A Recurrence type Characterisation of μ-Exponential Ergodicity for Markov Processes

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Abstract

The paper studies a characterisation of μ-exponential ergodicity through the total expected μ-rewards until absorption into a finite set, for regular, standard, conservative Markov processes on a countable state space. This total expected reward condition is closely related to Tweedie's [26] necessary and sufficient condition for exponential ergodicity, which uses the intensity matrix. Our condition can also be interpreted as a condition on the μ-norm of the taboo transition probability matrix.

If the Markov process is uniformisable, the properties of our interest can be studied through an approximating discrete time Markov chain. The applicability of the total μ-reward condition is illustrated by it's verification for the Jackson network with an arbitrary, but finite number of servers in each node.

Keywords

Markov process, exponential ergodicity, exponential recurrence, taboo probabilities, Jackson network.

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1. Introduction

Consider a Markov Process (MP) $\{X(t)\}_{t\geq 0}$ on a countable state space E. We denote the transition matrix by $P(t)$, with ij^{th} entry $P_{ij}(t) = \mathbb{P}\{X(t) = j \mid X(0) = i\}$. The intensity matrix $Q = (q_{ij})_{i,j\in E}$ is defined as

$$q_{ij} := \lim_{t\downarrow 0} \frac{\big(P(t) - I\big)_{ij}}{t},$$

where I is the identity matrix. Throughout this article we make the following standard assumption.

Assumption: The MP is standard, conservative with no instantaneous states, i.e. $\forall\, i \in E$

$$\left.\begin{array}{lll} q_{ij} & \geq 0, & j \neq i \\ q_i := \sum_{j\neq i} q_{ij} & = -q_{ii} < \infty & \\ q_i & < \infty. & \end{array}\right\} \tag{1.1}$$

It is well-known (cf. e.g. Chung [3]) that the process remains an exponentially (q_i) distributed amount of time in state i, after which it jumps to state j with probability ${}^{J}P_{ij} = \frac{1-\delta_{ij}}{q_i} q_{ij}$. The matrix ${}^{J}P$ with ij^{th} entry ${}^{J}P_{ij}$ defines the so-called jumpchain $\{{}^{\mathrm{J}}X(n)\}_n$ of the process.

In [26] Tweedie derives a necessary and sufficient condition on the Q-matrix for an irreducible MP to be exponentially or strongly ergodic. This condition essentially requires a contraction type of property for all states except for some finite set. For our derivations it is convenient to have one condition for *all* initial states, which uses the taboo intensity matrix ${}_{M}Q$ with taboo set M, and ij^{th} entry

$${}_{M}q_{ij} = \begin{cases} q_{ij},\ j \notin M \vee i = j \in M \\ 0,\ \textit{otherwise}. \end{cases}$$

${}_{M}Q$ is easily seen to be the intensity matrix of the taboo process with taboo set M and transition matrix ${}_{M}P(t)$, $t \geq 0$, defined by

$${}_{M}P_{ij}(t) = \begin{cases} \delta_{ij} e^{-q_i t} + \mathbb{P}\{S_i < t, X(s) \notin M\ \forall s \in (S_i, t), X(t) = j \mid X(0) = i\}, & j \notin M \\ e^{-q_i t}, & j = i \in M \\ 0, & \textit{otherwise}, \end{cases}$$

where S_i denotes the sojourn time in state i. This definition follows Chung [3] (cf. Theorem II.11.6). Note that $\{{}_{M}P(t)\}_{t\geq 0}$ is a transition semi-group, if $\{P(t)\}_{t\geq 0}$ is.

The representation of Tweedie's criterion that will be used throughout the paper, is formulated below (cf. section 2 for a more detailed discussion).

Tweedie's criterion for exponential ergodicity: There are a vector μ, with $\mu_i \geq 1\ \ \forall\, i \in E$, a positive constant λ_R and a finite set $M \subset E$, such that the following holds

$$\sum_j {}_{M}q_{ij}\mu_j \leq -\lambda_R \mu_i, \ \ i \in E. \tag{1.2}$$

Through this formulation, the criterion has the same format as the necessary and sufficient condition for geometric ergodicity of discrete time MCs, which has been introduced in [16]. As we will sometimes refer to the latter, we will include the discrete time condition here. Write P for the transition matrix of a countable MC and ${}_MP$ for the corresponding taboo probability matrix, that is ${}_MP_{ij} = P_{ij}$, if $j \notin M$, and 0 otherwise. Note, that the taboo for the MP is imposed *after* the first exit from a state in M, whereas for a MC it is imposed at the next observation instant.

Popov's criterion for geometric ergodicity: There are a vector μ, with $\mu_i \geq 1 \ \ \forall i \in E$, a positive constant $\beta_R < 1$, and a finite set M, such that the following holds

$$\sum_j {}_MP_{ij}\mu_j \leq \beta_R \mu_i, \ \ i \in E. \tag{1.3}$$

In a recent paper [13] Meyn & Tweedie extend the result in [26] considerably, whilst allowing MPs with a general state space. For countable state MPs their result reads as follows.

PROPOSITION A: *Suppose there is a vector μ solving Tweedie's criterion for an irreducible* MP, *such that $\mu_i \to \infty$ as $i \to \infty$. Then there are positive constants λ_E and c, such that*

$$\|P(t) - \Pi\|_M = \sup_{i \in E} \mu_i^{-1} \sum_j |P_{ij}(t) - \Pi_{ij}|\mu_j \leq c e^{-\lambda_E t}, \ t \geq 0, \tag{1.4}$$

where $\Pi_{ij} = \lim_{t\to\infty} P_{ij}(t)$ is the stationary probability on being in state j, when the initial state is i. $\|\cdot\|_M$ denotes the weighted supremum norm with weighting vector μ. To make this more explicit, let $x = (x_i)_{i \in E}$ be a vector on E and A a matrix on $E \times E$. Then $\|x\|_M := \sup_{i\in E} \mu_i^{-1}|x_i|$ is said to be the μ-norm of x and $\|A\|_M := \sup_{i \in E} \mu_i^{-1} \sum_{j \in E} |A_{ij}|\mu_j$ is the associated μ-norm of A.

We will refer to condition (1.4) as *μ-exponential ergodicity* ($\mu-$EE). Clearly, for $\mu_i \geq 1$, $\mu-$EE is a special case of exponential ergodicity, which requires the existence of a parameter $\lambda > 0$, such that $\sum_j |P_{ij}(t) - \pi_j| = \mathcal{O}(e^{-\lambda t})$ as $t \to \infty$ (cf. [26]). By virtue of the equivalence result in [26] and Proposition A, exponential ergodicity implies $\mu-$EE for some vector μ, with $\mu_i \geq 1 \ \forall i \in E$. If we take $\mu \equiv 1$, $\mu-$EE reduces to strong ergodicity.

If we assume the MP to be irreducible, the stationary matrix Π has equal rows. This motivates formulating (1.4) as a condition on weighted variation norms, which in fact has been done in [13]. However, we would rather adopt a matrix norm formulation. In conjunction with our formulation of Tweedie's criterion, this allows us to interpret all convergence results in terms of convergence of bounded linear operators. In this way we may also apply theory for bounded linear operators, so as to derive spectral theorems (cf. e.g. [8], [20]). Using matrix norms has been introduced in [5] for the analysis of Markov decision chains, and lateron used for MCs in e.g. [7]. Proposition A is in fact a partial continuous time extension of the Key Theorem in the latter reference. A similar result has been derived in [20] for *uniformisable* MP's with a countable state space.

In this paper we will study to some extent the $\mu-$EE property with respect to the recurrence time distributions to a finite set. This involves the following concept.

A MP is said to be *μ-weak exponentially recurrent* ($\mu-$WER), if there are a finite set $M \subset E$, and positive constants c, λ_R, such that

$$\|{}_MP(t)\|_M \leq c e^{-\lambda_R t}, \ t \geq 0. \tag{1.5}$$

REMARK 1 Note, that $\mu-$WER for the finite set M, constant $c = 1$ and parameter λ_R, is equivalent to μ being a λ_R-invariant vector for the taboo process with transition matrix ${}_MP(t)$, $t \geq 0$. Similarly, condition (1.4) is the same as requiring μ to be a λ_R-invariant vector for ${}_MQ$ (cf. [1], p. 192).

If $\mu \geq 1$, clearly the $\mu-$WER property implies the distribution of the recurrence time to a finite set to have expontially fast decaying tails. In fact λ_R lower bounds the decay parameter of the taboo process. When viewing the MP as a reward process, $\mu-$WER requires the expected μ-reward at time t, given that the process never jumped to M before t, to decrease exponentially fast. As a consequence the total expected μ-reward until absorption into set M is finite. Section 2 contains a discussion of related recurrence properties as well as the proof of the relation between the properties discussed in the foregoing, which is formulated in the following theorem.

THEOREM 1.1 *Let $\{X(t)\}_{t\geq 0}$ be a regular MP and let μ be a vector with $\mu_i \geq 1$, $i \in E$. Then*

$$\mu-\textsc{wer}\ (1.5) \quad \Longleftrightarrow \quad \mu-\textsc{ee}\ (1.4).$$

In the special case of $\sup_{i\in E}\mu_i < \infty$, Theorem 1.1 states the well-known result (cf. [7]) of strong ergodicity being equivalent to the first hitting times of a finite set having exponential tails that are uniformly bounded in the initial states.

Simultaneously to the present research Meyn & Tweedie have proved in [13], that $\mu-$WER implies $\mu-$EE for MPs with a general state space, if μ is a *moment*, that is, $\mu_i \to \infty$, as $i \to \infty$; this is not formulated explicitly, but is implicit in the proof of Theorem 6.1. The reverse implication does not seem to be known yet for MPs. The discrete time version of the equivalence has been shown first in [7] for general μ vectors that are bounded below by a positive constant. In this paper we do not use the extra condition in Proposition A on μ being a moment, which is mainly imposed to ensure regularity of the process. We have chosen to require regularity as a separate condition, since most applications are regular by nature, and we would like to keep μ as general as possible, so that the analogous results for strongly ergodic MPs are contained in the analysis.

The close relation of $\mu-$WER and $\mu-$EE can be of interest for analysing reward processes. A more general setting for such conditions are Markov decision processes, where the discrete time versions of the above conditions have been shown to yield existence of optimal strategies for a wide range of criteria (cf. [5]). Another application of the $\mu-$WER property is it's offering an alternative means for checking $\mu-$EE. Much to our surprise, $\mu-$WER turned out to be simpler to check for the Jackson network than Tweedie's criterion, which is essentially a Lyapunov function criterion.

In the light of practical applications, another topic of the paper is to study conditions on special MCs implying the desired properties for the MP in section 3. In practice, processes will often evolve at discrete time instants, or be observed at discrete time instants. It is of importance to know, to what extent ergodicity properties of the observed discrete time process relate to ergodicity properties of the underlying Markov process.

Various types of MCs are often associated with a specific MP. One is the skeleton chain; this has been extensively studied in e.g. [15] and we will not further pursue this. Another one is the jump chain. Trivially, a vector μ solving the Popov's criterion for the jump chain solves Tweedie's criterion for exponential ergodicity, if the expected sojourn times in a state are bounded as a function of the states. The reverse need not hold, examples of which can be found in [25] and references therein.

If the MP is uniformisable, it's evolution in time can be fully described by considering discrete time approximating MCs. As we can associate uniformisable MPs with a large class of queueing processes, we will explore the relation between MP and approximating MC quite extensively. Due to the fact that the transition probabilities of the MP can be expressed in terms of the transition probabilities of the approximating MC, it is quite straightforward to show a close connection between recurrence and ergodicity properties of both processes. As a corollary we obtain an expression for the relation between the convergence rates in the exponential (and geometric) ergodicity property of MP and approximating MC. This is of special interest to our present research, which focuses on methods for calculating these convergence rates in [8].

Finally section 4 illustrates the applicability of the criteria used by two examples. $\mu-$EE (1.4) of a *stochastic population model* is shown by checking Popov's criterion for the jump chain. The second example is the *Jackson network* with m nodes. $\mu-$EE (1.4) is established by essentially showing $\mu-$WER (1.5) for the approximating MC. A simple coupling argument for the sample paths of the process suffices.

In [21] we constructed a Lyapunov function in Popov's criterion for the Jackson network with two nodes; in spite of much effort I have not been able to extend this to networks with more than two nodes unto this very moment. Much to my interest, Fayolle et al. [6] recently succeeded in constructing a Lyapunov function, say f, in Foster's criterion for the Jackson network. Together with a boundedness result on the f, exponential ergodicity of the process follows by virtue of a result from [10].

The present result extends the result in [6] in two directions. We allow any finite number of servers in each node. Moreover, the μ-vectors we use for all applications, grow exponentially fast in the state variable. As a consequence, we do not only obtain geometric convergence of the marginal distributions, but geometric convergence of all moments. A different proof of the same result will be given in a subsequent paper [23], where the use of coupling arguments for showing geometric ergodicity of queueing networks is explored more fully.

In the paper we use the following general notation: $\mathbb{P}_i\{\cdot\}$ denotes the probability of event $\{\cdot\}$, when the starting state is i. Similarly, $\mathbb{E}_i\{\cdot\}$ is the expectation operator conditionally on the process being in i initially. δ_{ij} is the Kronecker delta; $1_{\{\cdot\}}$ equals 1, if event $\{\cdot\}$ occurs, and 0 otherwise; "$\leq$" stands for componentwise ordering of vectors and "1" for the vector consisting of 1's only, the dimension of which will always be clear from the context.

We close this section by remarking that the literature on geometric and exponential ergodicity is too vast to give proper credit to all authors, and we have restricted to mentioning only those papers that are directly related to the analysis here. For further references and overviews of the subject field we refer to [11], [12], [13], [7], [20], [21] and other papers that will be mentioned in this article.

2. Recurrence properties of the taboo probabilities

In the sequal only *irreducible* MPs will be taken into account; this is no restriction, in that all proofs can be extended to the more general case of requiring a finite number of closed classes. Apart from Assumption (1.1) we will assume *regularity* in this section. As a consequence of regularity, the MP $\{X(t)\}_{t\geq 0}$ is the Feller minimal process. Hence, $P(t)$ satisfies the so-called "first jump" and "last jump" identities (cf. [3], [26]). In a similar way, "first entrance" and "last exit" decomposition, which is a well-known concept for discrete time MCs (see [3]), is applicable to $P(t)$. Note that last exit decomposition of $P_{ij}(t)$ for a taboo state j does not have any meaning.

Lemma 2.1. *If the* MP *is regular, $P(t)$ satisfies "first entrance" decomposition, i.e.*

$$P_{ij}(t) = {}_MP_{ij}(t) + \int_0^t \sum_k {}_MP_{ik}(s) \sum_{\substack{m\in M\\ m\neq k}} q_{km}P_{mj}(t-s)ds \tag{2.1}$$

and "last exit" decomposition for $j \notin M$, i.e.

$$P_{ij}(t) = {}_M P_{ij}(t) 1_{\{i \notin M\}} + \int_0^t \sum_{m \in M} P_{im}(s) \sum_{k \neq m} {}_M q_{mk} {}_M P_{kj}(t-s) ds. \tag{2.2}$$

Proof: The main arguments and notation can be found in [3] pp. 249-250. Define $N(t)$ to be the number of jumps of the MP in $(0,t)$ and

$$P_{ij}^{\langle n \rangle}(t) := \mathbb{P}_i\{X(t) = j \cap N(t) = n\}.$$

As $\{X(t)\}_{t \geq 0}$ is the Feller minimal process,

$$P_{ij}(t) = \sum_{n=0}^{\infty} P_{ij}^{\langle n \rangle}(t), \quad \forall\, t \geq 0.$$

A similar expression obviously holds for the taboo process as well (this can be shown be e.g. introducing an extra absorbing state, towards which all disappearing mass is directed). Then

$$P_{ij}^{\langle 0 \rangle}(t) = \delta_{ij} e^{-q_i t}, \quad \forall\, i,j,\ \forall\, t \geq 0.$$

and it is sufficient to show for $n = 1, 2, \ldots$ and $i \in E$, that

$$P_{ij}^{\langle n \rangle}(t) = {}_M P_{ij}^{\langle n \rangle}(t) + \int_0^t \sum_{\nu=0}^{n-1} \sum_k {}_M P_{ik}^{\langle \nu \rangle}(s) \sum_{\substack{m \in M \\ m \neq k}} q_{km} P_{mj}^{\langle n-\nu-1 \rangle}(t-s) ds, \qquad j \in E \tag{2.3}$$

as well as

$$P_{ij}^{\langle n \rangle}(t) = {}_M P_{ij}^{\langle n \rangle}(t) 1_{\{i \notin M\}} + \int_0^t \sum_{\nu=0}^{n-1} \sum_{m \in M} P_{im}^{\langle \nu \rangle}(s) \sum_{k \neq m} {}_M q_{mk} {}_M P_{kj}^{\langle n-\nu-1 \rangle}(t-s) ds,\ j \notin M. \tag{2.4}$$

By taking the summation over all n, and using the monotone convergence theorem (cf. [18], p. 265), we then obtain (2.1) and (2.2). The proof of (2.1) and (2.2) is by induction and using the first jump identities. Since this is quite standard, cf. for example [1], p. 211, we will omit details. **QED**

Next we derive a relation between the $\mu-$WER property (1.5) and solutions to (1.2). Write q for the vector with i^{th} component q_i, $i \in E$. In the sequal assume, that μ is a *non-negative* vector.

THEOREM 2.2

i) $\mu-$WER (1.5) holds for the finite set M, constant $c = 1$ and parameter λ_R if and only if μ is a solution to (1.2) for the finite set M and parameter λ_R.

ii) If $\mu-$WER (1.5) holds for the finite set M, constant $c > 0$ and parameter λ_R, then $\mu^*(\lambda)$, with

$$\mu_i^*(\lambda) = \int_0^{\infty} e^{\lambda t} \sum_{j \in E} {}_M P_{ij}(t) \mu_j dt,\ i \in E,$$

satisfies

$$\sum_k {}_M q_{ik} \mu_k^*(\lambda) = -\mu_i - \lambda \mu_i^*(\lambda), \tag{2.5}$$

and $\|\mu^*(\lambda)\|_M \leq \frac{c}{\lambda_R - \lambda}$, for any $\lambda < \lambda_R$. $\mu^*(\lambda) \|q\|_M^{-1}$ is a solution of (1.2) with components greater or equal to 1, for the finite set M and parameter $\lambda_R^* = \lambda + \frac{\lambda_R - \lambda}{c}$, if $\|q\|_M$ is finite.

iii) Assume the existence of a constant c, for which

$$\mu_i^*(0) = \int_0^\infty \sum_{j\in E} {}_M P_{ij}(t)\mu_j dt \le c\mu_i,\ i \in E. \tag{2.6}$$

Then $\sum_j {}_M P_{ij}(t)\mu_j$ is finite and continuous as a function of t; moreover, for $\lambda < c^{-1}$

$$\mu_i^*(\lambda) = \int_0^\infty e^{\lambda t} \sum_j {}_M P_{ij}(t)\mu_j \le c \cdot \frac{1-(c-1)\lambda}{1-c\lambda}\mu_i,\ i \in E.$$

Remark 2 From Theorem 2.2i) we conclude that the existence of a solution μ to (1.2) for the finite set M is equivalent to $\{{}_M P(t)\}_t$ being a *contraction semi-group* with respect to the μ-norm.

Proof of Theorem 2.2 For the proof of i) note first that either condition implies $q_i \ge \lambda_R$. By remark 1 in section 1 we have to show that μ is a λ_R-invariant vector for the taboo probabilities ${}_M P(t)$, $t \ge 0$, iff μ is a λ_R-invariant vector for ${}_M Q$. Necessity is shown similarly to Proposition 5.4.1 in [1], which states the same equivalence under the assumption of E being a communicating class. To show sufficience, we assume that $\mu-\textsc{wer}$ holds for the finite set M, constant $c = 1$ and parameter λ_R. Then

$$\begin{aligned}\frac{1}{t}(e^{-\lambda_R t}-1)\mu_i \qquad\qquad & \ge \frac{1}{t}\big({}_M P_{ii}(t)-1\big)\mu_i + \frac{1}{t}\sum_{j\ne i} {}_M P_{ij}(t)\mu_j \\ & \ge \frac{1}{t}\big({}_M P_{ii}(t)-1\big)\mu_i + \sum_{j\ne i} {}_M q_{ij}\mu_j \Big\{ 1_{\{q_j \ne q_i\}} \frac{e^{-q_j t}-e^{-q_i t}}{t(q_i-q_j)} + 1_{\{q_j = q_i\}} e^{-q_i t} \Big\},\end{aligned}$$

bounding ${}_M P_{ij}(t)$ below by ${}_M P_{ij}^{\langle 1\rangle}(t)$ for $j \ne i$ in the second inequality. Taking the liminf on both sides, and using Fatou's lemma, we obtain from the foregoing inequality

$$-\lambda_R\mu_i \ge {}_M q_{ii}\mu_i + \sum_{j\ne i} {}_M q_{ij}\mu_j \liminf_{t\downarrow 0} \{ 1_{\{q_j \ne q_i\}} \frac{e^{-q_j t}-e^{-q_i t}}{t(q_j-q_i)} + 1_{\{q_j = q_i\}} e^{-q_i t} \} = \sum_j {}_M q_{ij}\mu_j,$$

so that μ solves (1.2).

For the proof of ii) first notice, that

$$\mu_i^*(\lambda) \le \int_0^\infty e^{(\lambda-\lambda_R)t} dt \cdot c\mu_i = \tfrac{c}{\lambda_R-\lambda}\mu_i. \tag{2.7}$$

Then

$$\begin{aligned}\sum_k {}_M q_{ik}\mu_k^*(\lambda) &= \sum_k {}_M q_{ik} \int_0^\infty e^{\lambda t} \sum_j {}_M P_{kj}\mu_j(t)dt = \sum_j \{ \int_0^\infty \sum_k {}_M q_{ik}\, {}_M P_{kj}(t)dt \}\mu_j \\ &= \sum_j \{ \int_0^\infty e^{\lambda t} \tfrac{d}{dt} {}_M P_{ij}(t)dt \}\mu_j \\ &= -\mu_i - \lambda\mu_i^*(\lambda),\end{aligned}$$

so that $\mu^*(\lambda)$ satisfies (2.5). If $\|q\|_M < \infty$,

$$\mu_i^*(\lambda) \ge \int_0^\infty e^{\lambda t} {}_M P_{ii}(t)dt\mu_i \ge \int_0^\infty e^{-(q_i-\lambda)t} dt\mu_i \ge q_i^{-1}\mu_i \ge \|q\|_M^{-1}.$$

The proof of ii) is then completed by combining (2.5) and (2.7) and by observing, that $\mu^*(\lambda)$ solves (1.2) iff $c'\mu^*(\lambda)$ solves (1.2) for a positive constant c'.

For iii) note, that ${}_MP_{ij}(t)$ is continuous in t. Write $f_i(t)$ for $\sum_j {}_MP_{ij}(t)\mu_j$ and suppose, that $f_i(\hat{t})$ diverges for some $\hat{t}$. Then $f_i(\hat{t}+x) \geq e^{-q_i x} f_i(\hat{t})$, for $x \geq 0$, so that $f_i(t)$ diverges for $t \geq \hat{t}$. Consequently the integral cannot converge. Hence, $f_i(t)$ must be finite for any $t \geq 0$.

Next we show continuity. Let $t_n \to t$, $n \to \infty$. By Fatou's lemma $\liminf_{n\to\infty} f_i(t_n) \geq f_i(t)$. On the other hand, if $t_n \uparrow t$, then $f_i(t) \geq {}_MP_{ii}(t-t_n) f_i(t_n)$, and thus $f_i(t) \geq \limsup_{n\to\infty} f_i(t_n)$, so that $f_i(t)$ is continuous to the left. It is then straightforward to prove that $f_i(t) \to 0$, for $t \to \infty$.

Let $t_n \downarrow t$, and suppose that $\liminf_{n\to\infty} f_i(t_n) \geq f_i(t) + \epsilon$, for some $\epsilon > 0$. Then for $x > 0$, $\liminf_{n\to\infty} f_i(t_n + x) \geq {}_MP_{ii}(x)\epsilon + f_i(t+x)$. Hence, $f_i(\hat{t}_n)$ strictly increases in n for some sequence $\hat{t}_n \to \infty$, $n \to \infty$. This contradicts convergence of $f_i(t)$ to 0, for $t \to \infty$.

Clearly, (2.5) implies, that for $i \in E$,

$$\sum_k {}_Mq_{ik}\mu_k^*(0) = -\mu_i \leq -\tfrac{1}{c}\mu_i^*(0), \tag{2.8}$$

so that $\mu^*(0)$ satisfies (1.2) for the finite set M and parameter c^{-1}. Thus, for $\lambda < c^{-1}$,

$$\int_0^\infty \sum_j e^{\lambda t}{}_MP_{ij}(t)\mu_j^*(0)dt \leq \tfrac{1}{c^{-1}-\lambda}\mu_i^*(0). \tag{2.9}$$

Observing, that

$$\begin{aligned}\int_0^\infty \sum_j e^{\lambda t}{}_MP_{ij}(t)\mu_j^*(0)dt &= \int_0^\infty \sum_j e^{\lambda t}{}_MP_{ij}(t)\Big\{\int_0^\infty \sum_k {}_MP_{jk}(u)\mu_k du\Big\}dt \\ &= \int_0^\infty e^{\lambda t}\int_t^\infty \sum_k {}_MP_{ik}(u)\mu_k du\, dt \\ &= \tfrac{1}{\lambda}\int_0^\infty e^{\lambda u}\sum_k {}_MP_{ik}(u)\mu_k du - \tfrac{1}{\lambda}\mu_i^*(0),\end{aligned}$$

we find that $\mu_i^*(\lambda) \leq c + c\lambda/(1-c\lambda)\mu_i$, by which the theorem is proved. **QED**

A simple example shows, that in general $\mu-WER$ (1.5) does not imply $\mu^*(\lambda)$ to be bounded away from 0. Let $E = \mathbb{N}_0$, $q_i = 2^i = q_{i0}$, for $i \geq 1$, $q_0 = 1$ and $q_{0i} = 2^{-i}$, $i \geq 1$. Take $M = \{0\}$, then

$$\mu_i^{-1}\sum_j {}_MP_{ij}(t)\mu_j = \begin{cases} e^{-2^i t}, & i \geq 1 \\ e^{-t} + \mu_0^{-1}\displaystyle\sum_{j\geq 1} 2^{-j}\frac{e^{-t}-e^{-2^j t}}{2^j - 1}\mu_j. \end{cases} \tag{2.10}$$

The summation in (2.10) converges if $\mu_j/(4-\epsilon)^j \to 0$, for $j \to \infty$, for some $\epsilon \in (0,4)$. If we choose $\mu \equiv 1$, then clearly the MP satisfies $\mu-WER$ (1.5) for this μ and set $M = \{0\}$; furthermore $q_j/\mu_j \to \infty$, as $j \to \infty$, but $\mu_j^*(\lambda) = (2^j - \lambda)^{-1} \to 0$ for $j \to \infty$.

This is a poignant constrast with similar results for MCs, where $\mu^*(\lambda) \geq \mu$ holds (cf. [7]). This type of inequality is also quite crucial in showing for MCs, that (2.6) implies $\mu-WER$ (1.5). For MPs this is still an open problem. We can conclude from Theorem 2.2iii), that indeed the decay parameter of $\mu_i^{-1}\sum_j {}_MP_{ij}(t)\mu_j$ (hence of the taboo probabilities) is at least c^{-1}, but so far we didn't succeed in showing the existence of one uniform (exponential) bound for all initial states.

It is illustrative to spend a few words on our representation of Tweedie's criterion and the original formulations in [26] and [13]. In [26] the criterion has been formulated as follows:

There are a vector y on E, a finite set $M \subset E$, and a positive constant $\lambda < q_i, \ \forall i \in E$, such that

$$\left.\begin{array}{ll} y_i = 0, & i \in M \\ \sum_k q_{ik} y_k \leq -\lambda y_i - 1, & i \notin M \\ \sum_k q_{ik} y_k < \infty, & i \in M \end{array}\right\} \tag{2.11}$$

Let $\mu \geq 1$ be a solution to (1.2) for the finite set M and parameter λ_R. For positive $\epsilon < \lambda_R$, y, defined by $y_i = (\lambda_R - \epsilon)^{-1}\mu_i$, $i \notin M$, and $y_i = 0$, for $i \in M$, is a solution to (2.11) for the finite set M and parameter $\lambda = \epsilon$.

Conversely, let y be a solution of (2.11) for the finite set M and parameter λ. It is straightforward to check, that μ, with $\mu_i = \lambda y_i + 1$, if $i \notin M$, and $\mu_i \geq \max\{(q_i - \lambda)^{-1} \sum_{k \neq i} {}_M q_{ik}\mu_k, 1\}$, if $i \in M$, is a solution to (1.2) with $\mu \geq 1$, for the finite set M and parameter $\lambda_R = \lambda$.

Let τ_A^+ (cf. [1], p.206) denote the first hitting time of set A, *after* the first jump of the MP. It is known from [26] (cf. also [1], p. 210), that the minimal non-negative solution of (2.11) for set M and parameter λ is given by $y_i = \lambda^{-1}(\mathbb{E}_i e^{\lambda \tau_M^+} - 1)$, $i \notin M$. As a consequence $\mu_i = \mathbb{E}_i e^{\lambda \tau_M^+}$, $i \in E$, is the minimal solution to (1.2) on $E \setminus M$ with $\mu \geq 1$. In general a solution μ to (1.2) need not be the minimal non-negative solution. In the foregoing example $\mathbb{E}_i e^{\lambda \tau_0^+} = 2^i/(2^i - \lambda) \to 1$ as $i \to \infty$, but $\mu_i^*(0) = 2^{-i}$ is a solution to (1.2) as well.

[13] shows, that the existence of a vector V on E and constants $c > 0$, C with

$$\left.\begin{array}{l} V_i \to \infty, \ i \to \infty \\ \sum_j q_{ij} V_j \leq -cV_i + C, \ i \in E \end{array}\right\} \tag{2.12}$$

implies $\mu-$EE (1.4) for $\mu = V + 1$. Obviously, under this moment condition $(V + \tilde{c}) -$ EE holds for any constant $\tilde{c}$.

Suppose, that V solves (2.12) for the constants c, C. Select a finite set S with $E \setminus S \subset \{i \mid -cV_i + C \leq -\frac{c}{2}(V_i + 1)\}$. Next, let M be a finite set, such that $\sum_j {}_M q_{ij}(V_j + 1) \leq -\frac{c}{2}(V_i + 1)$ for $i \in S$. Then $\mu = V + 1 \geq 1$ solves (1.2) for the finite set M and parameter $\lambda_R = c/2$.

The converse is even more immediate. If μ is a solution to (1.2) for the finite set M and parameter λ_R, then V with $V_i = \mu_i + 1$, $i \notin M$, and $V_i = 1$, $i \in M$, $c = \lambda_R$ and $C \geq \max\{\lambda_R, \lambda_R + \max_{m \in M} \sum_j q_{mj} V_j\}$ solves (2.12). Note, that no moment condition on μ is required for the latter implication, whereas the reverse cannot be obtained without the moment condition on V. This illustrates the importance of this moment condition in the very weak formulation (2.12) of Tweedie's criterion in [13].

The following theorem relates the $\mu-$WER (1.5) property to the expected μ-rewards until absorption into a *state*. It will be useful for proving Theorem 1.1.

THEOREM 2.3 *Let $\mu \geq 1$.*

i) $\mu-$WER (1.5) implies, that the MP *is positive recurrent and exponentially ergodic.*

ii) $\mu-$WER (1.5) for the finite set M is equivalent to the existence of a state $0 \in M$, a constant c^*

and a parameter λ^*, such that $\mu-$WER holds for the finite set $\{0\} \subset M$, i.e.

$$\|{}_0P(t)\|_M \le c^* e^{-\lambda^* t},\ t \ge 0. \tag{2.13}$$

Proof Suppose, that $\mu-$WER holds for the finite set M. As $\mu \ge 1$, (2.5) implies the vector y, defined as $y_i = \mu_i^*(\lambda)$, $i \notin M$, and $y_i = 0$, $i \in M$, to satisfy Tweedie's criterion. Invoking Theorem 3 in [26], positive recurrence directly follows by putting $\lambda = 0$, and exponential ergodicity by $\lambda > 0$. This proves i).

μ–WER for the set $\{0\}$ trivially implies μ–WER for any set $M \supset \{0\}$. So, let us assume that μ–WER holds for the finite set M, constant c and parameter λ_R. Write ${}_MF_{im}(t) = \mathbb{P}_i\{\tau_M^+ \le t, X(\tau_M^+) = m\}$, $m \in M$, for the probability that the process enters set M through state m after the first jump; then clearly ${}_MF_{im}$ has density function $\sum_k {}_MP_{ik}(t) q_{km} 1_{\{k \ne m\}}$. Note, that we allow $i = m$, contrary to [3], Theorem II.11.8. Let $0 \in M$; we will apply Nummelin's technique of "first entrance-last exit" decomposition (cf. [14], Theorem 4.1) of set $M \setminus \{0\}$ to the taboo process with taboo state 0. That is,

$$\begin{aligned}
{}_0P_{ij}(t) &= {}_MP_{ij}(t) + \sum_{\substack{m\in M\\ m\ne 0}} \int_0^t {}_0P_{mj}(t-s) d_MF_{im}(s)\\
&= {}_MP_{ij}(t) + \sum_{\substack{m\in M\\ m\ne 0}} \int_0^t \sum_{\hat m\in M} \Big[1_{\{j=\hat m\}}\, {}_0P_{m\hat m}(t-s) +\\
&\qquad \int_0^{t-s} 1_{\{j\ne \hat m\}}\, {}_0P_{m\hat m}(u) \sum_{k\ne \hat m} {}_Mq_{\hat m k}\, {}_MP_{kj}(t-s-u)du\Big] d_MF_{im}(s),
\end{aligned} \tag{2.14}$$

by using first entrance decomposition in the first equality and last exit decomposition for ${}_0P_{mj}(t-s)$ in the second. Multiplying by μ_j and taking the summation over j, we obtain

$$\begin{aligned}
\sum_j {}_0P_{ij}(t)\mu_j &= \sum_j {}_MP_{ij}(t)\mu_j + \sum_{\substack{m,\hat m\in M\\ m\ne 0}} \int_0^t {}_0P_{m\hat m}(t-s)\mu_{\hat m} d_MF_{im}(s)\\
&+ \sum_{\substack{m,\hat m\in M\\ m\ne 0}} \int_0^t \int_0^{t-s} {}_0P_{m\hat m}(u) \sum_{k\ne\hat m} {}_Mq_{\hat m k} \sum_j {}_MP_{kj}(t-s-u)\mu_j du\, d_MF_{im}(s).
\end{aligned} \tag{2.15}$$

These are all convolution integrals, so that integration over t finally yields

$$\begin{aligned}
\int_0^\infty \sum_j {}_0P_{ij}(t)\mu_j dt &= \int_0^\infty \sum_j {}_MP_{ij}(t)\mu_j dt + \sum_{\substack{m,\hat m\in M\\ m\ne 0}} \int_0^\infty {}_0P_{m\hat m}(t)\mu_{\hat m} dt \cdot \int_0^\infty d_MF_{im}(s)\\
&+ \sum_{\substack{m,\hat m\in M\\ m\ne 0}} \int_0^\infty {}_0P_{m\hat m}(u)du \cdot \sum_{k\ne\hat m} {}_Mq_{\hat m k} \int_0^\infty \sum_j {}_MP_{kj}(t)\mu_j dt \cdot \int_0^\infty d_MF_{im}(s).
\end{aligned} \tag{2.16}$$

$\int_0^\infty {}_0P_{m\hat m}(t)dt$ converges, since all states are transient in the taboo process (with taboo state 0). As M is a finite set and $\int_0^\infty d_MF_{im}(s) \le 1$, there is a constant $\hat c \ge \lambda_R^{-1}$ such that the righthandside of (2.16) is bounded above by $\hat c \mu_i$. Applying Theorem 2.2iii) yields the existence of a constant $c(\lambda)$, such that

$$\int_0^\infty e^{\lambda t} \sum_j {}_0P_{ij}(t)\mu_j dt \le c(\lambda)\mu_i, \tag{2.17}$$

for $\lambda < \hat{c}^{-1}$, so that $e^{\lambda t}{}_0P_{m\hat{m}}(t)\mu_{\hat{m}} \leq \hat{c}(\lambda)$, $m, \hat{m} \in M$, $t \geq 0$, for some constant $\hat{c}(\lambda)$. Moreover,

$$\int_0^\infty e^{\lambda s} d_M F_{im}(s) = \int_0^\infty d_M F_{im}(s) + \lambda \int_0^\infty \mathbb{P}_i\{\tau_M^+ > t, X(\tau_M^+) = m\} dt$$
$$\leq 1 + \lambda \int_0^\infty \sum_j {}_MP_{ij}(t) dt \leq (1 + \tfrac{\lambda}{\lambda_R})\mu_i, \qquad (2.18)$$

as $\mu \geq 1$ by assumption. Combination of (2.15), (2.17), (2.18) and (2.5) yields

$$\sum_j {}_0P_{ij}(t)\mu_j \leq c e^{-\lambda_R t}\mu_i + \hat{c}(\lambda)(1 + \tfrac{\lambda}{\lambda_R}) \cdot \sum_{\substack{m,\hat{m} \in M \\ m,\hat{m} \neq 0}} \{1 + q_{\hat{m}} \mu_{\hat{m}}^{-1} \mu_{\hat{m}}^*(\lambda)\} e^{-\lambda t}\mu_i.$$

The righthandside is then bounded by $c^* e^{-\lambda t}\mu_i$ for some constant c^* by virtue of inequality (2.7), since $\lambda < \lambda_R$. Choosing $\lambda^* = \lambda$ finally establishes the result. **QED**

Remark 3 The only place, where $\mu \geq 1$ is essentially used, is for the bound in (2.18), that is, to show that the first hitting time distributions of a finite set, rescaled with respect to μ, have exponential tails. This is needed for the proof of Theorem 1.1, even if we assume that M consists of one state only.

The proof of Theorem 1.1 in an earlier version of the paper used skeleton chains combined with the corresponding equivalence theorem for MCs from [7]. During a visit of the author to Fort Collins, Prof. Tweedie kindly pointed out a far more elegant and direct proof to me, which I am happy to report here.

Proof of Theorem 1.1 First assume property $\mu-EE$ for the constant c and a parameter λ_E as in (1.4). Then the stationary matrix has equal rows, say $\pi_j = \Pi_{ij}$, $\forall i \in E$, $\sum_j \pi_j\mu_j < \infty$ and $\|P(t)\|_M \leq c^* := \sum_j \pi_j\mu_j + c$, for $t \geq 0$.

Let $\epsilon \in (0,1)$. There is a finite set M, such that $\sum_{j \notin M} \pi_j\mu_j < \epsilon/3$. Choose t^* with $ce^{-\lambda_E t^*} < \epsilon/3$ and $e^{-q_i t^*} < \epsilon/3$, $i \in M$. Then

$$\sum_j {}_MP_{ij}(t^*)\mu_j \leq 1_{\{i \in M\}} e^{-q_i t^*} + \sum_{j \notin M} P_{ij}(t^*)\mu_j \leq \tfrac{\epsilon}{3} + \sum_{j \notin M} |P_{ij}(t) - \pi_j|\mu_j + \sum_{j \notin M} \pi_j\mu_j \leq \epsilon,$$

so that for $t \in [nt^*, (n+1)t^*)$

$$\|{}_MP(t)\|_M \leq \|{}_MP(t^*)\|_M^n \cdot \|{}_MP(t - nt^*)\|_M \leq \epsilon^n \|P(t - nt^*)\|_M \leq \epsilon^n c^*, \qquad (2.19)$$

and thus $\|{}_MP(t)\|_M \leq \hat{c}e^{-\lambda_R t}$ for the constant $\hat{c} = c^*\epsilon^{-1}$ and parameter $\lambda_R = -\frac{1}{t^*}\log\epsilon$.

Next assume, that $\mu-WER$ (1.5) holds for the finite set M. By virtue of Theorem 2.3 $\mu-WER$ holds for state $0 \in M$, say, for parameter λ_R and constant c. Combined "first entrance-last exit" decomposition for state 0, gives, similarly to (2.14)

$$P_{ij}(t) = {}_0P_{ij}(t) + 1_{\{j=0\}} \int_0^t {}_0P_{00}(t-s) d_0F_{i0}(s)$$
$$+1_{\{j \neq 0\}} \int_0^t \int_0^{t-s} P_{00}(t-s-u) \sum_{k \neq 0} {}_0q_{0k}\,{}_0P_{kj}(u) du\, d_0F_{i0}(s).$$

Furthermore, from the renewal reward theorem (cf. [17], p.78)

$$\pi_j = \pi_0 \sum_{k\neq 0} q_{0k} \int_0^\infty {}_0P_{kj}(u)du.$$

$\int_0^\infty d_0F_{i0}(s) = 1$, since the MP is positive recurrent, so that the foregoing equations taken together yield

$$\begin{aligned} P_{ij}(t) - \pi_j = {}_0P_{ij}(t) + 1_{\{j=0\}}\Big\{\int_0^t [P_{00}(t-s) - \pi_0]\, d_0F_{i0}(s) - \pi_0 \int_t^\infty d_0F_{i0}(s)\Big\} \\ +1_{\{j\neq 0\}}\Big\{\int_0^t \Big[\int_0^{t-u} [P_{00}(t-u-s)-\pi_0] d_0F_{i0}(s) - \pi_0 \int_{t-u}^\infty d_0F_{i0}(s)\Big]\cdot \sum_{k\neq 0} q_{0k\,0}P_{kj}(u)du \\ -\pi_0 \sum_{k\neq 0} {}_0q_{0k} \int_t^\infty {}_0P_{kj}(u)du\Big\}. \end{aligned}$$

Multiplying by μ_j and summing over j we get

$$\begin{aligned} \sum_j |P_{ij}(t) - \pi_j|\mu_j \leq \sum_j {}_0P_{ij}(t)\mu_j + \int_0^t |P_{00}(t-s) - \pi_0| d_0F_{i0}(s)\mu_0 + \pi_0 \int_t^\infty d_0F_{i0}(s)\mu_0 \\ + \Big\{\int_0^t \int_0^{t-u} |P_{00}(t-s-u) - \pi_0| d_0F_{i0}(s) \\ + \int_{t-u}^\infty d_0F_{i0}(s)\pi_0\Big\} \cdot \sum_{k\neq 0} q_{0k} \sum_j {}_0P_{kj}(u)\mu_j du \\ + \pi_0 \sum_{k\neq 0} q_{0k} \int_t^\infty \sum_j {}_0P_{kj}\mu_j(u)du. \end{aligned} \tag{2.20}$$

The first term in the righthandside of (2.20) is bounded by $ce^{-\lambda_R t}\mu_i$. As in the proof of Theorem 2.3 there is a $\lambda < \lambda_R$, such that (2.17) and (2.18) for $M = \{0\}$ are satisfied. Together with (2.5) this means the existence of a constant $c(\lambda)$ such that the third, fifth and sixth terms in (2.20) together are bounded by $c(\lambda)e^{-\lambda t}\mu_i$, since $\mu \geq 1$.

We can now proceed in two different ways. First, by last exit decomposition $\|{}_0P(t)\|_M$ is uniformly bounded in t. Then, since $(P(t) - \Pi)$ has the semi-group property, it is sufficient to prove that $\|P(t) - \Pi\|_M < 1$ for some value of t; this follows from the foregoing, if we use that $P_{00}(t) \to \pi_0$, as $t \to \infty$.

The second method uses, that μ−WER implies exponential ergodicity by Theorem 2.3i) for $\mu \geq 1$. Hence, $(P_{00}(t) - \pi_0)$ converges exponentially fast, say at rate λ_E, where we choose $\lambda_E \leq \lambda$. Similarly as before, this yields a bound, say $c(\lambda_E)e^{-\lambda_E t}\mu_i$, for the second and fourth terms in (2.20). Taking all bounds together, we find that the lefthandside of (2.20) is bounded by $c'e^{-\lambda_E t}\mu_i$ for some constant c' (since $\lambda_E \leq \lambda$). **QED**

Combination of Theorems 2.2 and 1.1 then gives the result of Proposition A for regular MPs and arbitrary vectors $\mu \geq 1$. Since the existence of a solution to Tweedie's criterion has been shown to be equivalent to exponential ergodicity in [26], we also conclude that exponential ergodicity is equivalent to μ−EE (1.4) for some vector $\mu \geq 1$.

3. Recurrence properties of associated Markov chains

This section analyses two types of MCs that are usually associated with a MP. Apart from assumption (1.1) we will only assume irreducibility. Regularity will be seen to follow from the premises. Since the analysis consists of studying MCs, it will be suitable to have the discrete time equivalents of $\mu-$EE (1.4) and $\mu-$WER (1.5) stated explicitly.

A MC is said to be *μ-geometrically ergodic ($\mu-$GE)*, if there are positive constants $\beta_E < 1$, c_E, such that

$$\|P^n - \Pi\|_M \leq c_E \beta_E^n, \; n = 0, 1, 2, \ldots \tag{3.1}$$

Write ${}_MP^n$ for the n^{th} iterate of ${}_MP$, and let ${}_MP^0 = I$.

A MC is said to be *μ-weak geometrically recurrent ($\mu-$WGR)*, if there are positive constants $\beta_R < 1$, c_R and a finite set $M \subset E$, such that

$$\|{}_MP^n\|_M \leq c_R \beta_R^n, \; n = 0, 1, 2, \ldots \tag{3.2}$$

3.1. The jump chain

In general no equivalence results on ergodicity properties of a MP and the associated jump chain can be obtained. This delicate relation between the two processes is illustrated in [25] and references therein. However, it is straightforward to see that nice properties of the jump chain do carry over to the MP.

Lemma 3.1: *Suppose the jump chain $\{X^J(n)\}_n$ satisfies Popov's criterion for the vector μ, the finite set M and parameter $\beta_R < 1$. Fix a positive $\lambda < \inf_i q_i$. Then the* MP *is regular and satisfies Tweedie's criterion for μ, the finite set M and parameter $\lambda_R = \lambda(1 - \beta_R)$.*

Proof: The jump chain's solving Popov's criterion implies it's solving Foster's criterion, hence it is ergodic. By virtue of Proposition 5 in [24], the MP is regular. The rest of the proof is immediate from the criteria. **QED**

Note, that if μ solves Popov's criterion for the finite set M and parameter β_R, then $\mu-$WGR (3.2) holds for the finite set M, constant $c_R = 1$ and parameter β_R. Combination of Lemma 3.1 with the results in the foregoing section and the Key theorem in [7] then yield the following result.

Corollary: *Under the conditions of Lemma 3.1, the jump chain has the $\mu-$WGR (3.2) property and the original* MP *has the $\mu-$WER property (1.5) for the same finite set M. Moreover, the jump chain and the* MP *have the $\mu-$GE (3.1) and $\mu-$EE (1.4) properties respectively.*

3.2. The approximating Markov chain

Next we will assume the MP to be *uniformisable*, i.e. $\bar{q} := \sup_i q_i < \infty$. Regularity immediately follows, cf. [4], p. 251. The results presented in this section are mainly from [20].

Let $h < \bar{q}^{-1}$ be the length of the time discretisation interval. Then the transition probabilities $P_{ij}(h)$ have first order approximations ${}^{\alpha(h)}P_{ij} = \big(I + hQ\big)_{ij}$, $i, j \in E$. This first order approximation defines the *h-approximation* (cf. [23]) $\{X^{\alpha(h)}(n)\}_n$ of the MP.

For our analysis it is quite crucial, that an explicit expression for the transition probabilities of a MP as a function of the transition probabilities of the h-approximation exist. Indeed, in e.g. [1], Proposition 2.2.10, one can find the following relation to hold

$$P_{ij}(t) = \big(e^{tQ}\big)_{ij} = \big(e^{th^{-1}({}^{\alpha(h)}P-I)}\big)_{ij} = \sum_{n=0}^{\infty} {}^{\alpha(h)}P_{ij}^{n} \frac{(th^{-1})}{n!} e^{-th^{-1}}, \; i,j \in E. \tag{3.3}$$

It is well-known that Π is the stationary matrix of the MP iff it is the stationary matrix of an h-approximation. Using (3.3) we can write the taboo probabilities of the MP as a function of the taboo probabilities of h-approximations.

$$_MP_{ij}(t) = \begin{cases} \sum\limits_{n\geq m\geq 0} \big({}^{\alpha(h)}P_{ii}\big)^m \cdot {}^{\alpha(h)}_{\;\;M}P_{ij}^{n-m} \cdot e^{-th^{-1}} \frac{(th^{-1})^n}{n!}, & i \in M \\ \sum\limits_{n\geq 0} {}^{\alpha(h)}_{\;\;M}P_{ij}^{n} \cdot e^{-th^{-1}} \frac{(th^{-1})}{n!}, & i,j \notin M \\ 0, & \textit{otherwise}. \end{cases} \tag{3.4}$$

Next we will explore the relations between recurrence and ergodicity properties of a MP and its h-approximations. As a first step we show that solutions to (1.2) for a MP coincide with solutions to (1.3) for an h-approximation.

Lemma 3.2: *μ is a solution to (1.2) for the finite set M and parameter λ_R iff μ is a solution to (1.3) for the h-approximation $\{X^{\alpha(h)}(n)\}_n$, with $h < \lambda_R^{-1}$, for set M and parameter $\beta_R = 1 - \lambda_R h$.*

Proof: Let us assume μ to be a solution to (1.2) and fix $h < 1/\bar{q}$. Then $q_i h < 1 \;\; \forall\, i \in E$, and since $\lambda_R \leq q_i$, we thus have $\lambda_R h < 1$. Multiplying both sides of (1.2) by h and adding μ_i gives the solution to (1.3) as desired. The proof of the converse goes along the same lines. **QED**

Combining the Key theorem and Lemma 3.1 from [7] with Theorem 2.2 and the foregoing lemma, we obtain equivalence of $\mu-$EE (1.4) and $\mu-$WER (1.5) of the MP to $\mu-$GE (3.1) and $\mu-$WGR (3.2) of h-approximations. Here we can take the same finite set M in both the $\mu-$WER and $\mu-$WGR properties. The next theorem shows these relations in a direct way, by using the representations (3.3) (and (3.4)) of the (taboo) transition probabilities. We use this as a first step towards gaining insight in how feasible convergence parameters in the $\mu-$EE property of MPs and the $\mu-$GE property of h-approximations relate to each other. This relation will be refined in a subsequent paper [8].

Theorem 3.3:

i) *$\mu-$WER (1.5) of a MP is equivalent to $\mu-$WGR (3.2) of an h-approximation, for some $h > 0$. We can take the same finite set in both properties.*

ii) *$\mu-$EE (1.4) of a MP is equivalent to $\mu-$GE (3.1) of an h-approximation, for some $h > 0$. If $\mu-$GE of an h-approximation holds for convergence parameter $\beta_E(h)$, then $\mu-$EE of the MP holds for convergence parameter $\lambda_E := \frac{1-\beta_E(h)}{h}$; vice versa, if λ_E is a convergence parameter for the $\mu-$EE property of the MP, then for all $\delta > 0$, there is an h_δ-approximation, such that $\beta(h_\delta) = 1-(\lambda_E-\delta)h_\delta$ is a convergence parameter in the μ–GE property of this h_δ-approximation. Suppose, that in addition $\beta(h) = 1-h\big[\frac{1-\beta(h_0)}{h_0}\big]$ is a convergence parameter in the μ–GE property of the h-approximation, $\forall h \leq h_0$, that is, the convergence parameters have a linear structure. Then $\beta(h) \in (1-\lambda_E h, 1)$ is a feasible convergence parameter for $\mu-$GE of the h-approximation, $h < h_0$.*

In a subsequent paper this linearity property of the convergence parameters in the μ–GE property of h-approximations will be shown for various queueing models. Actually, for these models the contraction factor in Popov's criterion gives this convergence parameter explicitly. Combination with the results here gives the "sharpest" convergence parameter for the underlying MP.

Proof of Theorem 3.3: i) Suppose the MP satisfies $\mu-$WER (1.5) for the finite set M, constant c and parameter λ_R. Let $h < 1/\bar{q}$. Using (3.4) we have for $t \geq 0$

$$ce^{-\lambda_R t}\mu_i \geq \sum_j {}_MP_{ij}(t)\mu_j = \begin{cases} \displaystyle\sum_j \sum_{n\geq 0} {}^{\alpha(h)}_{\;\;M}P^n_{ij} \cdot e^{-th^{-1}} \frac{(th^{-1})^n}{n!}\mu_j, & i \notin M \\ \displaystyle\sum_j \sum_{n\geq m\geq 0} \left({}^{\alpha(h)}P_{ii}\right)^m \cdot {}^{\alpha(h)}_{\;\;M}P^n_{ij} \cdot e^{-th^{-1}} \frac{(th^{-1})^n}{n!}\mu_j, & i \in M. \end{cases}$$

Integrating over $t \geq 0$, and reversing the order of summation and integration yields we obtain

$$\tfrac{c}{\lambda_R}\mu_i \geq \begin{cases} \displaystyle\sum_j \sum_{n\geq 0} {}^{\alpha(h)}_{\;\;M}P^n_{ij}\mu_j \cdot h, & i \notin M \\ \displaystyle\sum_j \sum_{n\geq m\geq 0} \left({}^{\alpha(h)}P_{ii}\right)^m \cdot {}^{\alpha(h)}_{\;\;M}P^n_{ij}\mu_j \cdot h, & i \in M, \end{cases} \tag{3.5}$$

where we use

$$\int_{t\geq 0} e^{th^{-1}} \frac{(th^{-1})^n}{n!} dt = h, \quad n = 0, 1, \ldots,$$

which can be easily shown by induction. As $\left({}^{\alpha(h)}P_{ii}\right)^0 = 1$, we can lowerbound the righthandside of (3.5) by considering only the terms where $m = 0$, so that

$$\tfrac{c}{h\lambda_R}\mu_i \geq \sum_{n\geq 0} {}^{\alpha(h)}_{\;\;M}P^n_{ij}\mu_j. \tag{3.6}$$

Finally combining (3.6) with Lemma 3.1 in [7] yields that the h-approximation satisfies the $\mu-$WGR (3.2) property for the finite set M, for any $h < 1/\bar{q}$.

To show to converse, assume that for some $h < 1/\bar{q}$ the h-approximation has the $\mu-$WGR (3.2) property for the finite set M, constant c and parameter β_R. Inserting the upperbounds from the $\mu-$WER property into (3.4), yields

$$\sum_j {}_MP_{ij}(t)\mu_j \leq \begin{cases} \displaystyle c\sum_{n\geq 0} \beta_R^n \cdot e^{-th^{-1}} \frac{(th^{-1})^n}{n!}\mu_i = ce^{-(1-\beta_R)th^{-1}}\mu_i, & i \notin M \\ \displaystyle c\sum_{n\geq m\geq 0} \left({}^{\alpha(h)}P_{ii}\right)^m \beta_R^{n-m} \cdot e^{-th^{-1}} \frac{\left(th^{-1}\right)^n}{n!}\mu_i & \\ \displaystyle \quad = c\sum_{n\geq 0} e^{-th^{-1}} \frac{\left(\beta_R th^{-1}\right)^n}{n!} \sum_{m=0}^{n} \left(\frac{{}^{\alpha(h)}P_{ii}}{\beta_R}\right)^m \mu_i, & i \in M. \end{cases} \tag{3.7}$$

By virtue of the irreducibility assumption ${}^{\alpha(h)}P_{ii} < 1$, for all i. Select $\beta_1 \in [\beta_R, 1)$, such that ${}^{\alpha(h)}P_{ii} < \beta_1$ for $i \in M$. Set $c_1 = \max_{i\in M} {}^{\alpha(h)}P_{ii}\beta_1^{-1}$. Obviously, all upper bounds in (3.7) only increase, if we replace β_R by the larger value β_1. As $c_1 < 1$, we thus obtain from (3.7) for all $i \in E$,

$${}_MP_{ij}(t)\mu_j \leq \frac{c}{1-c_1} e^{-(1-\beta_1)th^{-1}}\mu_i,$$

so that $\mu-$WER (1.5) is satisfied for set M, constant $c(1-c_1)^{-1}$ and parameter $(1-\beta_1)h^{-1}$.

Next we prove ii), and we assume μ–EE (1.4) first for parameter λ_E and constant c. We need $\|Q\|_M$ to be finite, which is easily shown as follows. Note, that by virtue of the irreducibility assumption $q_i > 0$, $i \in E$.

$$\begin{aligned} P_{ij}(t) \geq P_{ij}^{\langle 1\rangle}(t) &= \int_0^t e^{-q_i s} q_{ij} e^{-q_j(t-s)} ds \\ &\geq \int_0^t q_{ij} e^{-\bar{q}t} ds = q_{ij} t e^{-\bar{q}t}, \qquad t \geq 0. \end{aligned}$$

It is immediate, that $\mu-$EE implies the existence of a positive constant c', such that $\|P(t)\|_M \leq c'$. Fix $t > 0$, then

$$\sum_{j\neq i} q_{ij}\mu_j \leq \sum_{j\neq i} P_{ij}(t)\mu_j \cdot e^{\bar{q}t}t^{-1} \leq c' e^{\bar{q}t} t^{-1}\mu_i,$$

so that $\|Q\|_M \leq \tilde{c}$, for $\tilde{c} = c' e^{\bar{q}t} t^{-1} + \bar{q}$.

For proving $\mu-$GE for some h-approximation, it suffices to show the existence of $n_o \in \mathbb{N}$, and $h_o < 1/\bar{q}$, such that

$$\| {}^{\alpha(h_o)}P^{n_o} - \Pi \|_M \leq \epsilon, \tag{3.8}$$

for some $\epsilon \in (0,1)$, since we can then proceed similarly to (2.19). Fix $\epsilon \in (0,1)$ and t_o, such that $ce^{-\lambda t_o} \leq \epsilon/2$, then

$$\|P(t_o) - \Pi\|_M \leq c e^{-\lambda_E t_o} \leq \tfrac{\epsilon}{2}. \tag{3.9}$$

The proof (3.8) is complete, if we establish the existence of $N_o \in \mathbb{N}$, such that

$$\|P(t_o) - {}^{\alpha(t_o/n)}P^n\|_M = \|e^{t_o Q} - \big(I + \tfrac{t_o}{n}Q\big)^n\|_M \leq \tfrac{\epsilon}{2}, \;\; n \geq N_o. \tag{3.10}$$

Indeed, choosing $n_o \geq N_o$ with $h_o := t_o/n_o < 1/\bar{q}$, and then combining (3.9) and (3.10) yields (3.8). We study $|e^{tQ} - \big(I + \frac{t}{n}Q\big)^n|_{ij}$. Denote $(Q^m)_{ij}$ as q_{ij}^m, then

$$\begin{aligned} |e^{tQ} - \big(I + \tfrac{t}{n}Q\big)^n|_{ij} \quad &\leq \sum_{m\geq n+1} \frac{t^m |q_{ij}^m|}{m!} + \frac{t^2|q_{ij}^2|}{2!}\big(1-(1-\tfrac{1}{n})\big)\cdots \\ &\quad + \frac{t^n|q_{ij}^n|}{n!}\big(1-(1-\tfrac{1}{n})(1-\tfrac{2}{n})\cdots(1-\tfrac{n-1}{n})\big). \end{aligned}$$

Hence,

$$\begin{aligned} \sum_j |e^{tQ} - \big(I + \tfrac{t}{n}Q\big)^n|_{ij}\mu_j &\leq \sum_{m\geq n+1} \frac{t^m}{m!}\sum_j |q_{ij}^m|\mu_j + \frac{t^2}{2!}\sum_j |q_{ij}^2|\mu_j\big(1-(1-\tfrac{1}{n})\big) + \cdots \\ &\quad + \frac{t^n}{n!}\sum_j |q_{ij}^n|\mu_j\big(1-(1-\tfrac{1}{n})(1-\tfrac{2}{n})\cdots(1-\tfrac{n-1}{n})\big) \\ &\leq \Big\{ \sum_{m\geq n+1} \frac{t^m}{m!}\tilde{c}^m + \frac{t^2}{2!}\tilde{c}^2\big(1-(1-\tfrac{1}{n})\big) + \cdots \\ &\qquad + \frac{t^n}{n!}\tilde{c}^n\big(1-(1-\tfrac{1}{n})(1-\tfrac{2}{n})\cdots(1-\tfrac{n-1}{n})\big)\Big\}\mu_i \\ &= \big\{e^{t\tilde{c}} - \big(1+\tfrac{t}{n}\tilde{c}\big)^n\big\}\mu_i. \end{aligned} \tag{3.11}$$

As the righthandside of (3.11) converges to 0, as $n \to \infty$, for $t \geq 0$, there is an $N_o \in \mathbb{N}$, such that (3.10) is satisfied. This establishes (3.8).

The next step is to obtain estimates for the convergence parameters in the $\mu-GE$ property for h-approximations. Choose $\delta > 0$, and let t_o be such, that $\log 2c/t_o \leq \delta/2$. Furthermore, select $\epsilon' > 0$, for which $\epsilon' e^{\lambda_E t_o} \leq c$. Then, there exists N_o', such that $t_o/n_o < 1/\bar{q}$ for $n_o \geq N_o'$ and

$$\| P(t_o) - {}^{\alpha(t_o/n_o)}P^{n_o} \|_M \leq \epsilon'. \tag{3.12}$$

For $n \leq n_o$ clearly

$$\| {}^{\alpha(t_o/n_0)}P^n \|_M = \| \big(I + \tfrac{t_o}{n_o} Q\big)^n \|_M \leq \big(1 + \tfrac{t_o}{n_o}\tilde{c}\big)^n \leq e^{t_o \tilde{c}}. \tag{3.13}$$

Similarly to the derivation in (2.19) it is immediate from (3.12), (3.13) and the first bound in (3.9), that for $n_o \geq N_o'$

$$\| {}^{\alpha(t_o/n_o)}P^n - \Pi \|_M \leq \big(e^{t_0 \tilde{c}} + \| \Pi \|_M\big) e^{\lambda_E t_o} \big(c e^{-\lambda_E t_o} + \epsilon'\big)^{n/n_o}, \ n = 0, 1, 2, \ldots,$$

if we assume that $c \geq 1$. Hence, $\beta(t_o/n_o) := e^{-\lambda_E t_o/n_o}(c + \epsilon' e^{\lambda_E t_o})^{1/n_o}$ is a convergence parameter for the (t_o/n_o)-approximation. We have

$$\begin{aligned}\lim_{n_o \to \infty} \frac{1 - \beta(t_o/n_o)}{t_o/n_o} &= \lambda_E - \frac{\log(c + \epsilon' e^{\lambda_E t_o})}{t_o} \\ &\geq \lambda_E - \delta/2,\end{aligned}$$

because of our choice of t_o and ϵ'. Clearly for $n_o \geq N_o'$ large enough, $(1 - \beta(t_o/n_o)/(t_o/n_o) \geq \lambda_E - \delta$, so that $\beta(t_o/n_o) \leq 1 - (t_o/n_o)(\lambda_E - \delta)$. If we choose $h_\delta := t_o/n_o$, then certainly $\beta(h_\delta) = 1 - h_\delta(\lambda_E - \delta)$ is a feasible convergence parameter in the $\mu-GE$ property for the h_δ-approximation. If the convergence parameters for h-approximations have the linear structure from the statement of the theorem, for all $h < h_o$, then we find that $1 - h(\lambda_E - \delta)$ is a feasible convergence parameter for any $\delta > 0$ by similar arguments as in the foregoing.

Suppose finally that $\mu-GE$ holds for constant c and parameter $\beta_E(h)$ for some h-approximation. Then

$$\begin{aligned}\sum_j |P_{ij}(t) - \Pi_{ij}| &= \sum_j |\sum_{n=0}^{\infty} {}^{\alpha(h)}P_{ij}^n e^{-th^{-1}} \tfrac{(th^{-1})^n}{n!} - \Pi_{ij}|\mu_j \\ &= \sum_j |\sum_{n=0}^{\infty} \big({}^{\alpha(h)}P_{ij}^n - \Pi_{ij}\big) e^{-th^{-1}} \tfrac{(th^{-1})^n}{n!} |\mu_j \\ &\leq \sum_{n=0}^{\infty} c\beta_E^n e^{-th^{-1}} \tfrac{(th^{-1})^n}{n!} \mu_i \leq c \cdot e^{-(1-\beta_E)th^{-1}} \mu_i,\end{aligned}$$

so that $\| P(t) - \Pi \|_M \leq c e^{-\lambda_E t}$, for $\lambda_E = \big(1 - \beta_E(h)\big)h^{-1}$ and this λ_E is a convergence parameter for the MP.

QED

4. Applications

This section studies two examples, where the discrete time associated MCs described in the previous section, are used to study the long run behaviour of the underlying MP. The properties for the MCs are of interest in themselves, since often the observation or control of stochastic networks is in discrete time. For both examples we construct a μ-vector that increases exponentially with the state variable. This implies that both the marginal distributions converge at an exponential rate, as well as the expected marginal rewards, for immediate reward vectors that are polynomial in the state variable.

4.1. A stochastic population model

The model is from [17], p. 160. A Poisson process with parameter λ models mutant individuals entering a population. Such a mutant becomes the ancestor of a new family. Individuals give birth at an exponential rate ν and they die at an exponential rate μ, independently of the other individuals.

The stochastic process associated with the number of families of size k at time t, $k \in \mathbb{N}$, is a MP on a countable state space $E = \{\underline{i} \mid \underline{i} \in \mathbb{N}_0^\infty, |\underline{i}| < \infty\}$, where $|\cdot|$ denotes the ℓ_1-norm. The k^{th} component i_k of $\underline{i}$ denotes the number of families of size k. This MP is irreducible and for ergodicity it is sufficient to assume $\nu < \mu$.

As the transition rates are unbounded in the states, the MP is not uniformisable and therefore we will study the jump chain of the process. Write e_k for the k^{th} unit vector. Then the jump chain has the following transition probabilities

$$\begin{aligned}
{}^JP_{\underline{i},\underline{i}+e_1} &= \frac{\lambda}{\lambda + \sum_k k i_k(\nu+\mu)} \\
{}^JP_{\underline{i},\underline{i}-e_k+1_{\{k>1\}}e_{k-1}} &= \frac{k i_k \mu}{\lambda + \sum_k k i_k(\nu+\mu)} \\
{}^JP_{\underline{i},\underline{i}-e_k+e_{k+1}} &= \frac{k i_k \nu}{\lambda + \sum_k k i_k(\nu+\mu)}.
\end{aligned}$$

As the process satisfies a left skipfree type of property and the taboo set is finite, the transition probabilities are not affected by the taboo set for initial states $\underline{i}$ with $|\underline{i}|$ large, so that in (1.3) the summation is taken over *all* states for such $\underline{i}$. For $\mu_{\underline{i}} := e^{\epsilon \sum_k k i_k}$, $\epsilon > 0$,

$$\sum_{\underline{j}} {}^JP_{\underline{i}\underline{j}}\mu_{\underline{j}} \le \beta\mu_{\underline{i}} \iff \lambda e^\epsilon + \sum_k k i_k\big(e^{-\epsilon}\mu + e^\epsilon\nu\big) \le \beta\big[\lambda + \sum_k k i_k(\mu+\nu)\big]. \tag{4.1}$$

First consider the expression $\mu e^{-\epsilon} + \nu e^\epsilon$. As $\nu < \mu$, there is a $\beta < 1$, such that this expression is smaller than $\beta(\mu+\nu)$, for all sufficiently small $\epsilon > 0$. Consequently the difference $\sum_{k\ge1} k i_k\big\{\beta(\mu+\nu) - (e^{-\epsilon}\mu + e^\epsilon\nu)\big\}$ is strictly positive for $\underline{i} \neq 0$ and increases with increasing $\sum_{k\ge1} k i_k$. Hence, there is a constant $c^* \in \mathbb{N}$, say, such that (4.1) is satisfied for states $\underline{i}$ with $\sum_{k\ge1} k i_k \ge c^*$. Let $M := \{\underline{i} \in E \mid \sum_{k\ge1} k i_k \le c^*\}$. This is clearly a finite set. Then for all states $\underline{i}$, with $\sum_{k\ge1} k i_k \ge c^*$,

$$\sum_{\underline{j}} {}^J_MP_{\underline{i}\underline{j}}\mu_{\underline{j}} \le \sum_{\underline{j}} {}^JP_{\underline{i}\underline{j}}\mu_{\underline{j}} \le \beta\mu_{\underline{j}}.$$

From a state $\underline{i}$ with $\sum_k k i_k < c^*$ the process can never jump to a state $\underline{j} = (j_1, j_2, \ldots)$ with $\sum_k k j_k > c^*$, so that

$$\sum_{\underline{j}} {}^J_MP_{\underline{i}\underline{j}}\mu_{\underline{j}} = 0,$$

for such initial states $\underline{i}$. Combination with the corollary to Lemma 3.1 thus proves the following result.

LEMMA 4.1 *Consider the* MP *associated with the number of families of size* k, $k \in \mathbb{N}$, *in the stochastic population model described in the foregoing, with* $\nu < \mu$. *The jump chain satisfies (1.3) for the vector* μ *with* $\underline{i}^{th}$ *component* $e^{\epsilon \sum_k k i_k}$ *for some* $\epsilon > 0$ *and the finite set* M *as defined above. Hence, the jump chain has the* $\mu-$WGR *(3.2) and* $\mu-$GE *(3.1) properties and the* MP *has the* $\mu-$WER *(1.5) and* $\mu-$EE *(1.4) properties.*

4.2. The Jackson network

In this subsection we will show the $\mu-$EE (1.4) and $\mu-$WER (1.5) properties for the Jackson network, for a μ-vector that grows exponentially fast in the state variable, similar to the previous analysis.

The method used does not consist of constructing Lyapunov functions, but reduces to a coupling argument for the sample paths of the MP associated with the number of customers at each node. A drawback of our method is it's quite heavily using the stationary distribution to have exponentially fast decaying tails. [23] develops a method to show the $\mu-$EE property for exponential μ-vectors with similar coupling arguments, but without using the stationary distribution.

Let us describe the model. The network consists of m service centres. There are s_k servers in the k^{th} centre, and the service disciplines are assumed to be FCFS. The external arrival stream in the k^{th} centre is Poisson (λ_k) distributed. A customer at centre k requires an exponentially (ν_k) distributed amount of service; upon finishing service he routs to center l with probability r_{kl}, $k, l = 1, \ldots, m$. To ensure, that a customer eventually leaves the network, we assume the routing matrix $R = \left(r_{kl}\right)_{k,l=1,\ldots,m}$ to be the transition matrix of a transient MC.

The throughput γ_k in centre k is a solution to the following linear system:

$$\gamma = \lambda + R^T \gamma,$$

where γ and λ are column vectors with k^{th} component γ_k and λ_k respectively, and R^T is the transposed matrix R. As the state description we take $\underline{i} = (i_1, \ldots, i_m)$, where $i_k \in \mathbb{N}_0$ represents the number of customers at the k^{th} centre. The stochastic process associated with the number of customers present in each centre is a MP, and it is known to be ergodic, if and only if the following condition is satisfied (cf. [9], §2.4).

Stability condition: $\gamma_k < s_k \nu_k$, $k = 1, \ldots, m$.

Under the stability condition the stationary probability on state $\underline{i}$ is of product form, i.e.

$$\pi_{\underline{i}} = \pi_{\underline{0}} \prod_{k=1}^{m} \Big(\frac{\gamma_k}{s_k \nu_k}\Big)^{i_k} \Big\{ \frac{s_k^{i_k}}{i_k!} \cdot 1_{\{i_k \leq s_k\}} + \frac{s_k^{s_k}}{s_k!} \cdot 1_{\{i_k > s_k\}} \Big\}.$$

The MP clearly is uniformisable, so that μ–EE (1.4) and μ–WER (1.5) of the MP follow from μ–GE (3.1) or μ–WGR (3.2) of h-approximations. For the coupling argument it is simpler to use h-approximations. Choose the time-grid $h < \big(\sum_k \lambda_k + \sum_k s_k \nu_k\big)^{-1}$. The h-approximation has the following transition probabilities:

$$
\begin{aligned}
{}^{\alpha(h)}P_{\underline{i},\underline{i}+e_k} &= \lambda_k h \\
{}^{\alpha(h)}P_{\underline{i},\underline{i}+e_k-e_l} &= \Big[i_l \cdot 1_{\{0 < i_l < s_l\}} + s_l \cdot 1_{\{i_l \geq s_l\}}\Big]\nu_l r_{lk} h \\
{}^{\alpha(h)}P_{\underline{i},\underline{i}-e_l} &= \Big[i_l \cdot 1_{\{0 < i_l < s_l\}} + s_l \cdot 1_{\{i_l \geq s_l\}}\Big]\nu_l (1 - \sum_k r_{lk}) h \\
{}^{\alpha(h)}P_{\underline{i},\underline{i}} &= 1 - \sum \lambda_k h - \sum_l \Big[i_l 1_{\{0 < i_l < s_l\}} + s_l 1_{\{i_l \geq s_l\}}\Big] \cdot (1 - r_{ll}) h.
\end{aligned}
$$

Next choose $\mu_{\underline{i}} = (1+\epsilon)^{|\underline{i}|}$, such that $(1+\epsilon)\gamma_k / s_k \nu_k < 1$, $k = 1, \ldots, m$. Then we will prove the following: $\exists c > 0$, such that

$$\sum_n \sum_{\underline{j}} {}^{\alpha(h)}_{\underline{0}}P^n_{\underline{i}\underline{j}} \mu_{\underline{j}} \leq c \mu_{\underline{i}}, \quad \forall \, \underline{i}. \tag{4.2}$$

By lemma 3.1 in [7], (4.2) implies $\mu-$WGR of this h-approximation.

In words (4.2) means the following. Suppose that the system controller earns a reward $\mu_{\underline{j}}$, whenever the process is in state $\underline{j}$. Then (4.2) requires the vector of total expected μ-rewards until absorption into state $\underline{0}$ to be bounded by a constant times μ. For convenience we will write R_μ for the total μ-reward until the process is absorbed into state $\underline{0}$. So,

$$\mathbb{E}_{\underline{i}} R_\mu = \sum_n \sum_{\underline{j}} {}^{\alpha(h)}_{\underline{0}}P^n_{\underline{i}\underline{j}} \mu_{\underline{j}}.$$

The proof of (4.2) is in two steps. First we will show, that

$$\mathbb{E}_{\underline{i}} R_\mu < \infty, \quad \forall \, \underline{i}. \tag{4.3}$$

After that we will prove the following inequality

$$\mathbb{E}_{\underline{i}} R_\mu \leq (1+\epsilon) \mathbb{E}_{\underline{i}-e_k} R_\mu + \max_{l=1,\ldots,m} \mathbb{E}_{e_l} R_\mu, \tag{4.4}$$

for $i_k \geq 1$, by coupling sample paths.

Using (4.3) and (4.4), (4.2) then immediately follows. Indeed, let $\underline{i}(1) = \underline{i}$, $\underline{i}(2), \ldots, \underline{i}(|\underline{i}|) = e_{l^*}$ be a sequence of states, such that $\underline{i}(n) = \underline{i}(n-1) - e_k$, for some $k \in \{1, \ldots, m\}$. In each successive state one customer is removed from the system. By iterating (4.4) we obtain

$$
\begin{aligned}
\mathbb{E}_{\underline{i}} R_\mu &\leq (1+\epsilon) \mathbb{E}_{\underline{i}(2)} R_\mu + \max_l \mathbb{E}_{e_l} R_\mu \\
&\leq (1+\epsilon)^2 \mathbb{E}_{\underline{i}(3)} R_\mu + \big[1 + (1+\epsilon)\big] \max_l \mathbb{E}_{e_l} R_\mu = (1+\epsilon)^2 \mathbb{E}_{\underline{i}(3)} R_\mu + \epsilon^{-1}\big[(1+\epsilon)^2 - 1\big] \max_l \mathbb{E}_{e_l} R_\mu \\
&\;\;\vdots \\
&\leq (1+\epsilon)^{|\underline{i}|-1} \mathbb{E}_{\underline{i}(|\underline{i}|)} R_\mu + \epsilon^{-1}(1+\epsilon)^{|\underline{i}|-1} \max_l \mathbb{E}_{e_l} R_\mu \\
&\leq (1+\epsilon)^{|\underline{i}|} \cdot \epsilon^{-1} \max_l \mathbb{E}_{e_l} R_\mu = \mu_{\underline{i}} \cdot \epsilon^{-1} \max_l \mathbb{E}_{e_l} R_\mu.
\end{aligned}
$$

Consequently, (4.3) and (4.4) remain to be proved. For (4.3) we observe, that

$$\sum_{\underline{j}} \pi_{\underline{j}} \mu_{\underline{j}} = \pi_{\underline{0}} \sum_{\underline{j}} \sum_{n=0}^{\infty} {}^{\alpha(h)}_{\underline{0}}P^n_{\underline{0}\underline{j}} \mu_{\underline{j}} = \pi_{\underline{0}} \mathbb{E}_{\underline{0}} R_\mu, \tag{4.5}$$

by virtue of the renewal reward theorem (cf. [17], p.78). By our choice of μ, the lefthandside of (4.5) is finite, hence $\mathbb{E}_{\underline{0}}R_\mu < \infty$. As the state space is irreducible, $\mathbb{E}_{\underline{i}}R_\mu < \infty$ can be proved iteratively. This completes the proof of (4.3).

To prove (4.4) we need some notation. Let $\{X^s(n)\}_{n=0}^\infty$, $s = 1, 2$, be the MC with transition matrix ${}^{\alpha(h)}P$ and initial states $\underline{i} - e_{k^*}$ and $\underline{i}$ respectively (so, $i_{k^*} \geq 1$ by assumption). We construct two E-valued stochastic processes $\{\hat{X}^s(n)\}_n$, $s = 1, 2$, on a common probability space $(\Omega, \mathcal{F}, \mathbb{P}')$ with the following properties:

i) $\forall\, \omega \in \Omega, \forall\, n \in \mathbb{N}$ *there is an* $m(\omega, n) \in \{1, \ldots, m\}$, *such that* $\hat{X}^1(n)(\omega) \leq \hat{X}^2(n)(\omega) \leq \hat{X}^1(n)\omega + e_{m(n,\omega)}$.

ii) the probability laws of $\{X^s(n)\}_n$ *and* $\{\hat{X}^s(n)\}_n$ *agree,* $s = 1, 2$.

The construction of the processes $\{\hat{X}^s(n)\}_n$ is fairly standard (see for example [19]), but we give the proof here for completeness of the exposition.

Let $\{U(n)\}_n$ be a sequence of i.i.d. uniformly distributed random variables on $(\Omega, \mathcal{F}, \mathbb{P}')$ with support on $[0, 1)$. Partition the interval $[0, 1)$ into a set of half-open intervals (left closed, right open), which are denoted as $\mathcal{O}_1, \ldots, \mathcal{O}_m$, $\mathcal{O}_{k,l,r}$, $k = 1, \ldots, m$, $l = 0, \ldots, m$, $r = 1, \ldots, s_k$. The length of $\mathcal{O}_k$ equals $\lambda_k h$, and the length of $\mathcal{O}_{k,l,r}$ is $\nu_k r_{kl} h$ for $l \neq 0$, and $\nu_k(1 - \sum_n r_{kn})h$ for $l = 0$. Note, that the union of these intervals is strictly contained in $[0, 1)$. Set $\hat{X}^1(0)(\omega) = \underline{i} - e_{k^*}$, $\hat{X}^2(0)(\omega) = \underline{i}$, $\forall\, \omega \in \Omega$. Using the shorthand notation x and u for $\hat{X}^s(n)(\omega)$ and $U(n)(\omega)$ respectively, $\hat{X}^s(n+1)(\omega)$ are defined recursively as follows.

$$
\begin{aligned}
\hat{X}^s(n+1)(\omega) := x + \sum_{k=1}^m e_k \cdot 1_{\{u \in \mathcal{O}_k\}} + \sum_{k=1}^m \sum_{l \neq k} (e_l - e_k) \cdot 1_{\{x_k > 0\}} \cdot 1_{\{u \in \cup_{r=1}^{\min(x_k, s_k)} \mathcal{O}_{k,l,r}\}} \\
- \sum_{k=1}^m e_k \cdot 1_{\{x_k > 0\}} \cdot 1_{\{u \in \cup_{r=1}^{\min(x_k, s_k)} \mathcal{O}_{k,0,r}\}}
\end{aligned} \tag{4.6}
$$

The construction is such, that a departure in state x implies a departure in state $x + e_{k'}$ for any k', with probability 1. This ensures the process $\{\hat{X}^2(n)\}_n$ to "drift" towards the process $\{\hat{X}^1(n)\}_n$; as such our coupling is a monotone coupling of a special kind.

It is clear, that property ii) is satisfied. Property i) is easily shown inductively. Indeed, set $\hat{x} := \hat{X}^1(n)(\omega)$ and $x = \hat{X}^2(n)(\omega)$. The one step displacement vectors of both processes are the same, except in cases like the following: $x_k \leq s_k$, $x_k = \hat{x}_k + 1$ and $u \in \mathcal{O}_{k,l,x_k}$ for some $k \in \{1, \ldots, m\}$ and $l \neq 0, k$. This implies, however, that $\hat{X}^1(n+1)(\omega) = \hat{x}$ and $\hat{X}^2(n+1)(\omega) = \hat{x} + e_l$, so that property ii) is preserved. Similarly, if $u \in \mathcal{O}_{k,0,x_k}$, then $\hat{X}^1(n+1)(\omega) = \hat{x} = \hat{X}^2(n+1)(\omega)$ and the sample paths agree from time $n + 1$ on. Remark, that $\{\hat{X}^1(n), \hat{X}^2(n)\}_n$ is in fact a bivariate MC.

Next we define random variables $\hat{R}^s_\mu(\omega)$, $\hat{T}^s_\mu(\omega)$ on $(\Omega, \mathcal{F}, \mathbb{P}')$ to be the total μ-reward and the first hitting time of $\underline{0}$ in the corresponding sample path $\{\hat{X}^s(n)(\omega)\}_n$, $s = 1, 2$. By $\hat{T}^c(\omega)$ we denote the first instant n^c, for which $\hat{X}^1(n^c)(\omega) = \hat{X}^2(n^c)(\omega)$; this is the coupling time of the two processes. Obviously, for $\hat{T}^1(\omega) \geq \hat{T}^c(\omega)$

$$
\begin{aligned}
\hat{R}^2_\mu(\omega) &= (1+\epsilon) \sum_{n=0}^{\hat{T}^c(\omega)-1} (1+\epsilon)^{|\hat{X}^1(n)(\omega)|} + \sum_{n=\hat{T}^c(\omega)}^{\hat{T}^1(\omega)-1} (1+\epsilon)^{|\hat{X}^1(n)(\omega)|} \\
&\leq (1+\epsilon)\hat{R}^1_\mu(\omega),
\end{aligned} \tag{4.7}
$$

where the empty summation equals 0. Similarly, for $\hat{T}^1(\omega) < \hat{T}^c(\omega)$

$$\hat{R}^2_\mu(\omega) = (1+\epsilon)\hat{R}^1_\mu(\omega) + \sum_{n=\hat{T}^1(\omega)}^{\hat{T}^2(\omega)-1} (1+\epsilon)^{|\hat{X}^2(n)(\omega)|}. \tag{4.8}$$

Note that $\hat{X}^2\big(\hat{T}^1(\omega)\big)(\omega) = e'_k$ for some k'.

By the strong Markov property $\mathbb{E}\big(\sum_{n=\hat{T}^1}^{\infty}(1+\epsilon)^{|\hat{X}^2(n)(\omega)|} \mid \hat{X}^2(\hat{T}^1) = e'_k\big)$ is equal to the total expected μ-reward until hitting $\underline{0}$, when the process would have started in e'_k initially. By virtue of property ii) this quantity is equal to $\mathbb{E}_{e'_k} R_\mu$; furthermore, $\mathbb{E}\hat{R}^1_\mu = \mathbb{E}_{\underline{i}-e_{k^*}} R_\mu$ and $\mathbb{E}\hat{R}^2_\mu = \mathbb{E}_{\underline{i}} R_\mu$. Finally, combining (4.7) and (4.8), taking expectations and using property ii) as indicated, we obtain (4.4). **QED**

By combination with Theorem 3.3 we have finally obtained the following result.

LEMMA 4.2 *Consider the* MP *associated with the number of customers in the various service centers of a Jackson network with s_k server in centre k. Under the stability condition h-approximations have the $\mu-$WGR (3.2) property for μ a vector with $\mu_{\underline{i}} = (1+\epsilon)^{|\underline{i}|}$, for some $\epsilon > 0$, and finite set $M = \{\underline{0}\}$. Hence, the* MC *has the $\mu-$GE (3.1) property. Moreover the* MP *has the $\mu-$WER (1.5) (for the finite set $M = \{\underline{0}\}$) and $\mu-$EE (1.4) properties.*

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