.

Definition 2.1. Let D_1 and D_2 be distinct irriducible diviors then the Arakelov intersection of D_1 and D_2 is defined by:

- if $D_1 = X_\sigma$ for some $\sigma \in M_K^\infty$ and D_2 vertical: $[D_1, D_2] = 0$
- if $D_1 = X_\sigma$ for some $\sigma \in M_K^\infty$ and D_2 horizontal: $[D_1, D_2] = \varepsilon_\sigma \deg_K D_{2,K}$
- if D_1 and D_2 finite then $[D_1, D_2] = [D_1, D_2]_{fin} + [D_1, D_2]_{inf}$ with:
  - $[D_1, D_2]_{fin} = \sum \log \#\mathcal{O}_X, p/(f_{1,p}, f_{2,p})$ where $f_{i,p}$ is a local equation for D_i at p
  - if either D_1 or D_2 is vertical then $[D_1, D_2]_{inf} = 0$
  - if D_1 and D_2 are horizontal then

$$[D_1, D_2]_{inf} = \sum_{\sigma \in M_K^\infty} \varepsilon_\sigma \sum_{j_1, j_2} - \log G_\sigma(P^\sigma_{1,j_1}, P^\sigma_{2,j_2})$$

Where the j_i run over all embeddings of $K(D_i) \to \mathbb{C}$ that extend σ and the P^σ_{i,j_i} are the points in $X_\sigma(\mathbb{C})$ induced by D_i under this embedding.

Proposition 2.2. *The arakelov intersection pairing extends uniquely to a bilinear map*

$$[\_, \_] \colon \operatorname{Cl}_C X \times \operatorname{Cl}_C X \to \mathbb{R}$$

.

Proposition 2.3. *Let L/K and $\sigma : X_L^{reg} \to X$ the canonical map then*

$$[\sigma * (D_1), \sigma^*(D_2)] = [L : K][D_1, D_2]$$

Proposition 2.4. *Let $S : \operatorname{Spec} O_K \to X$ be a section of $X \to \operatorname{Spec} O_K$ then S is a cartier divisor. Put the canonical metric on $\mathcal{O}_X(S)$ and D be an Arakelov Divisor. Then:*

$$[S, D] = \deg S^* O_X(D)$$