Non-existence of a stochastic fluid limit for a cycling random walk

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Abstract

This papers studies a transient and irreducible face-homogeneous random walk on the lattice $\mathbf{Z}^2$ with cyclic fluid or Euler paths. The simplest practical models with cyclic paths such cycling seem to be priority queueing models (cf. [6]).

For this model, the time-space scaled process converges in distribution to the Euler path for initial points outside the origin. This is a standard result. In contrast, we will prove, that in general there is no convergence to the Euler path of the time scaled process for a *fixed* initial point. In particular, if there is convergence at all, then the limit distribution should be invariant with respect to the dynamical system associated with the Euler paths.

This fact implies that scattering cannot be properly defined when scaling linearly by time. The proofs show that instead one should scale back along the Euler paths. This work is the starting point of further research of scattering phenomena and connections with decomposing the process into atomic and non-atomic subsets.

Contents

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1 Introduction

The present research has been a long, tough and, by times, tedious one. Our utter conviction that the time-space scaled process associated with our model, would converge in distribution to a random variable on the set of fluid or Euler paths, made us blind to see the truth leaping to our eyes. Remains to confess that more perspicacious spirits than ours have erred likewise.

In the framework of a dynamical systems or fluid approximations approach to queueing networks we consider an example of continuous scattering. In other words, an example with an infinite number of Euler (or fluid) trajectories starting at the same point, 0 say. Our example is the simplest possible, but nevertheless non-trivial. This example is similar to the case of the random walk in $\mathbf{Z}_+^3$, (see [2]) with escape to infinity by cyclically rotating over all three faces of the octant. The well-known and simplest example in the queueing setting exhibiting this phenomenon, is a priority queueing network (cf. [6]). For terminological reasons we consider the two dimensional analogs of these cases.

Our model is a random walk $\{\xi_n\}_{n=0,1,\ldots}$ in $\mathbf{Z}^2$, with homogeneous jumps inside each of the four quarter planes (see Figure 1, where the drifts are shown). By virtue of the law of large numbers, we know that for any given state $x \in \mathbf{R}^2 \setminus \{0\}$ and sufficiently small time intervals of length t, the following limit (in distribution) of the space-time scaled process exists

$$\frac{\xi_{[tN]}([xN])}{N} \xrightarrow{\mathfrak{D}} u(x;t), \quad N \to \infty, \tag{1.1}$$

whenever the random walk starts at $\xi_0 = [xN]$, provided that x is a point in the interior of any of the quadrants. Here $u(x;t)$, $t > 0$, is the deterministic dynamical system defined by the drifts and the initial condition $u(0) = x$.

The ergodicity and transience conditions for this random walk in terms of the dynamical system are quite obvious: if the trajectories of the dynamical system go to infinity then the random walk is transient and if they converge to 0, then the random walk is ergodic. Such trajectories are called *Euler or fluid paths*.

In case the Euler paths converge to 0, it is clear that $\xi_N(x)/N$ converges to 0 in distribution and even a.s., for any *fixed* initial point x. Indeed, using techniques from [9] and [8] one can show that the walk is exponentially ergodic. Thus, once having reached a bounded set, the walk can only move outside it with exponentially small probability.

This paper studies the case of *diverging Euler paths*, where the situation is quite different. The random walk ξ_n starting at a given point x, turns out to spread out over all Euler trajectories. This is the reason why the time scaled process cannot converge in distribution. Indeed, if the process at macro-time is close to a certain Euler path, then scaling it linearly in time yields a cyclically rotating point along a closed curve that is homeomorphic to a circle. In this light, it is not surprising that for the limit distribution to exist, it has to be invariant with respect to the dynamical system associated with Euler paths.

Note that, as a consequence, in general there cannot be any convergence in distribution of the time scaled process $\{\xi_{[tN]}/N \,|\, 0 \le t \le T\}$ over macro-time intervals.

The main tool used here is an extension of Kolmogorov's inequality for i.i.d. random variables. This is a technical derivation, complicated by dispersion occuring whenever an axis is passed. In the way, we show that sets of cones defined by Euler paths, are so-called *sojourn sets.*

This paper is the starting point for further analysis of the space decomposition into closed sets (cf. Chung [1]) and scattering phenomena. The underlying idea for face-homogeneous random walks is the following. Decomposition into atomic sets should be equivalent to the occurrence of a discrete set of Euler paths, over which the process scatters in the long run. The scattering probabilities are then equal to the absorption probabilities of the atomic sets of the decomposition. More details can be found in a subsequent paper [5]. Ours will turn out to be an example of the state space being a single non-atomic set and there is continuous scattering over the (continuous) set of all Euler trajectories. In a subsequent paper ([7]) we will show that each trajectory turns out to be chosen randomly by a probability measure μ_q on the set of all trajectories starting at a given point q. Identifying each point of a circle around the origin with a trajectory, one can then say that the Poisson boundary is isomorphic to a circle.

2 The model

We consider an aperiodic irreducible Markov chain $\{\xi_n\}_{n=0,1,\dots}$ on the state space $\mathbf{Z}^2$ in discrete time. If the random walk starts at point $x \in \mathbf{Z}^2$, then this position at time n will be denoted by $\xi_n(x)$. We will assume the random walk to be *face-homogeneous*, i.e. the transition probabilities from two states, the components of which have the same sign (+,-, or 0) are equal. For any $u \in \mathbf{R}$, let $\mathrm{sgn}(u) = +, -, 0$ whenever $u > 0$, $u < 0$ or $u = 0$ respectively. Then we can denote the nine faces by Q^{ab}, $a, b \in \{+, -, 0\}$, where

$$Q^{ab} = \{x \in \mathbf{R}^2 \mid \mathrm{sgn}(x_1) = a, \mathrm{sgn}(x_2) = b\}.$$

Further, denote the mean drift in a point $x \in Q^{ab}$ by

$$m(x) \equiv m^{ab} = (m_1^{ab}, m_2^{ab}) = \sum_y (y - x)\mathsf{P}\{\xi_{n+1} = y \mid \xi_n = x\}.$$

Since the transitions on the half axes have a minor influence on the large time behaviour of the process, we will assume that the drifts from points on axes coincide with the counterclockwise quadrant following the half-axis in question.

Summarising, we assume that the one step transition probabilities $p_{x,x+k} = \mathsf{P}\{\xi_{n+1} = x + k \mid \xi_n = x\}$ satisfy

(i) homogeneity condition $p_{x,x+k} = p_k^{ab}$, for $x \in Q^{ab} \cap \mathbf{Z}^2$;

(ii) boundedness of jumps $p_k^{ab} = 0$ unless $-1 \le k_1, k_2 \le 1$.

(iii) drift condition $m^{+0} = m^{++}$, $m^{0+} = m^{-+}$, $m^{-0} = m^{--}$, $m^{0-} = m^{+-}$.

We also assume that these drifts are of the form shown in Figure 1 below.

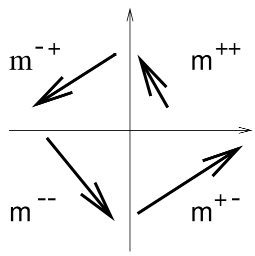

Figure 1

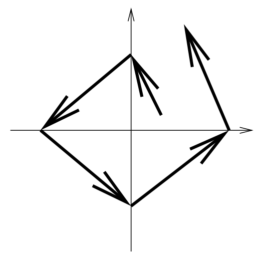

Figure 2

The figure evidences that the system to be studied, is transient: there is no smooth function to be constructed with all drifts pointing *inwards* the corresponding level sets. In particular, the transience and ergodicity regions in the parameter space are a.s. determined by the value of the constant

$$C = \frac{m_2^{++} m_1^{-+} m_2^{--} m_1^{+-}}{m_1^{++} m_2^{-+} m_1^{--} m_2^{+-}}. \tag{2.1}$$

Lemma 2.1 *The random walk is transient if $C > 1$. It is ergodic if $C < 1$.*

We will show this result in the next section. Our basic assumption can now be reformulated as follows.

Assumption 2.1 It holds that $C > 1$, and so the random walk is transient.

Next we define the continuous time dynamical system $u(x;t)$ associated with this Markov chain, mentioned in the introduction. It is a continuous mapping $u : \mathbf{R}^2 \setminus \{0\} \times \mathbf{R} \to \mathbf{R}^2$, defined by the mean drift vector field and initial condition $u(x; 0) = x$. More precisely, $u(x;t)$ is a *continuous* piecewise linear function of t with

$$\begin{cases} u(x;0) = x & x \neq 0, \\ \dfrac{d}{dt}u(x;t) = m(u(x;t)), & t \in \mathbf{R}, \text{ if } u(x;t) \in Q^{ab}, \ a, b = +, -. \end{cases} \tag{2.2}$$

Note that the dynamical system cycling off to infinity coincides with $C > 1$. Thus transience in this case is evident. Formally: $\|u(x;t)\|_2 \to \infty$, as $t \to +\infty$, $x \neq 0$ (see Figure 2). Similarly, $\|u(x;t)\|_2 \to 0$, as $t \to -\infty$.

An important property of the dynamical system is *homogeneity*:

$$u(\alpha x; \alpha t) = \alpha u(x;t), \quad x \neq 0, \alpha > 0. \tag{2.3}$$

At this point we will introduce some other notions that will be frequently used.

Euler or *fluid paths* are defined to be the trajectories of u. The Euler path Γ_x starting at $x \neq 0$ equals

$$\Gamma_x = \{u(x;t),\, t \in \mathbf{R}\}.$$

By the cycle time we understand the mapping $\tau : \mathbf{R}^2 \setminus \{0\} \to \mathbf{R}_+$ given by

$$\tau(x) = \min\{t > 0 \,|\, u(x;t) = \alpha x \text{ for some } \alpha > 0\}.$$

It is the time the dynamical system starting at x needs to pass precisely one cycle. By homogeneity of u, the cycle time is homogeneous as well:

$$\tau(\alpha x) = \alpha \tau(x), \quad \alpha > 0. \tag{2.4}$$

Finally, we introduce the 'Euler distance to 0'.

Definition 2.1 For $x \neq 0$ the Euler distance $r(x)$ to the point 0 is defined by

$$r(x) = \inf\{t > 0 \,|\, u(x; -t) = 0\}.$$

The set $\mathcal{I}(t) = \{x \in \mathbf{R}^2 \, : \, t = r(x)\}$, $t > 0$, will be called the *isochrone* at (Euler) distance t.

The mapping $x \to \Gamma_x$ clearly a bijection between $\mathcal{I}$ and the set of Euler paths. Note that $\mathbf{R}^2 \setminus \{0\} = \cup_x \Gamma_x$ and two Euler paths intersect only at the point 0. The following lemma establishes a simple relation between cycle time and Euler distance.

Lemma 2.2 *We have that*

$$r(x) = \frac{\tau(x)}{C-1},$$

with C defined in (2.1). Hence, points at the isochrone $\mathcal{I}(s)$ at distance s have cycle time $s(C-1)$.

Proof. Let us calculate the time $r(x)$ to reach zero from the point x. First note that $u(x;t) = y$ implies $u(y;-t) = x$, for any $x \in \mathbf{R}^2 \setminus \{0\}$ and $t > 0$.

Then from definition of $r(\cdot)$ and $\tau(\cdot)$ it follows directly that

$$r(x) = r(xC^{-1}) + \tau(xC^{-1}), \tag{2.5}$$

since the cycle time is measured forwards in time. As $r(y) \to 0$ as $y \to 0$, we get

$$r(x) = \sum_{k=1}^{\infty} \tau(xC^{-k}) = \tau(x) \sum_{k=1}^{\infty} C^{-k} = \tau(x) C^{-1} \frac{1}{1 - C^{-1}} = \frac{\tau(x)}{C-1}.$$

□

The following general result holds. It is the starting point for our analysis.

Proposition 2.1 *The Euler limit in distribution*

$$\lim_{N \to \infty} \frac{\xi_{[tN]}([xN])}{N} \stackrel{\mathfrak{D}}{=} u(x;t)$$

exists for any $x = (x_1, x_2) \neq 0$ and $t \geq 0$.

For small values of t, this follows from the law of large numbers. For larger t this statement is evident, although we do not know of any existing proof for this particular model. This is a general feature of convergence results of this type: in spite of its being an obvious fact, so far there are only proofs tuned to special classes of walks.

We will prove this result in the next section, as it will be the basis of our further analysis of the random walk starting at a *fixed* point,

3 Euler limit estimates for starting points $x \neq 0$

For any initial point $x \neq 0$ we will first estimate the distance $\|\xi_t(x) - u(r;x)\|_2$ and then we will prove Proposition 2.1. If we would have a completely homogenous random walk, than we could immediately apply (modulo generalisation to vector sequences with martingale components) the Azuma-Hoeffding inequality for martingales ([10], E14.2, p.237), that we recall here.

From now on by $\|\cdot\|$ we will understand the ℓ^2-norm $\|\cdot\|_2$.

Lemma 3.1 (Azuma-Hoeffding inequality ([10] p.237)) *Let $\{M_l\}_{l=1}^n$ be a zero-mean martingale, the increments $M_l - M_{l-1}$ (with $M_0 = 0$) of which are bounded in absolute value by c_l. Then for any $x > 0$ we have*

$$\mathsf{P}\{\sup_{1\le l\le n} M_l \ge x\} \le \exp\{-\frac{x^2}{2\sum_{l=1}^n c_l^2}\}. \tag{3.1}$$

In our case of only *face*-homogeneity, the sequence

$$M_{t+1} = \xi_{t+1}(x) - U(x;t),$$

with

$$U(x;t) = \sum_{n=0}^{t} m\big(\xi_n(x)\big)$$

is a sequence, the components of which form a martingale sequence. However, $U(x;t)$ itself is random. So, in order to get bounds on the distance $\|\xi_t(x) - u(x;t)\|$ by using the Azuma-Hoeffding-inequality, we have to bound $\|U(x;t) - u(x;t)\|$. Unfortunately, our investigations did not result in essentially stronger results than the bounds from Lemma 3.2 below. Since this was our earlier approach, we have decided to leave it intact, and use the martingale approach lateron, where we will need it. We would like to mention, that this setting will be a useful one for connecting atomic sets and discrete scattering properties (cf. [5]).

The main problem in this derivation is the following. Starting at a point far away from the origin, it takes a long time before any axis is crossed. Inside quadrants, the walk behaves as a sum of i.i.d. random variables, and the law of large numbers applies.

However, the chosen drifts guarantee that sooner or later some axis is hit. For some time the walk is controlled by two different distributions, until (exponentially quick!) absorption into the next quadrant. Again after some time the law of large number regime applies, based on another jump distribution than before. The time between passing from one law of large number regime to another allows for some dispersion that should be controlled.

Boundedness of the number of face-transitions in Lemma 3.2 is mainly used in order to control the total amount of dispersion. An additional problem will arise later, since Euler paths are not parallel, but their distance blows up a factor C each cycle.

For any $x \in \mathbf{R}^2 \setminus \{0\}$ we define

$$n(x) = \inf_{t<\tau(x)} \|u(x;t)\|.$$

In the following lemma we estimate the probability for the random walk to be inside a tube of radius v of the path $u(x,s), 0 \le s \le t$, where $t > 0$ is any fixed number. The radius v should be sufficiently small :

(i) $v < n(x)$, meaning that the v-tube does not contain 0 ;

(ii) $2v < n(x)\cdot|C-1| = \inf_{s<\tau(x)} \|Cu(x,s) - u(x,s)\|$,
meaning that "neighbouring parts" of the v-tube do not intersect.

This yields the following upper bound for v

$$v < n(x)\cdot \min\{\frac{|C-1|}{2}, 1\}. \tag{3.2}$$

For convenience we reformulate this upper bound in terms of the cycle time $\tau(x)$. Just note that there exists a positive constant g such that

$$n(x) > g\cdot\tau(x) \quad \text{for all } x \in \mathbf{R}^2 \setminus \{0\}$$

and therefore there exists a constant $\theta = \theta(C) > 0$ such that

$$v < \theta \cdot \tau(x) \tag{3.3}$$

implies (3.2).

Lemma 3.2 (Extension of Kolmogorov's inequality) *There exists a constant $c > 0$ such that for any constant v satisfying (3.3), the following holds*

$$\mathsf{P}\{\max_{0\le k\le t} \|\xi_k(x) - u(x;k)\| \le v\} \ge 1 - c \cdot \frac{t}{v^2}, \tag{3.4}$$

for any time t and any sufficiently big initial state $x \in \mathbf{Z}^2$. The constant c depends only on the number of face-transitions of the trajectory $u(x;s), 0 \le s \le t$.

Clearly, inequality (3.4) is trivially true whenever $v \le \sqrt{c \cdot t}$.

Proof. It is sufficient to prove the assertion for $x \in Q^{++}$. Let

$$t_0 = \frac{x_1}{|m_1^{++}|}$$

be the first time that the dynamical system $u(x;\cdot)$ hits Q^{0+}. Similarly, denote by

$$t_1 = \frac{x_2 + t_0 m_2^{++}}{|m_2^{-+}|} + t_0$$

the first time that the dynamical system hits Q^{-0}. In the time interval $[0, t_1)$ the dynamical system starting at x has precisely one face-transition.

Fix a constant $w > 1$. First we consider the case of $x_1, x_2 > w$ and x_1 is sufficiently large. The proof will establish the Euler limit estimate for this given x, $t < t_1$ and $w > 1$ such that the disc $\{y \in \mathbf{R}^2 \mid \|u(x;t) - y\| \le w \cdot \beta\}$ is contained in the interior of Q^{-+}, for some constant $\beta > 1$. The estimate for general t then follows by a finite glueing procedure applied to subpaths ending and starting inside a quadrant.

We consider three cases $t \le t_0 - w|m_1^{++}|^{-1}$, the dispersion situation $t_0 - w|m_1^{++}|^{-1} < t \le t_0 + 4\gamma^{-1}w$, where $\gamma = \min\{|m_1^{++}|, |m_1^{-+}|\}$, and $t_0 + 4\gamma^{-1}w < t < t_1$.

1. Case of $t < t_0 - w|m_1^{++}|^{-1}$.

Before the first face-transition occurs, the random walk inside a quadrant Q^{ab}, $a, b = +, -$, behaves as the sum of i.i.d. random vectors, i.e.

$$\xi_l(x) = x + \sum_{n=1}^{l} \eta_n^{ab},$$

where η_n^{ab} are i.i.d. random vectors with distribution

$$\mathsf{P}\{\eta_n^{ab} = q\} = p_q^{ab}$$

and with expectation $\mathsf{E}\eta_n = m^{ab}$ and variance vectors D^{ab}, $a, b = +, -$. Let

$$\sqrt{D} = \max\{\|D^{++}\|, \|D^{-+}\|, \|D^{--}\|, \|D^{+-}\|\}.$$

Denote $S_k^{ab} = (S_{k,1}^{ab}, S_{k,2}^{ab}) = \sum_{n=1}^{k} \eta_n^{ab}$. By Kolmogorov's inequality for random vectors we have for any $w > 0$ and any integer $t < t_0 - w|m_1^{++}|^{-1}$

$$\mathsf{P}\{\|S_k^{ab} - km^{ab}\| \le w, \text{ for all } k \le t\} \ge 1 - D \cdot \frac{t}{w^2}, \tag{3.5}$$

for the constant D and for any $a, b = +, -$.

This tube of radius w is contained in Q^{++}: indeed $u(x; t_0 - w|m_1^{++}|^{-1}) = x + m^{++}(t_0 - w|m_1^{++}|^{-1})$ has first component equal to $x_1 + m_1^{++}t_0 + w = w$ and the second component is increasing in the time variable.

Thus the probability of all trajectories inside this tube for the unrestricted S_l-proccess and our random walk are equal. We find that

$$\mathsf{P}\{\max_{k\leq t}\|\xi_k(x)-u(x;k)\|\leq w\}\geq 1-D\cdot\frac{t}{w^2}. \tag{3.6}$$

So for $t<t_0-w|m_1^{++}|^{-1}$ the statement of the lemma holds for $c=D$ and $v=w$.

2. Case of $t_0+4\gamma^{-1}w\geq t\geq t_0-w|m_1^{++}|^{-1}$.

Since the jumps have size at most one both horizontally and vertically, the ℓ^2 norm of the jump size is bounded by $\sqrt{2}$. Hence, in time

$$w(|m_1^{++}|^{-1}+4\gamma^{-1})\leq w\cdot 5\gamma^{-1},$$

the covered distance has norm at most $5\sqrt{2}\gamma^{-1}w$. Consequently, (3.6) implies that

$$\mathsf{P}\{\max_{k\leq t}\|\xi_k(x)-u(x;k)\|\leq w+5\sqrt{2}w\gamma^{-1}\}\geq 1-D\cdot\frac{t_0-w|m_1^{++}|^{-1}}{w^2}>1-D\cdot\frac{t}{w^2}.$$

Putting $v=(1+5\sqrt{2}\gamma^{-1})w$, this implies

$$\mathsf{P}\{\max_{k\leq t}\|\xi_k(r)-u(r;k)\|\leq v\}\geq 1-D\cdot\frac{t}{w^2}\geq 1-c\cdot\frac{t}{v^2}, \tag{3.7}$$

with $c=D\cdot(1+5\sqrt{2}\gamma^{-1})^2$.

3. Case of $t_1>t>t_0+4\gamma^{-1}w$.

Denote by

$$\tau_0=\min\{s>0\,|\,\xi_s(x)\in Q^{0+}\}$$

the first hitting time of the axis Q^{0+} for the random walk and let

$$\tilde{\tau}_0=\min\{k\,|\,x+S_k^{++}\in Q^{0+}\}.$$

Clearly,

$$\{\max_{k\leq t_0+w|m_1^{++}|^{-1}}\|S_k^{++}-km^{++}\|\leq w\}\subset\left\{\begin{array}{l}\max_{k\leq\tilde{\tau}_0}\|S_k^{++}-km^{++}\|\leq w,\\ t_0-w|m_1^{++}|^{-1}\leq\tilde{\tau}_0\leq t_0+w|m_1^{++}|^{-1}\end{array}\right\}.$$

Since τ_0 and $\tilde{\tau}_0$ are equal in distribution, we find by Kolmogorov's inequality that

$$\mathsf{P}\{C(x,w)\}\geq 1-D\cdot\frac{t_0+w|m_1^{++}|^{-1}}{w^2} \tag{3.8}$$

where

$$C(x,w)=\left\{\begin{array}{l}\max_{k\leq\tau_0}\|\xi_k(x)-u(x,k)\|\leq w\\ t_0-w|m_1^{++}|^{-1}\leq\tau_0\leq t_0+w|m_1^{++}|^{-1}\end{array}\right\}.$$

The event $C(x,w)$ implies that

$$\xi_{\tau_0}(x)\in C^*(x,w)=\{y\in\mathbf{Z}^2\,|\,y_1=0,\,|y_2-u_2(x;t_0)|\leq w(1+|m_1^{++}|^{-1})\}.$$

This is because of the event $C(x,w)$ implying that τ_0 occurs earliest at time $t_0-w|m_1^{++}|^{-1}$. At that time, the difference between the second coordinates of $\xi_{\tau_0}(x)$ and $u(x;\tau)$ is at most w and the maximum occurred deviation in the time interval till t_0 is at most $w|m_1^{++}|^{-1}$.

Consider the random walk starting at $q\in C^*(x,w)$. We will show that the random walk starting at point q leaves the axis Q^{0+} at exponential speed. To this end, consider the event

$$\{\xi_{[\alpha w],1}(q)>-w\},$$

with α to be determined lateron. Consider the process $S_k=\xi_{k,1}(q), k=0,1,2...$ We have

$$\mathsf{E}(S_{k+1}-S_k|S_k)\leq-\gamma$$

with γ as above. By virtue of [2] Theorem 2.1.7, for any positive $\delta_1 < \gamma$ there exist constants $h > 0$, $\delta_2 > 0$ such that for any k

$$\mathsf{P}\{\xi_{k,1}(q)) > -\delta_1 k\} \le \exp\{hq_1 - \delta_2 k\} = \exp\{-\delta_2 k\}, \tag{3.9}$$

the latter equality holding because of $q_1 = 0$. Choose $\delta_1 = \gamma/2$ and suitable corresponding constants h, δ_2. Set $\alpha = 3/(2\delta_1) = 3\gamma^{-1}$ and $k = [\alpha w]$. This choice ensures that $t > t_0 + (4/\gamma)w > \tau_0 + [\alpha w]$.

For all $w > 1$ we have

$$\delta_1[\alpha w] \ge \frac{\gamma}{2}\big(\frac{3}{\gamma}w - 1\big) = \frac{3}{2}w - \frac{\gamma}{2} \ge w,$$

since $\gamma \le 1$ by definition. This implies

$$\mathsf{P}\{\xi_{[\alpha w],1}(q) > -w\} \le \mathsf{P}\{\xi_{[\alpha w],1}(q) > -\delta_1[\alpha w]\} \le \exp\{-\delta_2[\alpha w]\}, \tag{3.10}$$

for $w > 1$. Clearly by decreasing δ_2, (3.10) can be made to be satisfied for *all* $w > 0$. By the boundedness of jumps we have

$$\mathsf{P}\{\xi_{[\alpha w]}(q) \in A(q,w)\} \ge 1 - \exp\{-\delta_2[\alpha w]\}, \tag{3.11}$$

with

$$A(q,w) = \left\{ y \in \mathbf{Z}^2 \;\middle|\; \begin{array}{l} -[\alpha w] \le y_1 \le -w \\ |y_2 - q_2| \le [\alpha w] \end{array} \right\} \subset Q^{-+}.$$

Next note that the event $\{\xi_{[\alpha w]}(q) \in A(q,w)\}$, $q \in C^*(x,w)$, implies

$$\{\max_{1\le k\le[\alpha w]} \|\xi_k(q) - u(q;k)\| \le w\sqrt{(2\alpha)^2 + (\frac{3}{2}\alpha)^2} = \frac{5}{2}\alpha w\}.$$

As a consequence, the event $C(x,w) \cap \{\xi_{\tau_0+[\alpha w]}(x) \in A(\xi_{\tau_0}(x),w)\}$ implies

$$\big\{\max_{\tau_0\le k\le \tau_0+[\alpha w]} \|\xi_k(x) - u(x;k)\| \le w\big(1 + \frac{5}{2}\alpha\big)\big\}, \tag{3.12}$$

so that the dispersion deviation is well-controlled.

Next we bound the maximum deviation between dynamical system and walk after time $\tau_0 + [\alpha w]$. Choose any point $p \in A(q,w)$ and let

$$\tau = t - \tau_0 - [\alpha w].$$

Given the event $C(x,w)$, we have $\tau \ge 0$ and

$$|\tau - (t - t_0 - [\alpha w])| = |\tau_0 - t_0| \le w|m_1^{++}|^{-1}. \tag{3.13}$$

So for any realisation s of τ we have (given $C(x,w)$)

$$\mathsf{P}\{\max_{k\le s} \|\xi_k(p) - (p + km^{-+})\| \le w\} \ge 1 - D\cdot\frac{t - t_0 - [\alpha w] + w|m_1^{++}|^{-1}}{w^2}. \tag{3.14}$$

We will now apply a gluing procedure. To this end, note that combining $p \in A(q,w)$ and $q \in C^*(x,w)$ yields the estimate

$$\begin{aligned}
\|p + sm^{-+} - u(x;t)\| &\le \|u(x;t_0) + sm^{-+} - u(x;t)\| + \|p - u(x;t_0)\| \\
&\le\ |t_0 + s - t|\|m^{-+}\| + \sqrt{\alpha^2w^2 + (w(1 + |m_1^{++}|^{-1}) + \alpha w)^2} \\
&\le\ w\Big\{(\alpha + |m_1^{++}|^{-1})\|m^{-+}\| + \sqrt{\alpha^2 + (1 + |m_1^{++}|^{-1} + \alpha)^2}\Big\},
\end{aligned} \tag{3.15}$$

for all w, given the event $C(x,w)$.

Recall $\alpha = 3\gamma^{-1} > 3$, where $\gamma = \min\{|m_1^{++}|, |m_1^{-+}|\}$. Therefore, $|m_1^{++}|^{-1} < \alpha$ and so (3.15) implies

$$\|p + sm^{-+} - u(x;t)\| \le w\big\{2\alpha\|m^{-+}\| + \sqrt{\alpha^2 + (1 + 2\alpha)^2}\big\}$$

Choose now $w = \beta^{-1}v$ with

$$\beta = \max\{2\alpha\|m^{-+}\| + \sqrt{\alpha^2 + (1+2\alpha)^2}\,,\, 1 + \frac{5}{2}\alpha\}. \tag{3.16}$$

Putting (3.8), (3.11), (3.12), (3.14) and (3.15) together, we find using the Markovian property

$$\begin{aligned}
&\mathsf{P}\{\max_{k\le t}\|\xi_k(x) - u(x;k)\| \le v\} \ge \\
&\ge \Big(1 - D\cdot\frac{t_0 + w|m_1^{++}|^{-1}}{w^2}\Big)\Big(1 - \exp\{-\delta_2\alpha w\}\Big)\times \\
&\quad\times\Big(1 - D\cdot\frac{t - t_0 - [\alpha w] + w|m_1^{++}|^{-1}}{w^2}\Big) \\
&\ge 1 - D\cdot\frac{t}{w^2} + D\frac{[\alpha w]}{w^2} - D\frac{2w|m_1^{++}|^{-1}}{w^2} - \exp\{-\delta_2\alpha w\}.
\end{aligned} \tag{3.17}$$

The constant D comes from the variance of the jump distributions, but clearly all previous inequalities continue to hold, if we make D larger. Thus we can assume that D is so large, that

$$D\frac{[\alpha w]}{w^2} - D\frac{2w|m_1^{++}|^{-1}}{w^2} - \exp\{-\delta_2\alpha w\} > 0$$

for any $w > 1$. Replacing w by $\beta^{-1}v$, we get

$$\mathsf{P}\{\max_{k\le t}\|\xi_k(x) - u(x;k)\| \le v\} \ge 1 - D\cdot\beta^2\cdot\frac{t}{v^2}.$$

So we put $c = c(x,t) = D\cdot\beta^2$.

Thus we have proved the assertion for the case that x_1, $x_2 > \beta^{-1}v$, $x \in Q^{++}$. The case of one of x_1, x_2 smaller than $\beta^{-1}v$ is treated similarly to the above analysis, case 3, when the random walk enters a neighbourhood of the order v of an axis. The case of x_1 and x_2 both smaller than $\beta^{-1}v$ cannot occur when the initial state x is sufficiently large.

This proves the assertion of the Lemma for $t < t_1$. □

Proof of Proposition 2.1

Since the point $u(x;t)$ is the outcome of a degenerate random vector, convergence in probability and convergence in distribution are equivalent. We check convergence in probability. Fix any $x \in \mathbf{R}^2 \setminus \{0\}$, any $t > 0$. We have to check for any (sufficiently smal) $\sigma > 0$ that

$$\mathsf{P}\{\|\frac{\xi_{[tN]}([xN])}{N} - u(x;t)\| > \sigma\} = \mathsf{P}\{\|\xi_{[tN]}([xN]) - u(xN;tN)\| > \sigma N\} \to 0, \quad N \to \infty.$$

In order to apply Lemma 3.2, we have to consider the point $u([xN];[tN])$ instead of $u(xN;tN)$. We will show that these are within uniformly bounded (in N) distance of each other and we will argue that the number of face transitions of the different trajectories of $u([xN];s)$ on $s \le [tN]$, do not differ more than a bounded number, for all sufficiently large N.

First we consider the difference $u(xN;tN)$ and $u([xN];[tN])$. The number of cycles $c(y;s)$ crossed in time s equals

$$c(y;s) = \max\{l \,|\, l + 1 \le \frac{\log\big(\frac{s(C-1)}{\tau(y)} + 1\big)}{\log C}\}.$$

Then

$$\lim_{N\to\infty} c([xN];[\tau N]) = \lim_{N\to\infty} c(xN;\tau N) = c(x;t) = \max\{l \,|\, l + 1 \le \frac{\log\big(\frac{t(C-1)}{\tau(x)} + 1\big)}{\log C}\}.$$

Choose N_0 with $c([xN];[\tau N]) = c(xN;\tau N) = c(x;t)$, $N \ge N_0$. After precisely $c(x;t)$ cycles the distance between the Euler paths starting at xN and $[xN]$ respectively is

$$\begin{aligned}
&\|u\big([xN];\frac{C^{c(x;t)} - 1}{C-1}\tau([xN])\big) - u\big(xN;\frac{C^{c(x;t)} - 1}{C-1}\tau(xN)\big)\| \\
&\qquad \le\ C^{c(x;t)}\|[xN] - xN\| \le C^{c(x;t)}\sqrt{2}.
\end{aligned}$$

If another cycle would be completed, then the distance would be at most $C^{c(x;t)+1}\sqrt{2}$. On $0 \le s \le [tN]$, the trajectory $u(xN,\cdot)$ 'lags behind' compared to the trajectory $u([xN];\cdot)$ (it did not yet complete the $c(x;t)$ cycles!). This lagging behind is proportional to the maximum distance that can be travelled by the dynamical system during the time that is equal to the difference in their respective $(c(x;t)+1)$-th cycle times. This time is bounded by $C^{c(x;t)+1}C'$ for some constant C'. Since the total travel times also differ by at most 1 unit time, the maximum difference between $u([xN];[tN])$ and $u(xN;tN)$ must be bounded by

$$C^{c(x;t)+1}\sqrt{2} + (C^{c(x;t)+1}C' + 1)C'',$$

for all $N \ge N_0$, where C'' is the maximum distance travelled during one unit time. One can therefore choose $N_1 > N_0$ such that for $N \ge N_1$ it holds that

$$\{\|\xi_{[tN]}([xN]) - u(xN;tN)\| > \sigma N\} \subset \{\|\xi_{[tN]}([xN]) - u([xN];[tN])\| > \frac{\sigma}{2}N\}.$$

To see that convergence is quicker than linear in N, choose any positive $\epsilon < 1/2$. Replace x and t in (3.4) by $[xN]$ and $[tN]$. Then the number of cycles passed by $u([xN];s)$ on $0 \le s \le [tN]$ is equal for any $N \ge N_1$. The number of face-transitions therefore can differ at most by 4.

Now let $N_2 > N_1$ be such that $N^{\epsilon+1/2} < \sigma N/2$, and (3.3) is satisfied for $N > N_2$. By Lemma 3.2 there exists a constant $c > 0$, such that

$$\begin{aligned}\mathsf{P}\Big\{\|\xi_{[tN]}([xN]) - u([xN];[tN])\| > \frac{\sigma}{2}N\Big\} &\le \mathsf{P}\{\,\|\xi_{[tN]}([xN]) - u([xN];[tN])\| > N^{\epsilon+1/2}\Big\} \\ &\le c\cdot\frac{[tN]}{N^{2\epsilon+1}} \le \frac{ct}{N^{2\epsilon}}, \quad N > N_2.\end{aligned}$$

The result follows from the fact that $\lim_{N\to\infty} ct/N^{2\epsilon} = 0$. □

4 Criteria for ergodicity and transience

This section will show the validity of Lemma 2.1. For proving ergodicity and transience, the construction of a suitable Lyapunov function suffices. In the case of non-zero drifts, one can often use the Euler distance to 0 as a Lyapunov function.

To this end, we need studying properties of the Euler distance $r(x)$ as a function of the initial point x, or equivalently, of the cycle time $\tau(x)$ (cf. Lemma 2.2).

4.1 Cycle time and isochrone

The cycle time is easily calculated explicitly. To this end, define the constants

$$\begin{aligned}
\tau_1 &= \frac{1}{|m_1^{++}|} + \frac{m_2^{++}}{m_1^{++}m_2^{-+}} + \frac{m_2^{++}|m_1^{-+}|}{m_1^{++}m_2^{-+}m_1^{--}} + \frac{m_2^{++}m_1^{-+}m_2^{--}}{m_1^{++}m_2^{-+}m_1^{--}m_2^{+-}},\\
\tau_2 &= \frac{1}{|m_2^{-+}|} + \frac{m_1^{-+}}{m_2^{-+}m_1^{--}} + \frac{m_1^{-+}|m_2^{--}|}{m_2^{-+}m_1^{--}m_2^{+-}} + \frac{m_1^{-+}m_2^{--}m_1^{+-}}{m_2^{-+}m_1^{--}m_2^{+-}m_1^{++}},\\
\tau_3 &= \frac{1}{m_1^{--}} + \frac{|m_2^{--}|}{m_1^{--}m_2^{+-}} + \frac{m_2^{--}m_1^{+-}}{m_1^{--}m_2^{+-}m_1^{++}} + \frac{m_2^{--}m_1^{+-}m_2^{++}}{m_1^{--}m_2^{+-}m_1^{++}|m_2^{-+}|},\\
\tau_4 &= \frac{1}{m_2^{+-}} + \frac{m_1^{+-}}{m_2^{+-}|m_1^{++}|} + \frac{m_1^{+-}m_2^{++}}{m_2^{+-}m_1^{++}m_2^{-+}} + \frac{m_1^{+-}m_2^{++}|m_1^{-+}|}{m_2^{+-}m_1^{++}m_2^{-+}m_1^{--}}.
\end{aligned}$$

Further denote $\tau^{++} = \tau^{+0} = (\tau_1, \tau_2)$, $\tau^{-+} = \tau^{0+} = (-\tau_3, \tau_2)$, $\tau^{--} = \tau^{-0} = (-\tau_3, -\tau_4)$, $\tau^{+-} = \tau^{0-} = (\tau_1, -\tau_4)$. It immediately follows that $\tau^{ab}\cdot m^{ab,T} = C - 1$. Here superscript T stands for the transposed vector, a column vector, and $\cdot$ for the inner product.

For calculating the cycle time, we need to associate with any $x \neq 0$ the sequence of succesive (deterministic) times $t_i(x)$ that the Euler path $u(x;\cdot)$ changes face. For instance, when $x \in Q^{++}$,

$$\left.\begin{array}{rcl} t_0(x) & = & \dfrac{1}{|m_1^{++}|}x_1 \\ t_1(x)-t_0(x) & = & \dfrac{1}{|m_2^{-+}|}(x_2+m_2^{++}t_0(x)) \\ t_2(x)-t_1(x) & = & \dfrac{|m_1^{-+}|}{m_1^{--}}\big(t_1(x)-t_0(x)\big) \\ t_3(x)-t_2(x) & = & \dfrac{|m_2^{--}|}{m_2^{+-}}\big(t_2(x)-t_1(x)\big) \\ t_4(x)-t_3(x) & = & \dfrac{m_1^{+-}}{|m_1^{++}|}\big(t_3(x)-t_2(x)\big) \\ t_{i+1}(x)-t_i(x) & = & C\big(t_{i-3}(x)-t_{i-4}(x)\big), \quad i \geq 4. \end{array}\right\} \tag{4.1}$$

Lemma 4.1 *The cycle time $\tau(x)$ is given by the continuous function*

$$\tau(x) = \tau^{ab}\cdot x^T, \quad x \in Q^{ab}, \quad ab \neq 00. \tag{4.2}$$

Moreover, $u(x;\tau(x)) = Cx$ with C as defined in (2.1).

As a consequence,

$$\begin{array}{rcl} \tau(x+m(x))-\tau(x) & = & \tau^{ab}\cdot m(x) \\ & = & \tau^{ab}m^{ab,T} = C-1 \text{ for any } x \in Q^{ab}, ab = +,-,0. \end{array} \tag{4.3}$$

Proof of Lemma 4.1. Let us prove (4.2). Suppose we start at some point $x \in Q^{++}$. Using (4.1), note that the dynamical system starting at x hits Q^{0+} for the first time at time $t_0(x) = x_1/|m_1^{++}|$ at the point $(0, x_2+t_0(x)m_2^{++})$, in short

$$u(x;t_0(x)) = \big(0, x_2+t_0(x)m_2^{++}\big) = \big(0, x_2+\frac{m_2^{++}}{|m_1^{++}|}x_1\big).$$

Similarly, at time

$$t_1(x) = t_0(x) + \frac{1}{|m_2^{-+}|}\big(x_2+\frac{m_2^{++}}{|m_1^{++}|}x_1\big)$$

the dynamical system $u(\cdot)$ hits Q^{-0} for the first time at the point

$$u(x;t_1(x)) = \big(\frac{m_1^{-+}}{|m_2^{-+}|}\big(x_2+\frac{m_2^{++}}{|m_1^{++}|}x_1\big),0\big).$$

At time

$$t_2(x) = t_1(x) + \frac{m_1^{-+}}{m_2^{-+}m_1^{--}}\big(x_2+\frac{m_2^{++}}{|m_1^{++}|}x_1\big)$$

Q^{0-} is hit for the first time at the point

$$u(x;t_2(x)) = \big(0, \frac{m_1^{-+}m_2^{--}}{m_2^{-+}m_1^{--}}\big(x_2+\frac{m_2^{++}}{|m_1^{++}|}x_1\big)\big).$$

Finally, at time

$$t_3(x) = t_2(x) + \frac{m_1^{-+}m_2^{--}}{|m_2^{-+}|m_1^{--}m_2^{+-}}\big(x_2+\frac{m_2^{++}}{|m_1^{++}|}x_1\big)$$

Q^{+0} is hit for the first time, at point

$$u(x;t_3(x)) = \big(\frac{m_1^{-+}m_2^{--}m_1^{+-}}{|m_2^{-+}|m_1^{--}m_2^{+-}}\big(x_2+\frac{m_2^{++}}{|m_1^{++}|}x_1\big),0\big).$$

We will calculate time and place where the line αx, $\alpha > 0$, is crossed for the first time by the dynamical system $u(\cdot)$ starting at the point $u(x, t_3(x))$. In order that $u\big(u(x, t_3(x)), \tau\big) = \alpha x$ for some $\alpha > 0$ and τ minimal,

$$\tau = \alpha \frac{1}{m_2^{++}} x_2.$$

Hence

$$\frac{m_1^{-+} m_2^{--} m_1^{+-}}{|m_2^{-+}| m_1^{--} m_2^{+-}} \big(x_2 + \frac{m_2^{++}}{|m_1^{++}|} x_1 \big) + \alpha \frac{m_1^{++}}{m_2^{++}} x_2 = \alpha x_1,$$

and so

$$\alpha = \frac{m_1^{-+} m_2^{--} m_1^{+-} m_2^{++}}{m_2^{-+} m_1^{--} m_2^{+-} m_1^{++}},$$

thus giving the constant C. Note that we can leave out absolute signs. Furthermore,

$$\tau = \frac{m_1^{-+} m_2^{--} m_1^{+-}}{m_2^{-+} m_1^{--} m_2^{+-} m_1^{++}} x_2.$$

By combination of the above relations we then find $\tau(x) = t_3(x) + \tau = \tau^{++} \cdot x^T$. The proof for the other cases goes similarly.

Continuity only needs to be checked at face-transitions. But this is evident. □

For convenience, we extend the cycle time τ as a continuous function on $\mathbf{R}^2$ by setting

$$\tau(0) = 0.$$

The form of the isochrone is now an easy consequence of the above combined with Lemma 2.2. We have

$$\begin{aligned} \mathcal{I}(t) &= \{x \in \mathbf{R}^2 \,|\, t = r(x)\} \\ &= \{x |\, \tau^{ab} \cdot x^T = t(C-1), x \in Q^{ab},\ a, b, = +, -, 0\}. \end{aligned} \tag{4.4}$$

This immediately yields the following result.

Lemma 4.2 *For any $t > 0$, the isochrone $\mathcal{I}(t)$ is closed and homeomorphic to a circle in $\mathbf{R}^2$.*

Proof. We will prove that isochrone is closed.

By virtue of Lemma 4.1, the set $t = \tau(x)/(C-1)$ is a straight line on each of the closures of faces $\bar{Q}^{++}$, $\bar{Q}^{-+}$, $\bar{Q}^{--}$, $\bar{Q}^{+-}$.

It is an easy computation to check that line segments from neighbouring quadrants intersect at one and the same point of an axis. □

4.2 Lyapunov functions

There is an interesting construction in [2] for Lyapunov functions for ergodicity and transience of face-homogeneous random walks on the lattice in $\mathbf{Z}_+^d$. Out of a Lipschitz-continuous and non-negative function that increases by at least ϵ in the direction of the generalised drift ("second vectorfield" in the terminology of this book), they show how to construct a Lyapunov function for Foster's criterion for ergodicity. If this function decreases, then it yields a Lyapunov function for transience. As mentioned before, a typical function with this property is the Euler distance.

The proofs go through in our case, but here we prefer to cite the ergodicity and transience criteria used and we will explicitly construct convenient Lyapunov functions.

We recall [2] Theorems 2.2.4 and 2.2.7.

Theorem 4.1 *The Markov chain ξ_t is ergodic if and only if there exists functions $f : \boldsymbol{S} \to \mathbf{R}_+$, $k : \boldsymbol{S} \to \mathbf{Z}_+$, a constant $\epsilon > 0$ and a finite set $A \subset \boldsymbol{S}$, such that*

$$\mathsf{E}\{f(\xi_{t+k(\xi_t)} \,|\, \xi_t = x\} \begin{cases} \leq f(x) - \epsilon k(x), & x \notin A, \\ < \infty, & x \in A \end{cases}$$

Theorem 4.2 *The Markov chain ξ_t is transient if there exist a function $f : \boldsymbol{S} \to \mathbf{R}_+$, a* bounded *function $k : \boldsymbol{S} \to \mathbf{Z}_+$, constants $\epsilon, C', d > 0$, such that the set $A_{C'} = \{x \,|\, f(x) > C'\} \neq \emptyset$; $|f(x) - f(y)| > d$ implies $\mathsf{P}\{\xi_{t+1} = y \,|\, \xi_t = x\} = 0$ and*

$$\mathsf{E}\{f(\xi_{t+k(\xi_t)} \,|\, \xi_t = x\} \geq f(x) + \epsilon, \quad x \in A_{C'}.$$

Proof of Lemma 2.1. Let us first prove the transience in case of $C > 1$.

Set $f(x) = r(x)$. For $x \in Q^{ab}$, $a, b \neq 0$, inside one of the quadrants, set the function $k(x) = 1$. We have

$$\mathsf{E}\{r(\xi_{t+1}) - r(\xi_t) \,|\, \xi_t = x\} = \frac{1}{C-1}\mathsf{E}\{\tau^{ab} \cdot (\xi_{t+1} - \xi_t) \,|\, \xi_t = x\} = \frac{1}{C-1}\tau^{ab} \cdot m^{ab} = 1,$$

by Lemma 4.1 and (4.3).

For $x \in Q^{ab}$, a or $b = 0$, but $\{a, b\} \neq \{0, 0\}$, on one of the axes, the situation is slightly more complicated and we have to invoke Lemma 3.2. Choose the constant $c > 0$ from this Lemma corresponding to paths with at most one face-transition.

Let $x \in Q^{+0}$, and choose $\gamma < 1/2$ and $t < x_1/|m_1^{++}|$. Consider the set $B = \{y \in \mathbf{R}^2 \,|\, \|y - u(x; t)\| < t^{\gamma+1/2}\}$. We have to lowerbound $r(y) - r(x)$ for $y \in B$.

Observe that $r\big(u(x; t)\big) = r(x) + t$. The 'worst' value of $r(y)$ one can get on B, is orthogonal to the isochrone $\mathcal{I}(r(x) + t)$, at the point $y = u(x; t) - d\tau^{++}$, with $d = t^{\gamma+1/2}/\|\tau^{++}\|$. Provided that $y \in Q^{++}$, we find that

$$r(y) = \frac{\tau(y)}{C-1} = \frac{\tau^{++} \cdot y^T}{C-1} = r\big(u(x; t)\big) - \frac{d\|\tau^{++}\|^2}{C-1} = r(x) + t - \frac{t^{\gamma+1/2}\|\tau^{++}\|}{C-1}.$$

Hence, for $y \in B$ we find that

$$r(y) \geq r(x) + t - \frac{t^{\gamma+1/2}\|\tau^{++}\|}{C-1}. \tag{4.5}$$

This gives a lower bound on a set with probability at least $1 - c/t^{2\gamma}$.

This 'worst' achievable value of $r(\xi_t)$ achieved outside this set, is at the point $(x_1 - t, 0)$. The Euler distance at this point equals $r(x_1 - t, 0) = r(x) - \tau_1^{++}t/(C-1)$.

Combination with (4.5) yields for any $x' = (x_1', 0) \in Q^{+0}$

$$\begin{aligned} \mathsf{E}\{r(\xi_t) - r(\xi_0) \,|\, \xi_0 = x'\} \quad &\geq \quad \mathsf{E}\{\big(r(\xi_t) - r(\xi_0)\big)\mathbf{1}_{\{\xi_t \in B\}} \,|\, \xi_0 = x'\} \\ &\qquad + \mathsf{E}\{\big(r(\xi_t - r(\xi_0)\big)\mathbf{1}_{\{\xi_t \notin B\}} \,|\, \xi_t = x'\} \\ &\geq \quad (1 - ct^{-2\gamma}) \cdot \big(t - \frac{t^{\gamma+1/2}\|\tau^{++}\|}{C-1}\big) - ct^{1-2\gamma}\frac{\tau_1^{++}}{C-1}, \end{aligned} \tag{4.6}$$

provided that $x_1 > t$, and $B \subset Q^{++}$.

By our choice of γ, the right-hand side of (4.6) is bigger than 1 for sufficiently large t. One can then choose $x \in Q^{+0}$, so that the previous requirements are fulfilled for this t. Choose $k(x') = t$ for $x' \in Q^{+0}$, $x' \neq 0$.

The same procedure can be applied to the other axes. Setting $\epsilon = 1$, one can find a constant C' such that the conditions of Theorem 4.2 are met.

The proof of ergodicity in case of $C < 1$, is analogous. □

5 Preliminary results for a fixed initial point

5.1 Convergence of the scaled Euler distance

The previous section uses the Euler distance $r(x)$ as a Lyapunov function. Using a technique of transforming (additive) Lyapunov functions for ergodicity and transience into (multiplicative) Lyapunov functions for exponential ergodicity and transience (cf. [9], [8]), one can get exponential bounds on the probability of deviating from a given level set of r. This can be used to show the following lemma.

Lemma 5.1 *For any initial point $x \in \mathbf{Z}^2$ the limit*

$$\lim_{N\to\infty} r(\frac{\xi_{[tN]}(x)}{N})$$

almost surely exists and equals t.

Instead of the line of proof sketched in the above, there are reasons for choosing a martingale approach and then to apply the Azuma-Hœffding inequality. To this end, we need to make a simplifying assumption. We would like to point out that the results go through in the more general case, but to a cost of increased tediousness of proofs.

In this subsection we therefore assume that

$$\mathsf{E}\{\tau(\xi_{n+1}) - \tau(\xi_n) \,|\, \xi_n = x\} = C - 1, \tag{5.1}$$

for any x. Note that this holds by (4.3) for any $x \in Q^{ab}$, $a, b \neq 0$. Indeed, for such point x

$$\mathsf{E}\{\tau(\xi_{n+1}) - \tau(\xi_n) \,|\, \xi_n = x\} = \tau^{ab} \cdot \mathsf{E}\{\xi_{n+1} - \xi_n \,|\, \xi_n = x\} = \tau^{ab} \cdot m^{ab} = C - 1.$$

One way to construct models where (5.1) is satisfied, is to impose that transitions from one quadrant to the next quadrant in clockwise direction, or in the direction of 0 on the axes, cannot occur (under the initial assumption of the drifts from an axis point and from points in the next counterclockwise quadrant being equal).

Proof of Lemma 5.1 onder condition (5.1)

We will prove the statement for $t = 1$. The general statement then easily follows. Indeed, $r\big(\xi_N(x)/N\big) \to 1$, a.s., implies the same for any subsequence and so $r\big(\xi_{[tN]}(x)/[tN]\big) \to 1$, a.s. Note that homogeneity of the function τ implies homogeneity of the function r. Hence, by multiplying by t, $r\big(t\xi_{[tN]}(x)/[tN]\big) \to t$, a.s. Clearly the sequence $r\big(\xi_{[tN]}/N\big)$, $N = 1, 2, \ldots$, has the same a.s. limit.

For the proof of the case $t = 1$, we will prove that the sequence

$$M_n = r(\xi_n) - r(\xi_0) - n \tag{5.2}$$

is a zero-mean martingale. We have

$$\mathsf{E}\{M_{n+1} - M_n \,|\, M_n\} = \frac{1}{C-1}\mathsf{E}\{\tau(\xi_{n+1}) - \tau(\xi_n) \,|\, \xi_n\} - 1.$$

By virtue of (5.1), we get

$$\mathsf{E}\{M_{n+1} - M_n \,|\, M_n\} = 0.$$

Recall that the jumps of ξ_t are bounded in absolute value. Then the increments $M_l - M_{l-1}$ are bounded in absolute value as well.

From martingale limit theory it follows immediately that $n^{-1}M_n \to 0$ almost surely (see for instance [4] Theorem 2.18) and so $r(\xi_n/n) \to 1$ a.s. □

A consequence of this result is that any limiting distribution of the sequence $\xi_{[tN]}/N$ is concentrated on the isochrone $\mathcal{I}(t)$. The following lemma provides an estimate on the speed of convergence of $r(\xi_N/N)$.

Lemma 5.2 *Let $\epsilon > 0$ and $\delta \in (1/2, 1)$. Under condition (5.1), there exists $\gamma > 0$ such that*

$$\mathsf{P}\{|r(\xi_N) - r(\xi_0) - N| \leq \epsilon N^\delta \text{ for all } N \geq M\} \geq \gamma, \tag{5.3}$$

for some M.

Proof. The zero-mean martingale $\{M_l\}_l$ from (5.2) satisfies the conditions of Lemma 3.1, for constants $c_l = c$, for some constant c. Take $x = \epsilon n^\delta$ in (3.1) with $\epsilon > 0$ and $\delta \in (1/2, 1)$. Then (3.1) implies

$$\mathsf{P}\{|M_n| \geq \epsilon n^\delta\} < 2\exp\{-\frac{\epsilon^2}{2c^2}n^{2\delta-1}\}. \tag{5.4}$$

Taking (5.4) into account we obtain for any m

$$\mathsf{P}\{|M_n| \le \epsilon n^\delta \text{ for all } n \ge m\} \ge 1 - \sum_{n\ge m} \mathsf{P}\{|M_n| \le \epsilon n^\delta\} \ge 1 - 2\sum_{n\ge m} \exp\{-\frac{\epsilon^2}{2c^2} n^{2\delta-1}\}.$$

Since $2\delta - 1 > 0$, the series converges. And so, there exists $\gamma > 0$ and m such that

$$2\sum_{n\ge m} \exp\{-\frac{\epsilon^2}{2c^2} n^{2\delta-1}\} < 1-\gamma.$$

$\square$

5.2 Absorption between any two Euler paths

Next we will prove that the random walk starting at a fixed point $x \in \mathbf{Z}^2$, will be absorbed between any two (non-identical) Euler paths with positive probability. This will be the crucial step in showing that the scaled process cannot have any limiting distribution in general.

We need some notation. Let $x, y \in \mathcal{I}(s)$, $s > 0$. By $[x \rightsquigarrow y]$ we denote the set of points $r \in \mathcal{I}(s)$, that we pass when moving *anticlockwise* from x to y along $\mathcal{I}(s)$, including x and y. Using a round bracket instead of a straight one, excludes the corresponding end point.

Similarly, the set $[\Gamma_x \rightsquigarrow \Gamma_y]$ denote the set of paths between Γ_x and Γ_y in anticlockwise direction, i.e.

$$[\Gamma_x \rightsquigarrow \Gamma_y] = \{u(z;t) \,|\, z \in [x \rightsquigarrow y], t \in \mathbf{R}\}.$$

Again we may replace (one of) the straight brackets by round ones, thus excluding the corresponding 'end' path.

Our goal is hence to show that for any $x, y \in \mathcal{I}$ and any initial state $p \in \mathbf{Z}^2$ there exists N such that

$$\mathsf{P}\{\xi_n \in (\Gamma_x \rightsquigarrow \Gamma_y) \text{ for all } n \ge N \,|\, \xi_0 = p\} > 0, \tag{5.5}$$

in other words, the set $(\Gamma_x \rightsquigarrow \Gamma_y)$ is a *sojourn set* in Feller's terminology (cf. [3], [1]). For proving this, we will use expanding tubes $\mathcal{T}^\gamma(p)$ containing a given Euler path $\{u(p;t), t \ge 0\}$ and contained between two Euler paths, $\mathcal{T}^\gamma(p) \subseteq (\Gamma_x \rightsquigarrow \Gamma_y)$. The parameter γ indicates the 'width' of the tube. For these tubes we will show

$$\mathsf{P}\{\xi_n \in \mathcal{T}^\gamma(p), n \ge 0 \,|\, \xi_0 = p\} > 0, \tag{5.6}$$

which clearly implies (5.5).

A connected subset $A \subseteq \mathcal{I}(s)$ will be called an *interval on the isochrone* $\mathcal{I}(s)$. Take any interval $A \subset \mathcal{I}(s)$, $s > 0$. All points of A have the same cycle time $(C-1)s$. Therefore, the dynamical system u maps the interval A to the interval $CA \subset \mathcal{I}(Cs)$. Below we will have to deal with neighbourhoods of an interval $A \subset \mathcal{I}(s)$.

The action of u on the disc

$$\mathcal{O}(A,\alpha) = \{x \in \mathbf{R}^2 \mid \inf_{y\in A} \|x-y\| < \alpha\}, \quad \alpha > 0,$$

transforms it in a rather complicated way. Most points of $\mathcal{O}(A,\alpha)$ have different cycle times and the shape of ℓ^2-balls is not consistent with the piecewise linearity of the dynamical system. To deal with this problem, we first construct an invariant under u that can be thought of as a kind of "angle" between two Euler paths. Fix any reference point $x_0 \ne 0$ and corresponding reference Euler path Γ_{x_0}. Let $x_s = \Gamma_{x_0} \cap \mathcal{I}(s)$. Define

$$\psi_{x_0} : \mathbf{R}^2 \setminus \{0\} \to [0, C-1]$$

by

$$\psi_{x_0}(x) = \inf\Big\{t \ge 0 \,|\, x = \frac{u(x_s; st)}{1+t}\Big\}, \quad x \in \mathcal{I}(s).$$

Let us prove that ψ_{x_0} is indeed invariant under the action of u.

Lemma 5.3 *For $x \in \mathcal{I}(s)$ we have*

$$\psi_{x_0}\big(u(x;t)\big) = \psi_{x_0}(x), \quad t > 0.$$

Proof. The function $t \to u(x_s; st)/(1+t)$ is continuous and periodic. Hence the infimum is a minimum, say it is assumed for t', i.e.

$$\psi_{x_0}(x) = \inf\Big\{t \geq 0 \,|\, x = \frac{u(x_s; st)}{1+t}\Big\}.$$

By the construction of this function, this value t' is the unique value on $[0, C-1)$ for which

$$x = \frac{u(x_s; st')}{1+t'}.$$

The assertion follows from

$$u(x;v) = u\Big(\frac{u(x_s; st')}{1+t'}; v\Big) = \frac{u\big(u(x_s; st'); v(1+t')\big)}{1+t'} = \frac{u\big(u(x_s; v); (s+v)t'\big)}{1+t'} = \frac{u\big(x_{s+v}; (s+v)t'\big)}{1+t'}.$$

□

Now we will construct a special (open) *time tube* of an interval $A \subset \mathcal{I}(s)$, which we will denote by $\mathcal{T}(A, \rho)$, $\rho < s$. Its pre-image is in fact a rectangle in the (ψ, r)-plane.

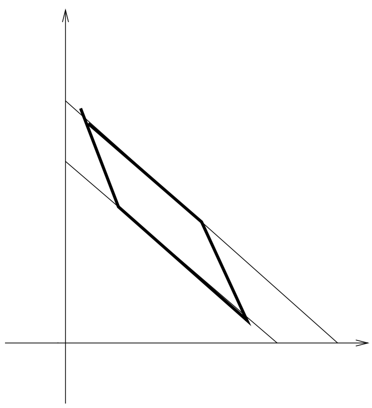

Figure 3.

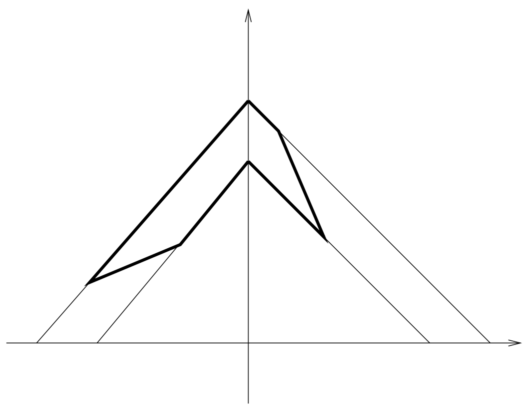

Figure 4.

For $A = \mathcal{I}(s)$ denote $\mathcal{T}(\mathcal{I}(s), \rho) = \{x \in \mathbf{R}^2 \mid |r(x) - s| < \rho\}$. For $A = [x \rightsquigarrow y]$,

$$\begin{aligned} \mathcal{T}(A, \rho) &= \{p \in \mathbf{R}^2 \,|\, 0 \leq \psi_x(p) \leq \psi_x(y), |r(p) - s| < \rho\} \\ &= \{p = u(q; w) \,|\, q \in A, |w| < \rho\}. \end{aligned}$$

If $\mathcal{T}(A, \rho) \subset Q^{ab}$, then it is a parallellogramme containing the set A (cf. Figure 3).

By invariance of ψ_x, time tubes remain time tubes under the action of u, i.e.

$$u\big(\mathcal{T}(A, \rho); t\big) = \mathcal{T}\big(u(A; t), \rho\big), \quad t > 0, \quad \text{any interval } A \subset \mathcal{I}(s). \tag{5.7}$$

In particular, $u\big(\mathcal{T}(A, \rho); \tau(A)\big) = \mathcal{T}(CA, \rho)$. Note that the pre-images (in the (ψ, r)-plane) of $\mathcal{T}(A, \rho)$ and $\mathcal{T}(CA, \rho)$ are identical!

Now let us see how these time tubes relate to open balls containing an interval $A \subset \mathcal{I}(s)$. Denote the restriction of $\mathcal{O}(A, \alpha)$ to $\mathcal{I}(s)$ by

$$\mathcal{O}_s(A, \alpha) = \mathcal{O}(A, \alpha) \cap \mathcal{I}(s).$$

This is clearly an interval on $\mathcal{I}(s)$.

Properties of time tubes
For any two intervals $A, B \subset \mathcal{I}(s)$ and sufficiently small $\rho > 0$ we have that
P1 there exist positive constants $\nu < 1 < \sigma$, not depending on s, such that for any $\alpha > 0$

$$\mathcal{T}\big(\mathcal{O}_s(A, \nu\alpha), \nu\alpha\big) \subset \mathcal{O}(A, \alpha) \subset \mathcal{T}\big(\mathcal{O}_s(A, \sigma\alpha), \sigma\alpha\big). \tag{5.8}$$

P2 $\mathcal{T}(\mathcal{O}_{\gamma s}(\gamma A, \gamma\alpha), \gamma\rho) = \gamma\mathcal{T}(\mathcal{O}_s(A, \alpha), \rho)$, for any $\gamma > 0$.

P3 $\mathcal{T}(A, \rho) \cap \mathcal{T}(B, \rho) = \mathcal{T}(A \cap B, \rho)$ and $\mathcal{T}(A, \rho) \cup \mathcal{T}(B, \rho) = \mathcal{T}(A \cup B, \rho)$.

As a consequence one can define also time tubes of unions of intervals on the same isochrone by the union of the corresponding time tubes.

Let $1/2 < \delta < 1$ be given, as well as the constants ν and σ from property **P1**. For constructing a time tube between two given Euler paths, we also need a deviation factor $\gamma > 0$. For any initial point $p \in \mathbf{R}^2$, $p \neq 0$, define

$$\mathcal{T}_k^\gamma(p) = \left\{ \mathcal{O}_{C^k r(p)+s}\big(u(x; s), \sigma\gamma \cdot \tau^\delta(C^k p)\big) \;\middle|\; \begin{array}{l} x \in \mathcal{O}_{C^k r(p)}(C^k p, \alpha_k^\gamma), \\ -\rho_{k+1}^\gamma < s < C^k \tau(p) + \rho_{k+1}^\gamma \end{array} \right\}$$

$k = 0, 1, \ldots$, with

$$\begin{aligned} \alpha_0^\gamma &= 0 \\ \alpha_k^\gamma &= C\alpha_{k-1}^\gamma + \sigma\gamma \cdot \tau^\delta(C^{k-1}p) \end{aligned}$$

and

$$\begin{aligned} \rho_0^\gamma &= 0, \\ \rho_k^\gamma &= \rho_{k-1}^\gamma + \sigma\gamma \cdot \tau^\delta(C^{k-1}p), \end{aligned}$$

for $k = 1, \ldots$. Note that

$$\alpha_k^\gamma = \sigma\gamma \cdot \tau^\delta(C^{k-1}p) \cdot (1 + C^{1-\delta} + \cdots + C^{(k-1)(1-\delta)}) = \sigma\gamma \cdot \tau^\delta(p) \frac{C^k - C^{k\delta}}{C - C^\delta} \tag{5.9}$$

$$\rho_k^\gamma = \sigma\gamma \cdot \tau^\delta(p) + \cdots C^{(k-1)\delta}\sigma\gamma \cdot \tau^\delta(p) = \sigma\gamma \cdot \tau^\delta(p) \frac{C^{k\delta} - 1}{C^\delta - 1}. \tag{5.10}$$

Additionally, we set $\mathcal{T}^\gamma(p) = \cup_{k=0}^\infty \mathcal{T}_k^\gamma(p)$.

A rough explanation of these quantities is the following. Compare the random walk and the dynamical system starting at a sufficiently large point p. The parameter α_k^γ is the cumulative dispersion during the first k cycles parallel to isochrones, and ρ_k^γ the associated cumulative (time)dispersion along the dynamical system having almost all probability mass. The first one blows up by a factor C, each time a cycle has been passed, in addition to 'noise' incurred while passing the last cycle, which scales by a factor in the power δ. The second only consists of the added 'noise' term, because of (5.7). The set $\mathcal{T}_k^\gamma(p)$ contains the set of 'most likely' realisations of the random walk *during* the $(k+1)$th cycle of the dynamical system.

We will first prove the next theorem.

Theorem 5.1 *Let $1/2 < \delta < 1$, $\gamma > 0$. Then there exist positive constants c', depending on δ, and d', depending on δ and γ, such that for any sufficiently big initial point $p \in \mathbf{Z}^2$ and any t*

$$\mathsf{P}\Big\{\xi_n \in \mathcal{T}^\gamma(p),\ \textit{for all } n < t \textit{ and } |r(p) + t - r(\xi_t)| < d'\sigma\gamma \cdot t^\delta \;\big|\; \xi_0 = p\Big\} \geq 1 - \frac{c'}{\gamma^2} \cdot \tau^{1-2\delta}(p), \tag{5.11}$$

with σ from Property **P1**.

Proof. Denote by $t_k = t_k(p)$ the time the dynamical system u requires for passing precisely k cycles when starting at p. Then $t_0 = 0$ and

$$t_k = t_{k-1} + C^{k-1}\tau(p) = \frac{C^k - 1}{C - 1}\tau(p) = (C^k - 1)r(p). \tag{5.12}$$

We will use an induction argument. To this end we first need some further notation. For notational convenience we will suppress the dependence on γ in our notation.

Let $A_0 = \{p\}$, and

$$A_k = \mathcal{O}_{C^k r(p)}(CA_{k-1}, \sigma\gamma\tau^\delta(C^{k-1}p)), \quad k \geq 1. \tag{5.13}$$

The underlying idea is that starting in a point from A_{k-1} after the $(k-1)$th cycle completion, the random walk can deviate from the dynamical system by a 'distance' of at most $\sigma\gamma \cdot \tau^\delta(C^{k-1}p)$ (with high probability). Moreover, the set $A_k \subset \mathcal{I}(C^k p)$ is the intersection of the set $\mathcal{T}_{k-1}(p)$, $k \geq 1$, with the isochrone of the point $C^k p$:

$$A_k = \mathcal{O}_{C^k r(p)}(C^k p, \alpha_k). \tag{5.14}$$

Furthermore, by (5.7)

$$u\big(\mathcal{T}(A_k, \rho_k); \tau(C^k p)\big) = \mathcal{T}\big(u\big(A_k; \tau(C^k p)\big), \rho_k\big) = \mathcal{T}(CA_k, \rho_k) \subset \mathcal{T}(A_{k+1}, \rho_k).$$

It suffices to show the following statement. There exist a constant c'' and $k \geq 1$ with $t_{k-1} < t \leq t_k$, such that

$$\begin{aligned} \mathsf{P}\left\{ \begin{array}{l} \xi_n(p) \in \cup_{l=0}^{k-1} \mathcal{T}_l(p), n \leq t \\ \xi_t(p) \in \mathcal{T}\big(\mathcal{O}_{C^{k-1}r(p)+s}\big(u(A_{k-1}; s), \sigma\gamma s^\delta\big), \rho_{k-1} + \sigma\gamma s^\delta\big) \end{array} \right\} & \\ \geq \; 1 - \frac{c''}{\gamma^2}\tau^{1-2\delta}(p) \cdot \frac{C^{k(1-2\delta)} - 1}{C^{1-2\delta} - 1}, & \end{aligned} \tag{5.15}$$

where $s = t - t_{k-1}$.

Let us first argue that (5.15) implies (5.11). Indeed,

$$\xi_t(p) \in \mathcal{T}\big(\mathcal{O}_{C^{k-1}r(p)+s}\big(u(A_{k-1}; s), \sigma\gamma s^\delta\big), \rho_{k-1} + \sigma\gamma s^\delta\big), \quad s = t - \rho_{k-1},$$

implies

$$|r(\xi_t) - r(p) - t| < \rho_{k-1} + \sigma\gamma s^\delta,$$

since $r(C^{k-1}p) + s = t_{k-1} + r(p) + s = t + r(p)$. Further, provided that $\tau(p) > 1$, we have $\tau^\delta(p) < \tau(p)$. So, for $k \geq 2$ we have by (5.10) and (5.12) that

$$\begin{aligned} \frac{\rho_{k-1}}{\sigma\gamma t_{k-1}^\delta} &= \frac{C^{(k-1)\delta} - 1}{C^\delta - 1} \cdot \frac{(C-1)^\delta}{(C^{k-1} - 1)^\delta} \\ &= \frac{(C-1)^\delta}{C^\delta - 1} \cdot \frac{1 - C^{-(k-1)\delta}}{(1 - C^{-(k-1)})^\delta} \to \frac{(C-1)^\delta}{C^\delta - 1}, \end{aligned}$$

as $k \to \infty$. Hence there is a constant, $d'' > 1$ say, such that $\rho_{k-1}/\sigma\gamma \leq d'' t_{k-1}^\delta$. Clearly, for $k = 1$, $\rho_{k-1} = 0 = t_{k-1} \leq d'' t_{k-1}$. As a consequence,

$$\rho_{k-1} + \sigma\gamma s^\delta < \sigma\gamma\big(\frac{\rho_{k-1}}{\sigma\gamma} + s^\delta\big) < \sigma\gamma\big(d'' t_{k-1}^\delta + s^\delta\big) < 2d''\sigma\gamma(t_{k-1} + s)^\delta \leq 2d''\sigma\gamma t^\delta.$$

Note that $1 - 2\delta < 0$, and so $C^{k(1-2\delta)} \downarrow 0$, as $k \to \infty$. Thus (5.15) implies for any t and $k \geq 1$ with $t_{k-1} < t \leq t_k$, that

$$\mathsf{P}\left\{ \begin{array}{l} \xi_n(p) \in \cup_{l=0}^{k-1} \mathcal{T}_l(p), \, n \leq t \\ \big|r\big(\xi_t(p)\big) - t - r(p)\big| < 2d''\sigma\gamma \cdot t^\delta \end{array} \right\} \geq 1 - \frac{c''}{\gamma^2(1 - C^{1-2\delta})} \cdot \tau^{1-2\delta}(p).$$

Putting $c' = c''/(1 - C^{1-2\delta})$ and $d' = 2d''$, proves that the assertion from the lemma follows from (5.15). We will now show the validity of (5.15) by induction to the number of cycles.

Let first $t \leq t_1$. Then u has at most 5 face transitions, when starting at p. By Lemma 3.2 (for convenience we use a version with $<$-sign instead of the given one with $\leq$-sign, but the proof is analogous) there exists a constant c, such that for any sufficiently large p and for any constant v satisfying (3.3)

$$\mathsf{P}\big\{ \max_{n \leq t} \|\xi_n(p) - u(p; n)\| < v \big\} \geq 1 - c \cdot \frac{t}{v^2}.$$

Put $v = \gamma t^\delta$. Obviously (3.3) is satisfied for initial state p sufficiently big. Thus

$$\big\{ \max_{0 \le n \le t} \|\xi_n(p) - u(p;n)\| < \gamma t^\delta \big\} = \cap_{n=0}^t \big\{ \xi_n(p) \in O\big(u(p;n), \gamma t^\delta\big) \big\}.$$

By **P1**,

$$O\big(u(p;n), \gamma t^\delta\big) \subset \mathcal{T}\big(\mathcal{O}_{r(p)+n}\big(u(p;n), \sigma\gamma t^\delta\big), \sigma\gamma t^\delta\big).$$

In turn, this implies that

$$\begin{aligned}\big\{ \max_{0 \le n \le t} \|\xi_n(p) - u(p;n)\| < \gamma t^\delta \big\} \quad &\subset \quad \cap_{n=0}^t \big\{ \xi_n(p) \in \mathcal{T}\big(\mathcal{O}_{r(p)+n}\big(u(p;n), \sigma\gamma t^\delta\big), \sigma\gamma t^\delta\big) \big\} \\ &\subset \quad \mathcal{T}_0(p) \cap \big\{ \xi_t(p) \in O\big(u(p;t), \gamma t^\delta\big) \big\}.\end{aligned}$$

Hence,

$$\mathsf{P}\left\{ \begin{array}{l} \xi_n(p) \in \mathcal{T}_0(p),\ n \le t \\ \xi_t(p) \in \mathcal{T}\big(\mathcal{O}_{r(p)+t}\big(u(p;t), \sigma\gamma t^\delta\big), \sigma\gamma t^\delta\big) \end{array} \right\} \ge 1 - \frac{c}{\gamma^2} \cdot t^{1-2\delta} \ge 1 - \frac{c}{\gamma^2} \tau^{1-2\delta}(p),$$

so that (5.15) holds for $t \le t_1$, when choosing $c'' = c$. Do note, that the term $1 - (c''/\gamma^2)\tau^{1-2\delta}(p) > 0$ for sufficiently large p.

Assume now that the statement holds for time periods $t \le t_K$ for some integer $K > 1$. We will show that (5.15) holds for $t_K < t \le t_{K+1}$ as well.

Note, that the event in the left-hand side of (5.15) implies the event

$$\xi_{t_K}(p) \in \mathcal{T}\big(\mathcal{O}_{C^K r(p)}\big(u(A_{K-1}; \tau(C^{K-1}p), \sigma\gamma\tau^\delta(C^{K-1}p)\big), \rho_{K-1} + \sigma\gamma\tau^\delta(C^{K-1}p)\big) = \mathcal{T}(A_K, \rho_K).$$

Suppose $\xi_{t_K} = q \in \mathcal{T}(A_K, \rho_K)$. For bounding the probability of a large deviation of the random walk between t_K and t from the dynamical system starting at q, we would like to apply Lemma 3.2 with the *same constant* c as for the case of $t \le t_1$. In particular, we would like to bound

$$\mathsf{P}\{ \max_{0 \le n \le s} \|\xi_n(q) - u(q;n)\| < \gamma s^\delta \},$$

where $s = t - t_k$.

The constant c depends on the number of face-transitions. Clearly, the number of face-transitions between time t_K and t is at most 5 like before. As an additional requirement we need to check that γs^δ satisfies (3.3).

Observe that $s \le \tau(C^K p)$ and $r(q) > r(C^K p) - \rho_K$. Moreover, by (5.10) we have

$$\begin{aligned}\rho_K \quad &= \quad \frac{(C-1)^\delta}{C^\delta - 1} \sigma\gamma r^\delta(p) \cdot (C^{K\delta} - 1) \\ &< \quad \frac{(C-1)^\delta}{C^\delta - 1} \sigma\gamma r^\delta(C^K p).\end{aligned}$$

Hence,

$$\frac{s}{\tau(q)} \le \frac{r(C^K p)}{r(C^K p) - \rho_K} \le \frac{1}{1 - \sigma\gamma r^{\delta-1}(C^K p) \cdot (C-1)^\delta/(C^\delta - 1)}.$$

For any $\epsilon > 0$, this is smaller than $1 + \epsilon$, provided that p is big enough. As a consequence, for $K \ge 1$, we have $\gamma s^\delta < \theta \cdot \tau(q)$, any $q \in \mathcal{T}(A_K, \rho_K)$, and $s = t - t_K \le t_{K+1} - t_K$, provided p is sufficiently big, for θ the constant from condition (3.3). Thus we can apply Lemma 3.2 with the same constant c as in the above. This yields that

$$\mathsf{P}\{ \max_{0 \le n \le s} \|\xi_n(q) - u(q;n)\| < \gamma s^\delta \} \ge 1 - \frac{c}{\gamma^2} \cdot s^{1-2\delta} \ge 1 - \frac{c''}{\gamma^2} \cdot \tau^{1-2\delta}(C^K p). \tag{5.16}$$

By (5.7)

$$u(q;n) \in u\big(\mathcal{T}(A_K, \rho_K); n\big) = \mathcal{T}(u(A_K;n), \rho_K), \quad 0 \le n \le s.$$

By **P1**

$$\mathcal{O}(u(q;n), \gamma s^\delta) \subset \mathcal{T}\big(\mathcal{O}_{C^K r(p)+n}\big(u(A_K;n), \sigma\gamma s^\delta\big), \rho_K + \sigma\gamma s^\delta\big) \subset \mathcal{T}_K(p).$$

As a consequence, given that $q \in \mathcal{T}(A_K, \rho_K)$,

$$\left\{ \begin{array}{l} \xi_n(q) \in \mathcal{T}_K(p), 0 \le n \le s \\ \xi_t(p) \in \mathcal{T}\big(\mathcal{O}_{C^K r(p)+s}\big(u(A_K; s), \sigma\gamma s^\delta\big), \rho_K + \sigma\gamma s^\delta\big) \end{array} \right\} \supset$$

$$\supset \bigcap_{n=0}^{s} \{\xi_n(q) \in \mathcal{O}(u(q;n), \gamma s^\delta)\} = \{\max_{0 \le n \le s} \|\xi_n(q) - u(q;n)\| < \gamma s^\delta\}$$

For $t_K < t \le t_{K+1}$ we finally have

$$\begin{aligned}
&\mathsf{P}\left\{ \begin{array}{l} \xi_n(p) \in \cup_{l=0}^K \mathcal{T}_l(p), n \le t \\ \xi_t(p) \in \mathcal{T}\big(\mathcal{O}_{C^K r(p)+s}\big(u(A_K; s), \sigma\gamma s^\delta\big), \rho_K + \sigma\gamma s^\delta\big) \end{array} \right\} \ge \\
&\ge \sum_{q \in \mathcal{T}(A_K, \rho_K)} \mathsf{P}\{\xi_{t_K}(p) = q,\, \xi_n(p) \in \cup_{l=0}^{K-1} \mathcal{T}_l(p), 0 \le n \le t_K\} \times \mathsf{P}\{\max_{0 \le n \le s} \|\xi_n(q) - u(q;s)\| \le \gamma s^\delta\} \\
&\ge \Big(1 - \frac{c''}{\gamma^2} \tau^{1-2\delta}(p) \cdot \frac{C^{K(1-2\delta)} - 1}{C^{1-2\delta} - 1}\Big) \times \Big(1 - \frac{c''}{\gamma^2} \tau^{1-2\delta}(p) \cdot C^{K(1-2\delta)}\Big) \\
&\ge 1 - \frac{c''}{\gamma^2} \tau^{1-2\delta}(p) \cdot \frac{C^{(K+1)(1-2\delta)} - 1}{C^{1-2\delta} - 1},
\end{aligned}$$

where (5.16) and the induction assumption were used for the second inequality. This shows (5.15), as we wanted. □

For showing (5.5) as well as for future reference, we will examine the sets $\mathcal{T}^\gamma(p)$ more closely. We use the notation from the proof of Theorem 5.1. Suppose that $p \in \mathcal{I}(s)$, $s \ge 1$, i.e. $r(p) = s$. Fix $\gamma > 0$ and use σ from **P1**. Define by T_n^s the projection of the time tube $\mathcal{T}_n^\gamma(p)$ *along Euler paths* onto the isochrone $\mathcal{I}(s)$ at distance s:

$$T_n^s = \{x \in \mathcal{I}(s) \,|\, \Gamma_x \cap \mathcal{T}_n^\gamma(p) \neq \emptyset\}.$$

Since $T_n^s \subset \mathcal{I}(s)$, and $\mathcal{I}(s)$ compact, $\limsup_{n \to \infty} T_n^s$ exists as an open subset of $\mathcal{I}(s)$. We will determine the limsup.

The projections T_n^s are determined by the set (cf. (5.14))

$$\mathcal{O}_{C^n r(p) - \rho_{n+1}^\gamma}\big(u(A_n; -\rho_{n+1}^\gamma), \sigma\gamma \cdot \tau^\delta(C^n p)\big).$$

Projecting $u(A_n; -\rho_{n+1}^\gamma)$ back to $\mathcal{I}(s)$ is the same as projecting A_n back along the Euler path to $\mathcal{I}(s)$. This map is given by (cf. (5.14))

$$\begin{aligned}
u\big(u(A_n; -\rho_{n+1}^\gamma); 1 - C^n r(p) + \rho_{n+1}^\gamma\big) &= u(A_n; 1 - C^n r(p)) = \mathcal{O}_s(p, \alpha_n^\gamma / C^n) \\
&= \mathcal{O}_s\Big(p, \sigma\gamma \cdot \tau^\delta(p) \frac{1 - C^{-n(1-\delta)}}{C - C^\delta}\Big) \\
&\uparrow \mathcal{O}_s\big(p, \sigma\gamma \cdot \tau^\delta(p) \frac{1}{C - C^\delta}\big), \quad n \to \infty.
\end{aligned} \tag{5.17}$$

Further, whenever $\rho_{n+1}^\gamma < \tau(C^{n-1}p)$, one has

$$u\Big(\mathcal{O}_{C^n r(p) - \rho_{n+1}^\gamma}\big(u(A_n; -\rho_{n+1}^\gamma), \sigma\gamma \cdot \tau^\delta(C^n p)\big); \rho_{n+1}^\gamma\Big) \subset \mathcal{O}_{C^n r(p)}\big(A_n, \sigma\gamma\tau^\delta(C^n p) \cdot C\big).$$

Hence,

$$\begin{aligned}
&u\Big(\mathcal{O}_{C^n r(p) - \rho_{n+1}^\gamma}\big(u(A_n; -\rho_{n+1}^\gamma), \sigma\gamma \cdot \tau^\delta(C^n p)\big); 1 - C^n r(p) + \rho_{n+1}^\gamma\Big) \\
&\qquad \subset \mathcal{O}_s\Big(u(A_n; 1 - C^n r(p)), \sigma\gamma \cdot \frac{\tau^\delta(C^n p)}{C^{n-1}}\Big) = \mathcal{O}_s\Big(p; \frac{\alpha_n^\gamma}{C^n} + \sigma\gamma \cdot \frac{\tau^\delta(C^n p)}{C^{n-1}}\Big) \\
&\qquad \subset \mathcal{O}_s\Big(p, \sigma\gamma \cdot \tau^\delta(p) \frac{1 + C^{-n(1-\delta)}(C^2 - C^{1+\delta} - 1)}{C - C^\delta}\Big) \\
&\qquad \to \mathcal{O}_s\big(p, \sigma\gamma \cdot \tau^\delta(p) \frac{1}{C - C^\delta}\big), \; n \to \infty.
\end{aligned}$$

This holds for all sufficiently large p. Similarly,

$$\begin{aligned} u\Big(\mathcal{O}_{C^n r(p)-\rho^\gamma_{n+1}}\big(u(A_n;-\rho^\gamma_{n+1}),\sigma\gamma\cdot\tau^\delta(C^n p)\big);1-C^n r(p)+\rho^\gamma_{n+1}\Big) \\ \supset\quad \mathcal{O}_s\Big(p,\sigma\gamma\cdot\tau^\delta(p)\frac{1+C^{-n(1-\delta)}(C-C^\delta-1)}{C-C^\delta}\Big) \\ \to\quad \mathcal{O}_s\big(p,\sigma\gamma\cdot\tau^\delta(p)\frac{1}{C-C^\delta}\big),\ n\to\infty. \end{aligned}$$

It follows that

$$\limsup_{n\to\infty} T^s_n = \mathcal{O}_s\Big(p,\sigma\gamma\cdot\tau^\delta(p)\frac{1}{C-C^\delta}\Big). \tag{5.18}$$

In particular, for all sufficiently large p

$$\begin{aligned} \mathcal{O}_s\big(p,\sigma\gamma\cdot\tau^\delta(p)\frac{1+\min\{0,C^{-n(1-\delta)}(C-C^\delta-1)\}}{C-C^\delta}\big)\subset \inf_{m\ge n} T^s_n \subset \\ \subset \sup_{m\ge n} T^s_n \subset \mathcal{O}_s\Big(p,\sigma\gamma\cdot\tau^\delta(p)\frac{1+\max\{0,C^{-n(1-\delta)}(C^2-C^{1+\delta}-1)\}}{C-C^\delta}\Big). \end{aligned} \tag{5.19}$$

As a consequence, for all sufficiently large p

$$\begin{aligned} \big\{\Gamma_x\,|\,x\in\mathcal{O}_s\big(p,\sigma\gamma\cdot\tau^\delta(p)\frac{1+\min\{0,C^{-n(1-\delta)}(C-C^\delta-1)\}}{C-C^\delta}\big)\big\}\cap\{q\,|\,r(q)>C^n s\}\subset \\ \subset \bigcup_{m\ge n} T^\gamma_m(p)\subset\big\{\Gamma_x\,|\,x\in\mathcal{O}_s\big(p,\sigma\gamma\cdot\tau^\delta(p)\frac{1+\max\{0,C^{-n(1-\delta)}(C^2-C^{1+\delta}-1)\}}{C-C^\delta}\big)\big\}. \end{aligned} \tag{5.20}$$

The construction together with Theorem 5.1 show the validity of the following Lemma.

Lemma 5.4 *For any $x,y\in\mathcal{I}$, $x\ne y$, and any point $p\in\mathbf{Z}^2$, there exists a time $N>0$, such that (5.5) is valid, that is, $(\Gamma_x\rightsquigarrow\Gamma_y)$ is a sojourn set.*

Proof. Let $x,y\in\mathcal{I}$ be given, $x\ne y$, as well as $p\in\mathbf{Z}$. Choose any $p'\in(x\rightsquigarrow y)$, $p'\ne x,y$. Let $1/2<\delta<1$ be given. Note that $\tau(p')=C-1$. Choose $\gamma>0$ such that

$$\mathcal{O}_1\big(p',\sigma\gamma\cdot(C-1)^\delta\max\{1/(C-C^\delta),C\}\big)\subset(x\rightsquigarrow y).$$

Then by (5.20) $\mathcal{T}^\gamma(p')\subset(\Gamma_x\rightsquigarrow\Gamma_y)$.

By Theorem 5.1 there exists a positive constant c', such that for any sufficiently big initial point q and any time t,

$$\mathsf{P}\{\xi_n\in\mathcal{T}^\gamma(q)\,|\,\xi_0=q\}\ge 1-\frac{c'}{\gamma^2}\cdot\tau^{1-2\delta}(q).$$

Choose $q=C^{k'}p'$ for some large enough k'. Then it is easily checked that $\mathcal{T}^\gamma(q)\subset\mathcal{T}^\gamma(p')\subset(\Gamma_x\rightsquigarrow\Gamma_y)$. Hence,

$$\mathsf{P}\{\xi_n\in(\Gamma_x\rightsquigarrow\Gamma_y)\,|\,\xi_0=q\}\ge 1-c'\cdot\tau^{1-2\delta}(q).$$

Because of irreducibility, there is a path of positive probability from the selected point p to q, say it has length m and the probability equals π. Then

$$\mathsf{P}\{\xi_n\in(\Gamma_x\rightsquigarrow\Gamma_y),n>m\,|\,\xi_0=p\}\ge\pi\cdot\big(1-c'\tau^{1-2\delta}(q)\big).$$

□

6 Non-existence of the limit for the time scaled process

6.1 Large initial points

Lemma 5.4 shows, that the random walk starting at a large but *fixed* point will end up with positive probability "close" to any Euler path. This suggests that scaling back along the Euler path will yield convergence in distribution, as will be the subject of the paper [7]. This scaling is given by

$$\xi_t \to u\big(\xi_t; t+1-r(\xi_t)\big).$$

The limit distribution provides the probability mass of sets of Euler paths that the process may end up in. In general, the scattering is called discrete or continuous whenever the limit distribution (provided it exists!) under this scaling is discrete or continuous.

The time scaled process can only converge, when with time scaled Euler paths $\{u(x;t)/t\}_{t\geq 0}$ one can associate precisely one point. In our case, time scaling yields a cycling set of points of the isochrone, as time goes by, thus accounting for non-convergence of the time scaled process.

We will prove the non-convergence property for all sufficiently large initial points p.

Theorem 6.1 *For all initial points $p \in \mathbf{Z}$, except possibly a compact set, the time scaled process $\xi_{[tN]}(p)/N$ does* not *converge in distribution for any macro time t.*

Proof. Fix a reference point $x_0 \in \mathcal{I}$. For $m \in \mathbf{N}$ to be determined later, split the isochrone $\mathcal{I}(s)$ into m 'equal' parts as follows: $\mathcal{I}(s) = \cup_{l=1}^m \mathcal{I}_l(s)$, with

$$\mathcal{I}_l(s) = \big\{x \in \mathcal{I}(s) \,\big|\, \frac{l-1}{m} \leq \frac{\psi_{x_0}(x)}{C-1} < \frac{l}{m}\big\},$$

where ψ_{x_0} is the 'angle' defined in §5.2. Note that $\mathcal{I}_l(s) = u\big(\mathcal{I}_l(1); s-1\big)$ by virtue of Lemma 5.3. Moreover, for $t = t_k + ((r-l)/m)C^k(C-1)$, $l \leq r < l+m$

$$\frac{1}{t+1} u\big(\mathcal{I}_l(1); t\big) = \mathcal{I}_{r\,(\text{mod } m)}(1),$$

with $t_k = C^k - 1$ is the time for an Euler path starting at a point of $\mathcal{I}$ to pass precisely k cycles.

Fix a point $q \in \mathcal{I}_1^\circ(1)$, where superscript $^\circ$ denotes the interior of a set. Again we use the notation from the proof of Theorem 5.1 and σ from **P1**. Let $\epsilon \ll 1$. Choose $\gamma > 0$ and $1/2 < \delta < 1$. By the proof of Theorem 5.1, there exists a constant c'' such that (5.15) holds for any sufficiently big initial point p. By (5.19), for all r large enough, one can take the generic point $p = \lfloor C^r q \rfloor$, such that

i) (5.15) holds, with

$$\mathsf{P}\big\{\xi_t(p) \in \mathcal{T}\big(\mathcal{O}_{C^{r+k}+s}\big(u(A_k; s), \sigma\gamma s^\delta\big), \rho_k^\gamma + \sigma\gamma s^\delta\big\} > 1-\epsilon, \quad t_{r+k} \leq t < t_{r+k+1}, \tag{6.1}$$

where $s = t - t_{r+k}$;

ii) $\mathcal{O} = \mathcal{O}_1\big(q, \sigma\gamma \cdot C^{r\delta - r}(C-1)^\delta \max\{1/(C-C^\delta), C\}\big) \subset \mathcal{I}_1^\circ(1)$, so that

$$\mathcal{T}^\gamma(p) \subset \big\{\Gamma_x \,|\, x \in \mathcal{O}_1(q, \sigma\gamma \cdot C^{r\delta-r}(C-1)^\delta \max\{1/(C-C^\delta), C\})\big\} \subset \big\{\Gamma_x \,|\, x \in \mathcal{I}_1^\circ(1)\big\}.$$

Define the infinite sequence $m_k \in \{0, 1, \ldots, m-1\}$, $k = 1, \ldots$. Then one can find an increasing sequence of times t'_k with $t_k < t'_k < t_{k+1}$, such that $u(\mathcal{O}; t'_k)/(1+t'_k) \subset \mathcal{I}_{m_k}^\circ(1)$. For k large enough, this implies for $\beta_k = \rho_k^\gamma + \sigma\gamma \cdot (t'_k - t_k)^\delta$ that

$$\bigcup_{t'_k - \beta_k < t' < t'_k + \beta_k} \frac{1}{1+t'} u(\mathcal{O}; t') \subset \mathcal{I}_{m_k}^\circ(1),$$

since $(t'_k \pm \beta_k)/t'_k \to 1$, as $k \to \infty$. Form the cones $C_l = \{\lambda \mathcal{I}_l(1), \lambda > 0\}$. In other words, k large enough

$$\bigcup_{t'_k - \beta_k < t' < t'_k + \beta_k} u(\mathcal{O}; t') \subset C_{m_k}^\circ.$$

As a result we have for k large enough

$$\mathsf{P}\{\xi_{t'_k}(p) \in C_{m_k}\} \geq 1 - \epsilon.$$

In words, the process keeps on cycling through different cones, as time goes by. But this implies that the sequence $\xi_{t'_k}(p)/t'_k$ cannot converge in distribution.

By using finitely many different choices of the reference point x_0, non convergence can be shown for all p outside a compact set. $\square$

6.2 Invariant measure

The question is left, whether there can be convergence at all, and under what conditions. It turns out, that if the scaled process converges in distribution, then the limiting distribution should be invariant with respect to the dynamical system. We prove the latter.

Let an initial point p be given. Let $\mathcal{B}^2$ denote the σ-algebra of Borel sets of $\mathbf{R}^2$. So, we assume that the sequence $\xi_N(p)/N$ converges in distribution to a random vector ξ on $(\mathbf{R}^2, \mathcal{B}^2)$ with distribution $\mu(A) = \mathsf{P}\{\xi \in A\}$ for any Borel-measurable subset of $\mathbf{R}^2$. Note further, that this is equivalent to the sequence $\xi_{[tN]}(p)/N$ converging in distribution to the vector $t\xi$.

Remind that the Euler distance $r(\xi_N(p)/N)$ of the time scaled process $\xi_N(p)/N$ a.s. converges to 1. Moreover, $\xi_N(p)/N \in A$ iff $\xi_N(p) \in NA$. This suggests to first study the measure μ of cone-type sets, defined by two non-intersecting curves starting at the origin. By virtue of Lemma 5.1, it is tempting to state that the measure μ is concentrated on the isochrone $\mathcal{I}$. One can identify it with a measure on $\mathcal{I}$ only if certain smoothness properties hold. This will follow from the analysis below.

With each interval $[x \rightsquigarrow y] \subset \mathcal{I}(s)$ one can associate a cone $A_{[x\rightsquigarrow y]} = \{\lambda[x \rightsquigarrow y] \,|\, \lambda > 0\}$. Analogously we define cones associated with open, or half open connected subsets of $\mathcal{I}$. Note that $q \in A_{[x\rightsquigarrow y]}$ if and only if $u\big(q; (t/s)\cdot r(q)\big) \in A_{u([x\rightsquigarrow y];t)}$. Further, write $\delta A = \bar{A} \setminus A^\circ$ for the boundary of set A.

We will work under condition 5.1, but this is only because of the necessary bounds already being available.

Lemma 6.1 *Assume that ξ_N satisfies condition 5.1. One has $\mu\big\{\delta A_{[x\rightsquigarrow y]}\big\} = 0$, for any cone $A_{[x\rightsquigarrow y]}$.*

Proof. Let $p = 0$ for simplicity. The case of arbitrary p complicates the choice of suitable constants in the estimates, but otherwise is not essentially different. Also, drop the dependence on p in the notation.

It is sufficient to prove that $\mu(\lambda \cdot x \,|\, \lambda > 0\} = 0$, for any $x \in \mathcal{I}$. Write $A_x = \{\lambda x \,|\, \lambda > 0\}$. Assume that $\mu\{A_x\} = \epsilon > 0$ for some $x \in \mathcal{I}$. We will prove that in that case $\mu\{A_y\} \geq \epsilon$ for any $y \in \mathcal{I}$. This contradicts finiteness of the measure μ.

Choose any $t < C - 1$. Write $B = A_{u(x;t)} = A_{u(x;t)/(1+t)}$. We will show that $\mu\{B\} \geq \mu\{A_x\}$. These two sets are related by the map $y \to u(y; t \cdot r(y))$. So, in order that $\xi_{[(1+t)N]}$ be sufficiently close to B, ξ_N needs to be close enough to A.

For our purpose it is sufficient to construct open sets A_γ, B_γ, with $A_\gamma \downarrow A_x$, and $\overline{B}_\gamma \downarrow B$, as $\gamma \to 0$, such that

$$\liminf_{N\to\infty} \mathsf{P}\{\xi_{[(1+t)N]} \in B_\gamma\} \geq \liminf_{N\to\infty} \mathsf{P}\{\xi_N \in A_\gamma\}. \tag{6.2}$$

Indeed, by assumption this implies

$$\begin{aligned}
\mu\{A_\gamma\} &\leq \liminf_{N\to\infty} \mathsf{P}\{\xi_N \in A_\gamma\} \\
&\leq \liminf_{N\to\infty} \mathsf{P}\{\xi_{[(1+t)N]} \in \overline{B}_\gamma\} \\
&\leq \limsup_{N\to\infty} \mathsf{P}\{\xi_{[(1+t)N]} \in \overline{B}_\gamma\} \\
&\leq \mu\{\overline{B}_\gamma\}.
\end{aligned}$$

As a consequence

$$\mu\{B\} = \lim_{\gamma\to 0} \mu\{\overline{B}_\gamma\} \geq \lim_{\gamma\to 0} \mu\{A_\gamma\} = \mu\{A_x\}.$$

Now, two factors for causing dispersion have to be taken into account in constructing suitable sets. The first is that the 'most likely paths' at point $q \in A_\gamma$, with $r(q)$ large, are 'close' to the Euler paths $u(q;\cdot)$ by Theorem 5.1, but not equal to them. These 'most likely paths' should end up in B_γ at time $[t \cdot r(q)]$.

Use the notation from the proof of Theorem 5.1 as well as σ, ν from Property **P1**. Choose $\gamma > 0$, $1/2 < \delta < 1$. Set $A_\gamma = \cup_{q \in A_x} \mathcal{O}_{r(q)}(q, \gamma)$. This set is open. For any point $q \in \mathbf{R}^2$, write $d(q,\gamma) = \sigma\gamma t^\delta \cdot r^\delta(q)$. Then by (5.15) the 'most likely paths' starting at point $q \in A_\gamma$, end up in the set $\mathcal{T}\big(\mathcal{O}_{r(q)(1+t)}\big(u(q; t \cdot r(q)), d(q,\gamma)\big), d(q,\gamma)\big)$ at time $t \cdot r(q)$.

The second fact to be taken into account, is that at time N, we only have that $r(\xi_N)$ is approximately equal to N. Starting at point $q \in A_\gamma$ with $r(q) \approx N$, one should therefore have that the 'most likely paths' end up in B_γ at time $[tN]$ instead of at time $[t \cdot r(q)]$. We will estimate the difference in position at the different times.

Let $1/2 < \eta < \delta$, and $\epsilon > 0$. Denote $C_N = \{x \in \mathbf{R}^2 \,|\, |r(x) - N| < \epsilon N^\eta\}$. By virtue of (5.4) using the martingale defined in (5.2), there exists a constant d, such that

$$\mathsf{P}\{\xi_N \in C_N\} \geq 1 - 2\exp\{-d\epsilon^2 N^{2\eta-1}\}. \tag{6.3}$$

For $q \in C_N$, $|[r(q)] - N| < \epsilon N^\eta$. The difference in positions at time $[tN]$ and $t \cdot r(q)$ is then bounded by

$$\|\xi_{[tN]}(q) - \xi_{[t\cdot r(q)]}(q)\| \leq \sqrt{2}\epsilon \cdot tN^\eta. \tag{6.4}$$

Then for $q \in A_\gamma \cap C_N$, the 'most likely paths' starting at q end up in

$$\mathcal{O}\big(u(q; t \cdot r(q)), \frac{1}{\nu} d(q,\gamma) + \sqrt{2}\epsilon t \cdot N^\eta\big)$$

at time $[tN]$. For $q \in C_N$ we have that $N^\eta \leq 2r^\eta(q)$. Combine this to define

$$B_\gamma = \cup_{q \in A_\gamma} \mathcal{O}\big(u(q; t \cdot r(q)), \frac{1}{\nu} d(q,\gamma) + 2\sqrt{2}\epsilon t \cdot r^\eta(q)\big).$$

B_γ is an open set, as can be deduced by explicitly writing the equation for its boundary. Observe that (6.3) implies

$$\mathsf{P}\{\xi_N \in A_\gamma \cap C_N\} \geq \mathsf{P}\{\xi_N \in A_\gamma\} - 2\exp\{-d\epsilon^2 N^{2\eta-1}\}.$$

Combination with (5.15) for $k = 1$ yields the existence of a constant c such that for all large values of N

$$\begin{aligned}
\mathsf{P}\{\xi_{[(1+t)N]} \in B_\gamma\} &\geq \sum_{q \in A_\gamma \cap C_N} \mathsf{P}\{\xi_N = q\} \mathsf{P}\{\xi_{[tN]}(q) \in B_\gamma\} \\
&\geq \sum_{q \in A_\gamma \cap C_N} \mathsf{P}\{\xi_N = q\} \Big(1 - \frac{c}{\gamma^2}(C-1)^{1-2\delta} r^{1-2\delta}(q)\Big) \\
&\geq \sum_{q \in A_\gamma \cap C_N} \mathsf{P}\{\xi_N = q\} \Big(1 - \frac{c}{\gamma^2}(C-1)^{1-2\delta}(N - \epsilon N^\eta)^{1-2\delta}\Big) \\
&\geq \Big(\mathsf{P}\{\xi_N \in A_\gamma\} - 2\exp\{-d\epsilon^2 N^{2\eta-1}\}\Big)\Big(1 - \frac{c}{\gamma^2}(C-1)^{1-2\delta}(N - \epsilon N^\eta)^{1-2\delta}\Big).
\end{aligned}$$

(6.2) immediately follows by taking $\liminf_{N\to\infty}$ on both sides. □

Desired invariance immediately follows.

Theorem 6.2 *Assume that ξ_N satisfies condition 5.1. The measure μ is invariant with respect to each cone $A_{[x \rightsquigarrow y]}$, $x, y \in \mathcal{I}$, i.e.*

$$\mu(A_{[x \rightsquigarrow y]}) = \mu\big(A_{[u(x;t) \rightsquigarrow u(y;t)]}\big), \quad t \geq 0. \tag{6.5}$$

The same applies for open and half open cones, by virtue of Lemma 6.1.

Proof of Theorem 6.2. It is sufficient to assume that $t \leq C - 1$. Assume $p = 0$, and select $x, y \in \mathcal{I}$. Write $A = A_{[x \rightsquigarrow y]}$ and $B = A_{[u(x;t) \rightsquigarrow u(y;t)]}$ and drop the dependence on p in the notation. Again conditioning on the state at time N, we can use precisely the same procedure as in the proof of the previous lemma, to obtain that $\mu(B) \geq \mu(A)$.

Now, write $x' = u(x;t)/(1+t)$ and $y' = u(y;t)/(1+t)$. Then note that we can map $B = A_{[x' \rightsquigarrow y']}$ along the dynamical system to A by

$$A = A_{[u(x';(C-1-t)/(1+t)) \rightsquigarrow u(y';(C-1-t)/(1+t))]}.$$

This implies that $\mu(A) \geq \mu(B)$. $\square$

It is now straightforward to compute the measure μ: it should be homogeneous with respect to the set of cones $A_{[x \rightsquigarrow y]}$. This can be seen, by splitting up the isochrone into k intervals that require the same time to be 'crossed'. By Theorem 6.2 their measures are equal and should be equal to $1/k$. The following corollary gives the general formula.

Corollary 6.3 *For $x, y \in \mathcal{I}$,*

$$\mu\{A_{[x \rightsquigarrow y]}\} = \frac{\psi_x(y)}{C-1}.$$

It is now an easy consequence, that the measure μ is smooth with respect to *any* cone. Take for instance a cone A' of the form

$$A' = \{\mathcal{O}_{r(q)}(q, \gamma r^\delta(q) \mid q \in A_{[x \rightsquigarrow y]}\},$$

for some positive γ and $0 < \delta < 1$. Choose monotone sequences $\{x_n\}_n$, $\{y_n\}_n \subset \mathcal{I}$, with $A_{[x_n \rightsquigarrow y_n]} \downarrow A_{[x \rightsquigarrow y]}$, $n \to \infty$. Then upto compact sets we have for any n

$$A_{[x \rightsquigarrow y]} \subset A_\gamma \subset A_{[x_n \rightsquigarrow y_n]}.$$

Hence,

$$\mu\{A_{[x \rightsquigarrow y]}\} \leq \mu\{A_\gamma\} \leq \mu\{A_{[x_n \rightsquigarrow y_n]}\}.$$

and by taking the limit $n \to \infty$ we obtain that $\mu\{A_{[x \rightsquigarrow y]}\} = \mu\{A_\gamma\}$. $\square$

A final remark should be made: if for some point p the time scaled process $\xi_N(p)/N$ converges in distribution to μ, then this μ is also the scattering measure on the set of Euler paths, for initial point p.

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