BASE POINTS OF CUBIC PENCILS, k-MINIMAL MODELS AND BAD FIBRES OF RATIONAL ELLIPTIC SURFACES

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1. Introduction

Let $\mathscr{E} \to B \simeq \mathbb{P}^1$ be a rational elliptic surface defined over a number field k, i.e. a smooth projective algebraic surface, geometrically birational to $\mathbb{P}^2$ endowed with a genus-one fibration with a section. Such a surface is geometrically, i.e. over the algebraic closure $\bar{k}$, isomorphic to the blow up of the nine base points of a linear pencil of cubics in $\mathbb{P}^2$. In [Mir80] Miranda establishes a connection between the possible bad fibres and the type of pencil that induces $\mathscr{E}$ (cf. Definition 2.1). Over the number field k those surfaces are not necessarily birational to $\mathbb{P}^2$. We will say that a smooth rational surface X is a k-minimal model of $\mathscr{E}$ if X contains no (-1) curves defined over k and is birational to $\mathscr{E}$ over k. The theory of minimal models over perfect fields was developed by Enriques, followed later by Manin and Tsfasman (see for example [MT86]).

The goal of this article is to relate three distinct features of rational elliptic surfaces: bad fibres, number of distinct points blown up and possible k-minimal models.

We will first briefly recall some arithmetic and geometric facts such as the construction of rational elliptic surfaces, the theory of minimal models over perfect fields and the Shioda-Tate formula for elliptic surfaces over number fields. In the third section we establish the connection between k-minimal models of rational elliptic surfaces and the points blown up in order to obtain it. Section 4 deals with the relations between those points and the bad fibres. Finally, in Section 5, we present some examples to illustrate what is discussed in the previous sections.

2. Preliminaries

Let k be a number field. Throughout this article an elliptic surface will be an projective algebraic surface $\mathscr{E}$ defined over k endowed with a flat morphism

$$\pi : \mathscr{E} \to B,$$

where is a projective curve B and $\pi^{-1}(t)$ is a genus-one curve for almost all $t \in B(\bar{k})$. Furthermore, we will assume that π admits a *zero section* σ_0:

$$\begin{array}{c} \mathscr{E} \\ \sigma_0 \uparrow\downarrow \pi \\ B \end{array}$$

defined over k. Such a surface is said to be *rational* if there exists a birational map $f : \mathscr{E} \to \mathbb{P}^2$ defined over $\bar{k}$. It is called k*-rational* if there is a rational map to $\mathbb{P}^2$ defined over k.

2.1. **Construction of rational elliptic surfaces.** Let F and G be two cubic homogeneous polynomials in $\overline{k}[X, Y, Z]$. Suppose F is smooth.

The linear pencil of cubics generated by F and G:

$$\Lambda = \{tF + uG = 0; \quad (t : u) \in \mathbb{P}^1\}$$

has nine base points (counted with multiplicities); the blow up of $\mathbb{P}^2$ in those points defines a rational surface $\mathscr{E}_\Lambda$, which, when endowed with the obvious morphism π

$$\begin{array}{ccc} \mathscr{E}_\Lambda & \xrightarrow{\quad\pi\quad} & \mathbb{P}^1 \\ \downarrow & \nearrow_{\alpha \mapsto (F(\alpha):G(\alpha))} & \\ \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^1 \end{array}$$

becomes an elliptic surface. This motivates the following definition.

Definition 2.1. Let Λ be a smooth pencil of cubics in $\mathbb{P}^2$ and let $\mathscr{E} \to \mathbb{P}^1$ be a rational elliptic surface. We will say that Λ *induces* $\mathscr{E}$ if there exists an isomorphism (of elliptic surfaces) between $\mathscr{E}$ and the elliptic surface $\mathscr{E}_\Lambda$ defined by the above construction.

The surface $\mathscr{E}_\Lambda$ is birational to the elliptic (eventually singular) surface:

$$\mathscr{E}'_\Lambda = \{([x, y, z], [t, u]) \in \mathbb{P}^2 \times \mathbb{P}^1 : tF(x, y, z) + uG(x, y, z) = 0\}.$$

Reciprocally, over an algebraically closed field we have:

Proposition 2.2. *[*[Mir80]*] Every smooth rational elliptic surface over $\overline{k}$ is induced by some smooth pencil of cubics in $\mathbb{P}^2$.*

For a proof of that fact see for example [Sal09, Chapitre 2].

2.2. **Minimal models of rational surfaces over perfect fields.** The theory in this subsection was developed by Enriques, Manin and Tsfasman (see [MT86], [Isk79], [Man86]). It extends the classical theory of minimal models of surfaces to the case of a perfect base field k. We recall that a surface is k-minimal, if there are no curves of self intersection -1 defined over k.

Notation: The *degree* of a surface is the self intersection of its canonical bundle. It will be denoted by $d_X := \omega_X^2$.

We only state the main results without proofs; for that purpose, we remind the reader that a *del Pezzo surface* X is a complete smooth surface whose anti-canonical bundle ω_X^{-1} is ample.

Theorem 2.3. *Let X be a k-minimal rational surface defined over a perfect field k. Then X is isomorphic to a surface in one of the following families:*

(**F1**) *A del Pezzo surface with* $\mathrm{Pic} X \simeq \mathbb{Z}$.
(**F2**) *A conic bundle with* $\mathrm{Pic} X \simeq \mathbb{Z}^2$.

*Every X in **F1** is minimal. If X belongs to **F2**, then it is* not *minimal if, and only if, $d_X = 3, 5, 6$ or $d_X = 8$ and X is isomorphic to the ruled surface $\mathbb{F}_1$. There are no minimal surfaces with $d_X = 7$.*

Certain surfaces endowed with a conic fibration are also del Pezzo surfaces, namely if $d_X = 3, 5, 6$ or $d_X = 1, 2, 4$ and X has two distinct fibrations, or $d_X = 8$ and $\bar{X} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown up in one point.

Over an algebraically closed field, a del Pezzo surface X of degree $1 \leq d_X \leq 7$ is isomorphic to $\mathbb{P}^2$ blown up in $9 - d_X$ distinct points in general position; if $d_X = 8$, then either $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, or X is $\mathbb{P}^2$ blown up at one single point (see [Man86, Theorem 24.3]).

Definition 2.4. *We will say that a surface X is k-birationally trivial, or k-rational if there is a birational map $\mathbb{P}^2 \to X$ defined over k.*

Theorem 2.5 ([MT86], Theorem 3.3.1)**.** *Let X be a del Pezzo surface or a conic bundle (over k). If $d_X \leq 4$, then X is not k-birationally trivial. If $d_X \geq 5$ and $X(k) \neq \emptyset$, then X is k-rational.*

Remark 2.6. A priori, most k-minimal rational surface with $d_X \geq 1$ with $X(k) \neq \emptyset$ can occur as a k-minimal model of a rational elliptic surface. The condition $X(k) \neq \emptyset$ is imposed by the existence of a section of $\mathscr{E} \to B$ defined over k that is contracted to a k-rational point. The condition $d_X \geq 1$ follows from $d_{\mathscr{E}} = 0$.

2.3. **The Shioda-Tate formula over a number field.** We will now recall the Shioda-Tate formula over an algebraically closed field as well as over an arbitrary number field k.

Let $\pi : \mathscr{E} \to B$ be an elliptic surface defined over a number field k with zero section σ_0. Let

$$F_v = \pi^{-1}(v) = n_1 C_1 + ... + n_s C_s$$

be the fibre over $v \in B(\bar{k})$, where C_i denote its irreducible components. Let

$$F_v = \bigoplus_i \frac{\mathbb{Z}C_i}{\mathbb{Z}F}.$$

Let G denote the absolute Galois group of k. The finite sum

$$\mathscr{F} = \bigoplus_{\text{reducible bad fibres}} F_v$$

is stable under the action of G. Taking the embedding of the Mordell-Weil group in the Néron-Severi group that sends a section σ to $\sigma - \sigma_0$ we have the isomorphism below.

Theorem 2.7 (Shioda-Tate formula)**.** *Let $\mathscr{E}$ be an elliptic surface. Identify the Mordell-Weil group $\mathscr{E}(\bar{k}(B))$ with its image in $\mathrm{NS}(\mathscr{E}|\bar{k})$ by means of the map $\sigma \mapsto \sigma - \sigma_0$. Then we have the following decomposition*

$$\mathrm{NS}(\mathscr{E}|\bar{k}) = \mathscr{E}(\bar{k}(B)) \oplus \langle \sigma_0, F \rangle \oplus \mathscr{F},$$

where $\langle \sigma_0, F \rangle \simeq \mathbb{Z}^2$ is the subgroup generated by a fibre and the zero section.

In particular we have:

Corollary 2.8. *Let $\mathscr{E}$ be an elliptic surface. Let ρ be the rank of its Néron-Severi group, r be the rank of its generic fibre over $\bar{k}$ and m_v the number of irreducible components of the fibre F_v then*

$$\rho = r + 2 + \sum_v (m_v - 1).$$

For the proof of the theorem above see [Shi99].

Over the number field k, we have an isomorphism of G-modules:

$$\mathrm{NS}(\mathscr{E}|\bar{k})^G \simeq \mathscr{E}(\bar{k}(B))^G \oplus \langle \sigma_0, F \rangle \oplus \mathscr{F}^G.$$

The obvious embedding $\mathrm{NS}(\mathscr{E}|k) \hookrightarrow \mathrm{NS}(\mathscr{E}|\bar{k})^G$ is not, in general, surjective. The former group has finite index inside the latter: tensoring with $\mathbb{Q}$ gives us an isomorphism of vector spaces $\mathrm{NS}(\mathscr{E}|k) \otimes \mathbb{Q} \cong \mathrm{NS}(\mathscr{E}|\bar{k})^G \otimes \mathbb{Q}$.

Corollary 2.9.

$$\mathrm{rank}(NS(\mathscr{E}|k)) = 2 + \mathrm{rank}(\mathscr{E}(k(B))) + \mathrm{rank}(\mathscr{F}^G).$$

3. k-MINIMAL MODELS AND BASE POINTS

Theorem 3.1. *Let $\mathscr{E}$ be a rational elliptic surface defined over a number field k, and X be a k-minimal model of $\mathscr{E}$. Then*

i) *X is a del Pezzo surface of degree one if and only if the generic rank of $\mathscr{E}$ is eight over $\bar{k}$ and zero over k,*
ii) *If $1 \leq d \leq 7$ then the generic rank over k is at most 7,*
iii) *Suppose $X \simeq \mathbb{P}^2_k$. The generic rank of $\mathscr{E}$ over k is eight if and only if $\mathscr{E}$ is obtained from X by blowing up nine distinct k-rational points in general position (no three collinear nor six in a conic).*

Proof: The proof consists of several applications of the Shioda-Tate formula.

i) Assume that X is a k-minimal del Pezzo surface of degree one then the Picard group of X is $\mathbb{Z}$ and thus $\mathrm{Pic}(\mathscr{E}) \simeq \mathbb{Z}^2$. By Corollary 2.9 we have that $\mathrm{rank}(\mathscr{E}|k) = 0$ and that $\mathscr{E}$ has no reducible bad fibre. Hence its geometric Mordell-Weil rank is as big as possible, i.e., eight.

ii) Note that if X has degree d since $\mathscr{E}$ is obtained by blowing up d points in X we have $\mathrm{rank}(\mathrm{Pic}(\mathscr{E}|k)) = \mathrm{rank}(\mathrm{Pic}(X|k)) + d \leq 2 + d \leq 9$. By the Shioda-Tate formula $\mathrm{rank}(\mathscr{E}|k) \leq 7$.

iii) Assume $\mathrm{rank}(\mathscr{E}|k) = 8$. We have to check that any nine points in the plane blown up in order to obtain $\mathscr{E}$ are k-rational, distinct and in general position. Suppose first that they are not in general position, say three in a line l. Consider the conic q through five among the other six base points. Then the cubic $l \cup q$ belongs to the pencil that induces $\mathscr{E}$ and produces a reducible bad fibre which contradicts the maximality of the rank. Now suppose the blown up points are not distinct (blow up infinitely near points), i.e., two cubics of the pencil intersect with multiplicity at least two in a base point P. Suppose for simplicity that $P = (0:0:1)$ and that the tangent direction is given by the line $x = 0$. This means that the cubics in the pencil are of the form $F_1 + \lambda F_2$ with $F_i : a_i y +$ terms of higher degree with $a_2 \neq 0$. The curve given by $F_1 + \lambda_0 F_2$ where $\lambda_0 = \frac{a_1}{a_2}$ is singular at P. Thus it induces, by blowing up the singular point, a reducible bad fibre in $\mathscr{E}$. To complete the proof of this direction we must show that the points are k-rational. Suppose the contrary. For simplicity assume that P_1 and P_2 are two conjugate points of order two so that $P_2 = \sigma(P_1)$ for some $\sigma \in \mathrm{Gal}(\bar{k}|k)$. Then since the base points are in general position the exceptional curves above P_1 and P_2 are independent in the Mordell-Weil group $\mathscr{E}(\bar{k}(B))$ and do not belong to $\mathscr{E}(k(B))$, it follows that $\mathrm{rank}(\mathscr{E}|k) < \mathrm{rank}(\mathscr{E}|\bar{k}) = 8$.

Now assume the blown up points are in general position then [Man64, Theorem 6] implies that the geometric rank is eight and thus there are no reducible bad fibres. Since all the points blown up to obtain $\mathscr{E}$ are k-rational, the exceptional curves above these points generate a finite index subgroup of $\mathscr{E}(k(B))$ and are all defined over k thus one has

$$\operatorname{rank}(Pic(\mathscr{E}|k)) = \operatorname{rank}(Pic(\mathbb{P}^2|k)) + 9 = 10.$$

Since there is no contribution of the bad fibres to the rank of the Picard group one has $\operatorname{rank}(\mathscr{E}|k) = 8$. □

Remark 3.2. In case $i)$ of the above theorem, Zariski density of k-rational points in X is tantamount to the existence of infinitely many fibres of positive rank in $\mathscr{E}$.

4. k-minimal models and bad fibres

Let $\mathscr{E}$ be a rational elliptic surface defined over k and X be a k-minimal model of degree d_X. Then according to Theorem 2.3 the Picard group $\operatorname{Pic}(\mathscr{E}|k)$ satisfies:

$$\operatorname{rank}(\operatorname{Pic}(\mathscr{E}|k)) \leq d_X + 1 \text{ or } d_X + 2.$$

We will place ourselves for simplicity in the first case, i.e. X is a del Pezzo surface with $\operatorname{Pic}(X|k) \simeq \mathbb{Z}$. This implies that $\operatorname{rank}(\mathscr{E}|k) \leq d_X - 1$, where the equality holds if and only if $\mathscr{E}$ has no reducible bad fibres or points blown up in X to obtain $\mathscr{E}$ are all distinct, k-rational and in general position. A closer analysis of the possible configuration of singular fibres and Galois action on it allows us to produce the following table:

d_X	$\operatorname{rank}(\operatorname{Pic}(\mathscr{E}|k))$	$\operatorname{rank}(\mathscr{E}|k)$	Possible types of bad fibres	$\operatorname{rank}(\mathscr{E}|\bar{k})$
1	2	0	I_1, II	8
2	3	0	$I_{n\leq 3}, II, III, IV$	≤ 7
2	3	1	I_1, II	8
3	4	0	$I_{n\leq 5}, II, III, IV, I_0^*$	≤ 6
3	4	1	$I_{n\leq 3}, II, III, IV$	≤ 7
3	4	2	I_1, II	8
4	5	0	$I_{n\leq 5}, II, III, IV, I_{n\leq 1}^*$	≤ 5
4	5	1	$I_{n\leq 5}, II, III, IV, I_0^*$	≤ 6
4	5	2	$I_{n\leq 3}, II, III, IV$	≤ 7
4	5	3	I_1, II	8
5	6	0	$I_{n\leq 5}, II, III, IV, I_{n\leq 2}^*, IV^*$	≤ 4
5	6	1	$I_{n\leq 5}, II, III, IV, I_{n\leq 1}^*$	≤ 5
5	6	2	$I_{n\leq 5}, II, III, IV, I_0^*$	≤ 6

d_X	rank(Pic($\mathscr{E}\|k$))	rank($\mathscr{E}\|k$)	Possible types of bad fibres	rank($\mathscr{E}\|k$)
5	6	3	$I_{n\leq 3}, II, III, IV$	≤ 7
5	6	4	I_1, II	8
6	7	0	$I_{n\leq 7}, II, III, IV, I^*_{n\leq 2}, IV^*$	≤ 3
6	7	1	$I_{n\leq 5}, II, III, IV, I^*_{n\leq 2}, IV^*$	≤ 4
6	7	2	$I_{n\leq 5}, II, III, IV, I^*_{n\leq 1}$	≤ 5
6	7	3	$I_{n\leq 5}, II, III, IV, I^*_0$	≤ 6
6	7	4	$I_{n\leq 3}, II, III, IV$	≤ 7
6	7	5	I_1, II	8
8	9	0	$I_{n\leq 9}, II, III, IV, I^*_{n\leq 4}, IV^*, III^*$	≤ 1
8	9	1	$I_{n\leq 9}, II, III, IV, I^*_{n\leq 3}, IV^*$	≤ 2
8	9	2	$I_{n\leq 7}, II, III, IV, I^*_{n\leq 3}, IV^*$	≤ 3
8	9	3	$I_{n\leq 7}, II, III, IV, I^*_{n\leq 2}, IV^*$	≤ 4
8	9	4	$I_{n\leq 5}, II, III, IV, I^*_{n\leq 1}$	≤ 5
8	9	5	$I_{n\leq 5}, II, III, IV, I^*_0$	≤ 6
8	9	6	$I_{n\leq 3}, II, III, IV$	≤ 7
8	9	7	I_1, II	8
9	10	$0 \leq r \leq 8$	$I_{n\leq 9}, II, III, IV, I^*_{n\leq 4} IV^*, III^*, II^*$	$0 \leq \bar{r} \leq 8$

We now give a sufficiency criterion for a rational elliptic surface to be k-rational.

Proposition 4.1. *Let $\pi : \mathscr{E} \longrightarrow B = \mathbb{P}^1$ be a rational elliptic surface defined over a number field k. If one of the bad fibres is of type* II* *or* III**, then $\mathscr{E}$ is k-rational.*

Proof: Due to Theorem 2.5, it suffices to show that the degree of a k-minimal model of $\mathscr{E}$ is at least six. The number of irreducible components on the union of all bad fibres is exactly 12. Hence, there exists *at most* one fibre F of type II* or III*; this entails that F is defined over k. Moreover, considering the configuration and multiplicity of the irreducible components of F, we see that each one of these is also defined over k. Each contributes to the Picard group Pic($\mathscr{E}|k$). Since a II^* (resp. III^*) fibre has nine (resp. eight) components we have that the rank of Pic($\mathscr{E}|k$) is at least nine and thus the degree of X is at least six. □

Remark 4.2. One can show even more: a k-minimal model of such a rational elliptic surface is isomorphic either to $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. For that it is required a closer look at the possible pencils that induce $\mathscr{E}$.

5. Bad fibres and base points

Theorem 5.1. *Let $\mathscr{E}$ be a rational elliptic surface and Γ a pencil of cubics that induces $\mathscr{E}$. Then Γ has one base point with multiplicity nine if and only if $\mathscr{E}$ has a fibre of type II^*.*

Proof: If $\mathscr{E}$ has a fibre of type II^* then a pencil that induces is unstable (cf. [Mir80]) and thus contains either a triple line or the union of a double line and another line. To obtain a fibre of type II^* one is forced to blow up the same point nine times in one of these to curves.

Now suppose that $\mathscr{E}$ is obtained by the blow up of the same point P nine times. Let C_1 and C_2 be two smooth cubics in the pencil. Since they intersect with multiplicity nine at P, this point is an inflection point. Since there is only one base point the reducible bad fibre is obtained by the blow up of the singular point P in a cubic C_3 in the pencil that has one single –possibly multiple– component (otherwise every member of the pencil would share a component). We have the following possibilities for C_3: a node, a cusp or a triple line. The first two cases never occur as the intersection of a smooth cubic with a cubic of these type contains at least two points. Hence a bad fibre is obtained by the blow up of a point in a triple line. This induces a fibre of type II^*.

Theorem 5.2. *Let $\mathscr{E}$ be a rational elliptic surface and Γ a pencil of cubics that induces $\mathscr{E}$. Let m be the number of distinct base points of Γ. The following hold:*

i) *If $m = 2$ then $\mathscr{E}$ has a bad fibre of type III^* or I_4^*.*
ii) *If $\mathscr{E}$ has a fibre of type IV^* then $m = 3$.*
iii) *If $\mathscr{E}$ has a fibre of type I_n with $n \geq 4$ then $m \leq 8$.*
iv) *If $\mathscr{E}$ has a fibre of type I_n^* with $n \leq 3$ then $3 \leq m \leq 6 - n$.*

Proof: We will analyze each case separately.

Case i) Let Γ be a pencil of cubics that induces $\mathscr{E}$. We will show that it contains either a triple line or the union of a double line and a simple line. This will be done in the following steps:

a) Γ contains at least one cubic whose singularity is a base point.
b) Γ does not contain a cubic given by the union of three distinct lines.
c) Γ does not contain a cubic given by the union of a line and a conic.
d) Γ does not contain an irreducible singular cubic whose singularity is a base point.

The proof of a) is the same as in iii) of Theorem 3.1.

If Γ contains a cubic given by the union of three distinct lines then at least two of the lines pass by only one base point (the same one) and thus each has to pass with multiplicity three (they are both inflection points). But this is an absurd since the points are smooth points in almost all curves of the pencil and thus have only one tangent direction.

The statement in c) follows from the fact that a smooth conic that passes through a singular point of a cubic intersects it in at least two other distinct points. The same type of argument applies to show that if the intersection of an irreducible singular cubic and a smooth cubic contains the singularity then it contains two other points. Thus the only possible singular cubic in the pencil is given by a triple line or by the union of a double line and another line. The blow up of the two base points in this type of singular cubic produces a III^* or an I_4^* fibre.

Case ii) It follows from [Mir80, Lemma 6.4] that a pencil that induces $\mathscr{E}$ contains a cubic given by either a triple line or by the union double line and another line. In the first case since the sum of the multiplicities of the base points is nine we have $m \leq 3$. By the previous case if $m = 2$ then $\mathscr{E}$ has a fibre of type III^* or I_4^*, an Euler number calculation shows that it cannot have a IV^* fibre. If $m = 1$ it follows from the previous theorem that $\mathscr{E}$ has a fibre of type II^* and by the same argument as above it cannot have a fibre of type IV^*. In the second case each line intersect a cubic in at most three points implying $m \leq 6$. If $m = 6$ then each component of the fibre given by this cubic has multiplicity one or two, but a IV^* fibre has a component with multiplicity three. If $m = 5$ the blow up of two points in the double line would give two multiplicity one components intersecting a double one, which once again does not occur in a fibre of type IV^*. Finally, if $m = 4$ then all the possible configurations of the base points induce either fibres of type I_0^* or I_1^*, or lead to inexistent fibre configurations.

Case iii) A semi-stable fibre is induced by a nodal cubic, or by a cubic given by the union of a line and a conic intersecting in two distinct points, or by three non concurrent lines. Hence if $n \geq 4$ we must blow up at least one singular point, which has multiplicity at least two and thus $m \leq 8$.

Case iv) A fibre of type I_n^* has $n+5$ components. Since a cubic has at most three components, to obtain such a fibre one has to blow up a point at least $3+n$ times. Hence $9-(3+3) \leq m \leq 9-(3+n) = 6-n$. □

6. Examples

Example 1: The following construction gives an example of a rational elliptic surface obtained by the blow up of nine distinct points in $\mathbb{P}^2(k)$, where $k = \mathbb{Q}[e^{\frac{2i\pi}{6}}]$, whose Mordell-Weil group has generic rank zero.

Let $F : x^3 + y^3 + z^3 = 0$ and $G : 3xyz = 0$. The intersection of these two cubic curves is given by the points $P_1, ..., P_9$ where $P_1 = (1 : -1 : 0), P_2 = (e^{\frac{2i\pi}{6}} : e^{\frac{2i\pi}{6}} : 0), P_3 = (1 : e^{\frac{2i\pi}{6}} : 0)$ and $P_4, ..., P_9$ are given by permutations of the coordinates of P_1, P_2 and P_3.

The generic fibre is described by the equation:

$$\mathscr{E}_\eta : x^3 + y^3 + z^3 - 3xyzt = 0, \quad \text{over } k(t).$$

The bad fibres are all of type I_3, two of them are defined over $\mathbb{Q}$ and the others over a quadratic field.

An application of the Shioda-Tate formula shows that the Mordell-Weil rank of $\mathscr{E}_\eta$ is zero over the algebraic closure, and thus over k as well. The Mordell-Weil group is isomorphic to $\frac{\mathbb{Z}}{3\mathbb{Z}} \times \frac{\mathbb{Z}}{3\mathbb{Z}}$ (cf. [Per90]).

Example 2: We will give an example of a rational elliptic surface defined over $\mathbb{Q}$ for which $\mathbb{P}^2$ is not a $\mathbb{Q}$-minimal model.

Consider the cubic surface X in $\mathbb{P}^3$ given by the following equation:

$$T_0^3 + T_1^3 + T_2^3 + 2T_3^3 = 0.$$

It is a $\mathbb{Q}$-minimal surface (see for example [Man86, Ex. 21.9]) such that $\mathrm{Pic}(X) \simeq \mathbb{Z}$ and $\mathrm{Pic}(X|\overline{\mathbb{Q}}) \simeq \mathbb{Z}^7$. Let $i : X \hookrightarrow \mathbb{P}^3$ be the closed immersion given by the anti-canonical bundle Ω_X^{-1}. The anti-canonical divisor satisfies:

$$\omega_X^{-1} \simeq i^*(\mathscr{O}_{\mathbb{P}^3}(1)).$$

A linear pencil in $|\omega_X^{-1}|$ has three base points.

We will consider a linear pencil such that one base point is defined over $\mathbb{Q}$ and the others are conjugate under $Gal(\overline{\mathbb{Q}}|\mathbb{Q})$.

The pencil is the following:

$$\{tF + uG = 0; (t : u) \in \mathbb{P}^1\}, \text{ where } F = T_0 - T_1 \text{ and } G = T_2.$$

The base points are

$$P_1 = (1 : 1 : 0 : -1), P_2 = (1 : 1 : 0 : -e^{\frac{2i\pi}{3}}) \text{ and } P_3 = (1 : 1 : 0 : -e^{\frac{4i\pi}{3}}).$$

The blow up of X on those three points gives rise to a (isotrivial) rational elliptic surface whose Weierstrass equation given by

$$\mathscr{E} : Y^2 = X^3 + (\frac{28}{27}t^6 + \frac{1}{216}t^3).$$

Its bad fibres are of type I_0^* and $3II$.

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