Suitability of Runge–Kutta methods

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Abstract: In this paper we introduce the concept of suitability, which means that the nonlinear equations to be solved in a Runge–Kutta method have a unique solution. We give several results about suitability, and prove that the Butcher I, II and III, and the Lobatto III A and IIIB methods are suitable.

Keywords: Initial value problems, implicit Runge–Kutta methods, suitability, transposed methods.

1. Introduction

We shall deal with the initial value problem

\[ \frac{dU}{dt} = f(t, U), \quad U(t_0) = u_0, \tag{1.1} \]

where \( U_0 \in \mathbb{R}^s \) and \( f(t, U) \) is a continuous function from \( \mathbb{R} \times \mathbb{R}^s \) to \( \mathbb{R}^s \). A numerical approximation of the solution \( U(t) \) of (1.1) can be computed by the implicit Runge–Kutta method \((c, b, A)\), which is usually denoted by the array

\[
\begin{array}{c|cccc}
  c_1 & a_{11} & a_{12} & \cdots & a_{1m} \\
  c_2 & a_{21} & a_{22} & \cdots & a_{2m} \\
   & \vdots & \vdots & \ddots & \vdots \\
  c_m & a_{m1} & a_{m2} & \cdots & a_{mm} \\
\end{array}
\begin{array}{c}
b_1 \\
b_2 \\
\vdots \\
b_m
\end{array}
\]

where \( c \) and \( b \) satisfy \( c_i = \sum_{j=1}^{m} a_{ij} \), \( 1 \leq i \leq m \) and \( \sum_{i=1}^{m} b_i = 1 \). We also assume that the method is nonconfluent, i.e. the abscissae \( c_i \), \( 1 \leq i \leq m \), are all different. The use of an implicit Runge–Kutta method to obtain a numerical solution of (1.1) requires the solution of a system of algebraic equations

\[ y_i = u_{n-1} + h \sum_{j=1}^{m} a_{ij} f(\tau_j, y_j), \quad 1 \leq i \leq m, \tag{1.2} \]

where \( \tau_i = t_{n-1} + c_i h \), \( y_i \in \mathbb{R}^s \), \( 1 \leq i \leq m \), and \( u_{n-1} \in \mathbb{R}^s \) is an approximation to the solution \( U(t_{n-1}) \). When system (1.2) allows a solution, and this solution has been obtained, an approximation to \( U(t_n) \), \( t_n = t_{n-1} + h \), can be computed from the formula

\[ u_n = u_{n-1} + h \sum_{i=1}^{m} b_i f(\tau_i, y_i). \tag{1.3} \]
Recently, conditions under which system (1.2) has a unique solution have been considered by various authors, e.g. Frank, Schneid and Ueberhuber [5], Hunds dorfer and Spijker [6], Dekker and Verwer [4, Chapter 5], Di Lena and Peluso [7]. The following result, formulated by Crouzeix, Hundsdorfer and Spijker [2] will be important in our analysis.

**Theorem 1.1.** If \( f(t, U) \) satisfies

\[
(f(t, \xi_1) - f(t, \xi_2), \xi_1 - \xi_2) \leq 0, \quad \forall t, \forall \xi_1, \xi_2 \in \mathbb{R}^s,
\]

and if there exists a positive definite diagonal matrix \( D \) such that \( DA + A^TD \) is positive definite, then system (1.2) has a unique solution.

Frank, Schneid and Ueberhuber [5] already proved that several important classes of implicit Runge–Kutta methods, viz. the Gauss–Legendre methods, the Radau IA and IIA-methods and the Lobatto IIIIC-method with \( m = 2 \) satisfy the condition of Theorem 1.1.

In this paper we shall give results for methods which do not satisfy this condition. First, we introduce the concept of *suitability* which is equivalent to system (1.2) having a unique solution if \( f(t, U) \) satisfies (1.4). Then, we prove that a lower triangular block matrix is suitable if and only if all diagonal blocks are suitable. We also prove that a matrix \( A \) is suitable if the method \((c, b, A)\) satisfies the simplifying conditions \( C(m) \) and \( B(2m - 2) \) (see e.g. [4, Chapter 3]) and \( c_1 = 0 \). Moreover, we show that these results are also applicable to the methods which are obtained by the process of transposition (cf. [8]). As a consequence system (1.2) has a unique solution for the Lobatto IIIA, IIIB and Butcher, I, II-methods (see [1]).

Finally, we prove that for the Butcher III-method [1] system (1.2) also has a unique solution.

2. Suitability

In this section we state the concept of suitability and we give the main theorem of the paper. For convenience we will denote \( hf(\tau_j, y_j) \) by \( f_j(y_j) \), \( 1 \leq j \leq m \), and we assume that \( f(t, U) \) satisfies condition (1.4), so for \( 1 \leq j \leq m \) there holds

\[
(f_j(\xi_1) - f_j(\xi_2), \xi_1 - \xi_2) \leq 0, \quad \forall \xi_1, \xi_2 \in \mathbb{R}^s.
\]

**Definition 2.1.** A matrix \( A \) is called *suitable* if the system

\[
y_i = \sum_{j=1}^{m} a_{ij} f_j(y_j), \quad 1 \leq i \leq m,
\]

has a unique solution \( y = (y_1, y_2, \ldots, y_m) \in (\mathbb{R}^s)^m \), whenever the functions \( f_j: \mathbb{R}^s \rightarrow \mathbb{R}^s \) are continuous and satisfy (2.1), \( j = 1, \ldots, m \).

**Remark 2.2.** Clearly suitability of \( A \) is equivalent to system (1.2) having a unique solution.

**Theorem 2.3.** Suppose that after a permutation of variables a matrix \( A \) is of the form

\[
A = \begin{pmatrix}
A_1 & 0 & \cdots & 0 \\
A_2 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
\cdots & \cdots & \cdots & A_k
\end{pmatrix}
\]
where $A_i$ is an $r_i \times r_i$ submatrix, $1 \leq i \leq k$, and $\sum_{i=1}^{k} r_i = m$. Then $A$ is suitable if and only if all $A_i$, $1 \leq i \leq k$, are suitable.

**Proof.** Suppose $A$ is suitable, then obviously $A_1$ is suitable. Now, choose $f_j(x) = 0$, $j = 1, \ldots, r_1$, and let $f_j(x)$, $r_1 + 1 \leq j \leq r_1 + r_2$, satisfy (2.1). Then the equations

$$y_i = \sum_{j=1}^{r_1} a_{ij} f_j(y_j) + \sum_{j=r_1+1}^{r_1+r_2} a_{ij} f_j(y_j) - \sum_{j=r_1+1}^{r_1+r_2} a_{ij} f_j(y_j), \quad r_1 + 1 \leq i \leq r_1 + r_2,$$

have a unique solution, as $A$ is suitable. Thus $A_2$ is suitable, and by induction it is proved that $A_i$ is suitable, $i = 1, 2, \ldots, k$. Next suppose all $A_i$, $1 \leq i \leq k$, are suitable. We shall prove that system (2.2) has a unique solution for all $f_1, f_2, \ldots, f_m$ satisfying (2.1). To this end we partition (2.2) as

$$y_i = \sum_{j=1}^{r_1} a_{ij} f_j(y_j), \quad i \leq i \leq r_1, \quad (2.3)$$

$$y_i = \sum_{j=r_1+1}^{r_1+r_2} a_{ij} f_j(y_j), \quad r_1 + 1 \leq i \leq r_1 + r_2, \quad (2.4)$$

$$y_i = \sum_{j=r_1+1}^{r_1+r_2} a_{ij} f_j(y_j), \quad p_{k-1} + 1 \leq i \leq p_k. \quad (2.5)$$

Here,

$$p_k = \sum_{i=1}^{k} r_i, \quad 1 \leq k \leq m.$$

Clearly (2.3) has a unique solution because $A_1$ is suitable. Now we define

$$\tilde{y}_i = y_i - e_i, \quad r_1 + 1 \leq i \leq r_1 + r_2,$$

$$g_i(\eta) = f_i(\eta + e_i), \quad r_1 + 1 \leq i \leq r_1 + r_2,$$

$$e_i = \sum_{j=1}^{r_1} a_{ij} f_j(y_j), \quad r_1 + 1 \leq i \leq r_1 + r_2,$$

where the $y_j$, $1 \leq j \leq r_1$, are already determined by (2.3). It is easily seen that (2.4) reduces to

$$\tilde{y}_i = \sum_{j=r_1+1}^{r_1+r_2} a_{ij} g_j(\tilde{y}_j), \quad r_1 + 1 \leq i \leq r_1 + r_2,$$

which has a unique solution as $A_2$ is suitable and the functions $g_j(\eta)$, $r_1 + 1 \leq j \leq r_1 + r_2$, satisfy (2.1). Therefore the equations (2.4) also have a unique solution, $y_i = \tilde{y}_i + e_i$, $r_1 + 1 \leq i \leq r_1 + r_2$. Continuing this procedure, we see that $A$ is suitable. \qed

**Remark 2.4.** Definition 2.1 and the corresponding Theorem 2.3 remain valid if the matrix $A = (a_{ij})$ is confluent (i.e. if $\sum_{k=1}^{m} a_{ik} = \sum_{k=1}^{m} a_{jk}$ for some $i \neq j$).
3. The Lobatto IIIA and Butcher I-methods

In this section we consider methods which satisfy the simplifying conditions $C(m)$ and $B(2m-2)$, i.e. (cf. e.g. [4])

\[
\sum_{j=1}^{m} a_{ij}c_{j}^{k-1} - \frac{1}{k} c_{i}^{k}, \quad 1 \leq k \leq m, \quad 1 \leq i \leq m, \quad (3.1)
\]

\[
\sum_{i=1}^{m} b_{i}c_{i}^{k-1} = \frac{1}{k}, \quad 1 \leq k \leq 2m-2. \quad (3.2)
\]

We also assume that the first abscissa $c_1 = 0$. From the equations (3.1) with $i = 1$ it is then obvious that the first row of $A$ contains zeros only. We shall prove the suitability of these matrices $A$ in a way analogous to Dekker [3].

**Theorem 3.1.** If a method $(c, b, A)$ satisfies the simplifying conditions $C(m)$ and $B(2m-2)$, $c_1 = 0$, and the other abscissae satisfy $0 < c_i \leq 1$, $2 \leq i \leq m$, then $A$ is suitable.

**Proof.** The method $(c, b, A)$ is given by

\[
\begin{bmatrix}
  c_1 & 0 & 0 & \cdots & 0 \\
  c_2 & a_{21} & a_{22} & \cdots & a_{2m} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  c_m & a_{m1} & a_{m2} & \cdots & a_{mm} \\
  b_1 & b_2 & \cdots & b_m
\end{bmatrix}
\]

We define the $(m - 1)$-stage submethod generated by

\[
\begin{bmatrix}
  c_1 & 0 & 0 & \cdots & 0 \\
  c_2 & a_{22} & a_{23} & \cdots & a_{2m} \\
  c_3 & a_{32} & a_{33} & \cdots & a_{3m} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  c_m & a_{m2} & a_{m3} & \cdots & a_{mm} \\
  b_2 & b_3 & \cdots & b_m
\end{bmatrix}
\]

Let

\[
\begin{bmatrix}
  c_2 & c_2^2 & \cdots & c_2^{m-1} \\
  c_3 & c_3^2 & \cdots & c_3^{m-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_m & c_m^2 & \cdots & c_m^{m-1}
\end{bmatrix}, \quad \begin{bmatrix}
  \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{m} \\
  \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{m+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  \frac{1}{m} & \frac{1}{m+1} & \cdots & \frac{1}{2m-2}
\end{bmatrix}
\]

$\tilde{B} = \text{diag}(b_2, b_3, \ldots, b_m)$, $\tilde{C} = \text{diag}(c_2, c_3, \ldots, c_m)$, $\tilde{S} = \text{diag}(\frac{1}{2}, \frac{1}{3}, \ldots, 1/m)$ and let $\tilde{v}_j$ stand for the $j$th unit vector ($1 \leq j \leq m - 1$). The simplifying conditions $C(m)$ and $B(2m-2)$ then become

\[
\tilde{A}\tilde{V} = \tilde{C}\tilde{V}\tilde{S}, \quad (3.3)
\]

\[
\tilde{V}^T\tilde{B}\tilde{C}^{-1}\tilde{V} = \tilde{I} \quad (3.4)
\]
Let $\tilde{D} = B\tilde{C}^{-2}$. Then $\tilde{D}$ is a positive diagonal matrix. From (3.3) and (3.4) we have for $1 \leq i, j \leq m - 1$

$$
\tilde{e}_i^T \tilde{V}^T \tilde{D} \tilde{A} \tilde{V} \tilde{e}_j = \frac{1}{j + 1} \frac{1}{i + j}, \quad \tilde{e}_i^T \tilde{V}^T \tilde{A}^T \tilde{D} \tilde{V} \tilde{e}_j = \frac{1}{i + 1} \frac{1}{i + j},
$$

$$
\tilde{e}_i^T \tilde{V}^T \tilde{A}^T B \tilde{C}^{-3} \tilde{A} \tilde{V} \tilde{e}_j = \frac{1}{i + 1} \frac{1}{j + 1} \frac{1}{i + j}.
$$

Hence

$$
\tilde{V}^T (\tilde{A}^T \tilde{D} + \tilde{D} \tilde{A} - 2 \tilde{A}^T B \tilde{C}^{-3} \tilde{A}) \tilde{V} = \tilde{S} \tilde{e} \tilde{e}^T \tilde{S} = \tilde{V}^T b b^T \tilde{V},
$$

where $\tilde{e} = (1, 1, \ldots, 1)^T$. Obviously, $b b^T$ is symmetric and positive semi-definite and $B\tilde{C}^{-3}$ is positive definite. Therefore,

$$
\tilde{D} \tilde{A} + \tilde{A}^T \tilde{D} = b b^T + 2 \tilde{A}^T B \tilde{C}^{-3} \tilde{A} \quad (3.5)
$$

is positive definite. According to Theorem 1.1 $\tilde{A}$ is suitable. Let $A_1 = (0)$, $A_2 = \tilde{A}$, then it follows from Theorem 2.3 that $A$ is suitable. □

**Corollary 3.2.** For the Lobatto IIIA-methods system (1.2) has a unique solution.

**Proof.** The Lobatto IIIA-methods satisfy the simplifying conditions $C(m)$ and $B(2m - 2)$ and the abscissae are those of the Lobatto quadrature formulas, so $c_1 = 0$ and $0 < c_i \leq 1$, $2 \leq i \leq m$ (see e.g. [4, Chapter 3]). □

**Corollary 3.3.** For the Butcher I-methods system (1.2) has a unique solution.

**Proof.** The Butcher I-methods satisfy the conditions of Theorem 3.1 (see [1]). □

### 4. The Lobatto IIIB and Butcher II-methods

Scherer and Törrke [8] have shown that some Runge–Kutta methods are related to each other. They established interesting interrelations between various methods by means of two mappings, viz. reflection and transposition. The latter one is a helpful tool in our analysis, so we shall give its definition. Let $B = \text{diag}(b_1, b_2, \ldots, b_m)$, $C = \text{diag}(c_1, c_2, \ldots, c_m)$, $e = (1, 1, \ldots, 1)^T$.

**Definition 4.1.** Let $(c, b, A)$ be a Runge–Kutta method and $c_r = e - c$, $A_r = B^{-1} A^T B$, $b_r = b$, then the Runge–Kutta method $(c_r, b_r, A_r)$ is called the transposed method.

It is seen that the abscissae of the transposed method are ordered in reverse by this definition. We will assume that they are reordered by a suitable permutation, so that

$$
c_r = P(e - c), \quad A_r = P B^{-1} A^T B P, \quad b_r = Pb,
$$
where \( P \) is the permutation matrix, and we define
\[
B_r = PBP, \quad C_r = P(I - C)P, \quad I = \text{diag}(1, 1, \ldots, 1).
\]

We shall now formulate results which are relevant for the suitability of a method. In the sequel we assume that \( D \) is a positive diagonal matrix.

**Theorem 4.2.** Suppose that \( B \) is a positive diagonal matrix. Then the following implications hold for \( D_r = PD^{-1}B^2P \):

(a) \( D_rA_r + A_r^TD_r \) is positive (semi) definite iff \( DA + A^TD \) is positive (semi) definite.

(b) If \( A \) is nonsingular, then \( D_rA_r^{-1} + A_r^{-T}D_r \) is positive (semi) definite iff \( DA^{-1} + A^{-T}D \) is positive (semi) definite.

**Proof.** The proof follows directly from the definitions of \( D_r \) and \( B_r \), and the fact that \( B \) and \( D \) are positive:

\[
D_rA_r + A_r^TD_r = PD^{-1}B^2PPB^{-1}ATBP + PBAB^{-1}PPB^2D^{-1}P
\]

\[
= PD^{-1}BA^TB + PBAB^{-1}P = PD^{-1}B(A^TD + DA)BD^{-1}P,
\]

and

\[
D_rA_r^{-1} + A_r^{-T}D_r = PD^{-1}B(DA^{-1} + A^{-T}D)BD^{-1}P.
\]

From [8], we have the following:

**Lemma 4.3.** The Butcher I and Lobatto IIIA-methods are changed into the Butcher II and Lobatto IIIB-methods by transposition respectively.

Let us denote the Butcher I, II and Lobatto IIIA, IIIB-methods by \((c_1, b_1, A_1), (c_II, b_II, A_II), (c_{IIIA}, b_{IIIA}, A_{IIIA}), (c_{IIIB}, b_{IIIB}, A_{IIIB})\) respectively. Then the following relations hold

\[
(c_{II}, b_{II}, A_{II}) = (P(e - c_1), Pb_1, PB_1^{-1}A_1^TB_1P),
\]

\[
(c_{IIIB}, b_{IIIB}, A_{IIIB}) = (P(e - c_{IIIA}), Pb_{IIIA}, PB_{IIIA}^{-1}A_{IIIA}^TB_{IIIA}P),
\]

where

\[
P = \begin{pmatrix} 0 & \cdots & 1 \\ 1 & \cdots & 0 \end{pmatrix}.
\]

From these relations it is easily seen that the Butcher II (and similarly the Lobatto IIIB) methods have the form

\[
\begin{array}{ccccccccc}
 c_1 & a_{11} & a_{12} & \cdots & a_{1m-1} & 0 \\
 c_2 & a_{21} & a_{22} & \cdots & a_{2m-1} & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 c_m & a_{m1} & a_{m2} & \cdots & a_{mm-1} & 0 \\
 b_1 & b_2 & \cdots & b_{m-1} & b_m
\end{array}
\]

with \( c_m = 1 \).
Theorem 4.4. For the Butcher II-methods system (1.2) has a unique solution.

Proof. Let \( \tilde{D}_I = \tilde{B}_I \tilde{C}_I^{-2} \) as in the proof of Theorem 3.1. Then \( \tilde{D}_I, \tilde{A}_I + \tilde{A}_I^T \tilde{D}_I \) is positive definite by (3.5). Let \( \tilde{A}_{II}, \tilde{B}_{II}, \tilde{C}_{II}, \tilde{I} \) denote the matrices which are obtained by eliminating the last row and column from \( A_{II}, B_{II}, C_{II} \) and \( I \), respectively. Using Lemma 4.3 we obtain

\[
\tilde{A}_{II} = \tilde{P} \tilde{B}_{II}^{-1} \tilde{A}_{II}^T \tilde{P}, \quad \tilde{B}_{II} = \tilde{P} \tilde{B}_{II} \tilde{P}, \quad \tilde{C}_{II} = \tilde{P}(\tilde{I} - \tilde{C}_I) \tilde{P},
\]

where \( \tilde{P} \) is the matrix which originates from \( P \) by deleting the first row and last column. Now let \( \tilde{D}_{II} = \tilde{P} \tilde{B}_{II}^2 \tilde{D}_I^{-1} \tilde{P} = \tilde{B}_{II} (\tilde{I} - \tilde{C}_I)^2 \). According to Theorem 4.2 \( \tilde{D}_{II} \tilde{A}_{II} + \tilde{A}_{II}^T \tilde{D}_{II} \) is positive definite, so \( \tilde{A}_{II} \) is suitable by Theorem 1.1. It follows from Theorem 2.3 that \( A_{II} \) is suitable. \( \square \)

Theorem 4.5. For the Lobatto IIIIB-methods system (1.2) has a unique solution.

Proof. The structure of the Lobatto IIIIB-methods is the same as in (4.1). Let \( \tilde{A}_{III}, \tilde{B}_{III}, \tilde{C}_{III} \) and \( \tilde{I} \) be defined in a similar way as in the previous theorem, and let \( \tilde{D}_{III} = \tilde{B}_{III} (\tilde{I} - \tilde{C}_I)^2 \). From the proof of Theorem 3.1 we know that \( \tilde{D}_{III} \tilde{A}_{III} + \tilde{A}_{III}^T \tilde{D}_{III} \) is positive definite. Hence, according to Lemma 4.3 and Theorem 4.2, \( \tilde{D}_{III} \tilde{A}_{III} + \tilde{A}_{III}^T \tilde{D}_{III} \) is positive definite, and \( \tilde{A}_{III} \) is suitable by Theorem 1.1. It is seen from Theorem 2.3 that also \( A_{III} \) is suitable. \( \square \)

Remark 4.6. The positive definiteness of \( \tilde{D}_{III} \tilde{A}_{III} + \tilde{A}_{III}^T \tilde{D}_{III} \) could also be concluded from [3, Example 5.6]. \( \square \)

5. The Butcher III-methods

The Butcher III-methods, introduced by Butcher [1], satisfy the simplifying conditions \( C(m - 1) \) and \( B(2m - 2) \), i.e.

\[
\sum_{j=1}^{m} a_{ij} c_{j}^{k-1} = \frac{1}{k} c_{i}^{k}, \quad 1 \leq k \leq m - 1, \quad 1 \leq i \leq m, \quad (5.1)
\]

\[
\sum_{j=1}^{m} b_{ij} c_{j}^{k-1} = \frac{1}{k}, \quad 1 \leq k \leq 2m - 2, \quad (5.2)
\]

and they are characterized by the array

\[
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
c_2 & a_{21} & a_{22} & \cdots & a_{2m-1} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
c_m & a_{m1} & a_{m2} & \cdots & a_{mm-1} & 0 \\
b_1 & b_2 & \cdots & b_{m-1} & b_m
\end{pmatrix}
\]

with \( c_m = 1 \).

Theorem 5.1. For the Butcher III-methods system (1.2) has a unique solution.
Proof. We define the \((m - 2)\) stage submethods by

\[
\begin{pmatrix}
\hat{b}_1 & \hat{a}_{12} & \hat{a}_{13} & \cdots & \hat{a}_{1m-1} \\
\hat{b}_2 & \hat{a}_{22} & \hat{a}_{23} & \cdots & \hat{a}_{2m-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\hat{b}_{m-1} & \hat{a}_{m-2} & \hat{a}_{m-3} & \cdots & \hat{a}_{m-1m-1}
\end{pmatrix}
\]

Let \(\hat{B} = \text{diag}(b_2, b_3, \ldots, b_{m-1})\), \(\hat{C} = \text{diag}(c_2, c_3, \ldots, c_{m-1})\), \(\hat{e}_j\) stand for the \(j\)th unit vector, \(1 \leq j \leq m - 2\), and

\[
\hat{\nu} = \begin{pmatrix}
c_2 & c_2^2 & \cdots & c_2^{m-2} \\
c_3 & c_3^2 & \cdots & c_3^{m-2} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m-1} & c_{m-1}^2 & \cdots & c_{m-1}^{m-2}
\end{pmatrix}
\]

Then we have from (5.1) and (5.2),

\[
\hat{e}_i^T \hat{V}^T \hat{B} \hat{A} \hat{V} \hat{e}_j = \frac{1}{j+1} \left( \frac{1}{i+j+2} - b_m \right), \quad 1 \leq i, \ j \leq m - 2,
\]

\[
\hat{e}_i^T \hat{V}^T \hat{B} \hat{C}^{-1} \hat{A} \hat{V} \hat{e}_j = \frac{1}{j+1} \left( \frac{1}{i+j+1} - b_m \right), \quad 1 \leq i, \ j \leq m - 2,
\]

\[
\hat{e}_i^T \hat{V}^T \hat{B} \hat{C}^{-2} \hat{A} \hat{V} \hat{e}_j = \frac{1}{j+1} \left( \frac{1}{i+j} - b_m \right), \quad 1 \leq i, \ j \leq m - 2.
\]

Let \(\hat{D} = \hat{B}(\hat{C}^{-1} - I)^2\), then again \(\hat{D}\) is a positive diagonal matrix, because the abscissae \(c_i, 2 \leq i \leq m - 1\), all lie in the interval \((0, 1)\), and we have

\[
\hat{e}_i^T \hat{V}^T(\hat{D} \hat{A} + \hat{A}^T \hat{D}) \hat{V} \hat{e}_j = \frac{2}{(i+1)(j+1)(i+j)(i+j+1)}, \quad 1 \leq i, \ j \leq m - 2.
\]

A straightforward calculation also shows that

\[
\hat{e}_i^T \hat{V}^T \hat{A}^T \hat{B} \hat{C}^{-2} \hat{A} \hat{V} \hat{e}_j = \frac{1}{i+1} \frac{1}{j+1} \left( \frac{1}{i+j+1} - b_m \right), \quad 1 \leq i, \ j \leq m - 2,
\]

and

\[
\hat{e}_i^T \hat{V}^T \hat{A}^T \hat{B} \hat{C}^{-3} \hat{A} \hat{V} \hat{e}_j = \frac{1}{i+1} \frac{1}{j+1} \left( \frac{1}{i+j} - b_m \right), \quad 1 \leq i, \ j \leq m - 2.
\]

A combination of these equalities yields

\[
\hat{D} \hat{A} + \hat{A}^T \hat{D} = 2 \hat{A}^T \hat{B} (\hat{C}^{-3} - \hat{C}^{-2}) \hat{A}.
\]

Consequently, \(\hat{D} \hat{A} + \hat{A}^T \hat{D}\) is positive definite, and there follows from Theorem 1.1 that \(\hat{A}\) is suitable. Let \(A_1 = (0), A_2 = \hat{A}\) and \(A_3 = (0)\), then it is obvious from Theorem 2.3 that \(\hat{A}\) is suitable. \(\square\)
Remark 5.2. It is also possible to choose $\tilde{D} = \tilde{B}(\tilde{C}^{-1} - \tilde{I})$ in the proof given above. It turns out that for this matrix $\tilde{D}$, $\tilde{D}A + A^T\tilde{D}$ is also positive definite.

Remark 5.3. It is interesting to mention that for the methods considered, the Butcher I, II, III and the Lobatto IIIA, IIIB-methods, system (1.2) has a unique solution, even though they are not algebraically stable, and the Butcher I, II, III-methods are not $A$-stable.

References