The diameter of the circumcircle of a Heron triangle

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A *Heron triangle* is a triangle with integral sides and integral area. The *circumradius* and *circumdiameter* of a triangle are respectively the radius and the diameter of its circumcircle. In this article we will give a necessary and sufficient condition for a positive integer to be the circumradius or the circumdiameter of some Heron triangle. It is stated in the following theorem.

Theorem 1 *Let k be a positive integer. Then the following are equivalent.*

(i) *There is a Heron triangle with circumradius k.*
(ii) *There is a Heron triangle with circumdiameter k.*
(iii) *There is a prime divisor p of k satisfying $p \equiv 1 \bmod 4$.*

This theorem generalizes the results of Kramer and Luca [**2**, Proposition 6] and Aassila [**1**, pp. 145–146]. They give identical proofs for the implication (iii) ⇒ (i). For the converse, Aassila had already conjectured that no prime p satisfying $p \equiv 11 \bmod 12$ is the circumradius of a Heron triangle. This conjecture was proven by Kramer and Luca, who showed that in fact no power of a prime p with $p = 2$ or $p \equiv 11 \bmod 12$ is the circumradius of a Heron triangle.

Question The question of which *rational* numbers occur as the circumdiameter of a Heron triangle remains open. It follows from Theorem 1 that if d is such a number, then there is a prime divisor p of the numerator of d that satisfies $p \equiv 1 \bmod 4$. We leave it as an exercise to the reader to show that every positive integer is the denominator of the circumdiameter of some Heron triangle.

To prove the implication (iii) ⇒ (i) of Theorem 1, papers [**1**] and [**2**] take any positive integer k with a prime divisor p as in (iii). Both first show that there is a triangle with circumradius p that is Pythagorean, i.e., right-angled and with integral sides. After scaling by a factor k/p they obtain a Pythagorean triangle with circumradius k. It is easy to show that every Pythagorean triangle has integral area and is thus a Heron triangle. The new implication (i) ⇒ (iii) of Theorem 1 tells us that among all Heron triangles there are no more integral circumradii than among just the Pythagorean triangles.

The construction above only yields a triangle with relatively prime sides if the integer k is prime. This leaves a rather unsatisfactory feeling, which is resolved by the following theorem. It gives necessary and sufficient conditions for an integer to be the circumdiameter of some Heron triangle with sides that are relatively prime to each other. Theorem 1 will follow as a consequence.

Theorem 2 *Let d be a positive integer. Then the following are equivalent.*

(i) *There is a Pythagorean triangle with circumdiameter d and sides a, b, and c satisfying* $\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1$.
(ii) *There is a Heron triangle with circumdiameter d and sides a, b, and c satisfying* $\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1$.
(iii) *There is a Heron triangle with circumdiameter d and sides a, b, and c satisfying* $\gcd(a, b, c, d) = 1$.

(iv) *There is a triangle with circumdiameter d and integral sides $a, b,$ and c satisfying* $\gcd(a, b, c, d) = 1$.

(v) *We have $d > 1$ and every prime divisor p of d satisfies $p \equiv 1 \bmod 4$.*

(vi) *There are positive integers x and y with x even and* $\gcd(x, y) = 1$, *such that d can be written as $d = x^2 + y^2$.*

All we will use is some elementary number theory and two formulas. Heron's formula

$$(4A)^2 = (a + b + c)(a + b - c)(a - b + c)(-a + b + c) \tag{1}$$

relates the area A of a triangle to the lengths a, b, and c of its sides. The circumdiameter d is related to these quantities through the equation

$$2Ad = abc. \tag{2}$$

Remark In Lemma 3 we will deduce from Heron's formula (1) that any triangle with integral sides and rational area is a Heron triangle. Together with (2) this proves the implication (iv) $\Rightarrow$ (iii) of Theorem 2. The fact that (ii) implies (i) is a priori not obvious. Neither is the implication (iii) $\Rightarrow$ (ii), as a triangle with sides a, b, c, and circumdiameter d may satisfy $\gcd(a, b, c, d) = 1$ even if a, b, and c have a divisor in common. The triangle with sides 15, 15, 24, and area 108, for example, has circumdiameter 25. By Theorem 2 the existence of this example does imply that there is a triangle with circumdiameter 25 whose sides are relatively prime. In this case, for $d = 25$, the triangle with sides 7, 24, and 25 satisfies all conditions of (i), and therefore also those of (ii).

The following lemmas will be useful in proving Theorem 1 and Theorem 2.

Lemma 3 *Any triangle with integral sides and rational area is a Heron triangle.*

Proof. The quantity in (1) is both an integer and the square of a rational number. Hence, it is the square of the integer $4A$. Equation (1) can be rewritten as $(4A)^2 = 4b^2c^2 - (a^2 - b^2 - c^2)^2$. Since -1 is not a square modulo 4, the quantity $a^2 - b^2 - c^2$ is even. Therefore, so are the four factors in the right-hand side of (1) and the quantity in (1) is divisible by 16. It follows that A^2, and thus A, is integral. □

For any prime p and any nonzero integer x, let $v_p(x)$ denote the largest integer m for which p^m divides x. Then v_p extends uniquely to a group homomorphism from the group $\mathbb{Q}^*$ of nonzero rational numbers to the group $\mathbb{Z}$ of integers.

Lemma 4 *Let p be a prime satisfying $p \not\equiv 1 \bmod 4$. Let s, t, and m be integers with $m \geq 0$. Then p^{2m} divides $s^2 + t^2$ if and only if p^m divides both s and t.*

Proof. The "if"-part is obvious. For the "only if"-part, we may replace s, t, and m by αs, αt, and $m + v_p(\alpha)$ for some $\alpha \in \mathbb{Q}^*$, and assume $p \nmid \gcd(s, t)$, say $p \nmid s$. By choosing α appropriately, we may even assume $s \equiv 1 \bmod p^2$. If p^2 divided $s^2 + t^2$, then we would have $t^2 \equiv -s^2 \equiv -1 \bmod p^2$. This contradicts the fact that -1 is neither a square modulo 4, nor modulo primes q satisfying $q \equiv 3 \bmod 4$, see [**3**, Theorem 3.1.d]. We conclude that p^2 does not divide $s^2 + t^2$, so we have $m = 0$. The result follows immediately. $\square$

We are now ready to prove our theorems.

Proof of Theorem 2. Reduction modulo 4 shows that at least one of the sides of a Pythagorean triangle other than the hypotenuse is even. Therefore, every Pythagorean triangle has integral area, which proves implication (i) $\Rightarrow$ (ii). The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious. For (iv) $\Rightarrow$ (v), let a, b, and c be the sides of a triangle as in (iv). Because the equilateral triangle with sides 1 does not have integral circumdiameter, we get $\max(a, b, c) > 1$. The largest chord of a circle being a diameter, we find $d \geq \max(a, b, c) > 1$. Let p be any prime divisor of d and suppose p satisfies $p \not\equiv 1 \bmod 4$. Without loss of generality, we may assume $v_p(a) \geq v_p(b) \geq v_p(c)$. As p does not divide $\gcd(a, b, c)$, we have $v_p(c) = 0$. By Lemma 3 the rational area $A = abc/(2d)$ is integral. This implies that p divides abc, so p divides a. Set $m = v_p(2a) > 0$. After rewriting (1) as $(4A)^2 + (a^2 + b^2 - c^2)^2 = 4a^2b^2$, we see that p^{2m} divides $(4A)^2 + (a^2 + b^2 - c^2)^2$. By Lemma 4 this implies $p^m | 4A$ and $p^m | a^2 + b^2 - c^2$. Hence, p divides $b^2 - c^2$, and we find $v_p(b) = v_p(c) = 0$. We obtain the contradiction $m = v_p(2a) = v_p(2abc) = v_p(4Ad) > v_p(4A) \geq m$ and conclude that p satisfies $p \equiv 1 \bmod 4$. The implication (v) $\Rightarrow$ (vi) is a well-known number theoretic result, see [**3**, Theorem 5.15 and its proof]. For (vi) $\Rightarrow$ (i), let x and y be as described. Then the triangle with sides $a = 2xy$, $b = |x^2 - y^2|$, and $c = x^2 + y^2$ satisfies the requirements of (i). $\square$

Corollary 5 *Suppose the integer d is the circumdiameter of a triangle with integral sides a, b, and c. Let p be a prime satisfying $p \not\equiv 1 \bmod 4$ and set $m = v_p(d)$. Then p^m divides $\gcd(a, b, c)$.*

Proof. After scaling the triangle if necessary, we may assume $\gcd(a, b, c, d) = 1$. Then Theorem 2 gives $m = 0$. $\square$

Proof of Theorem 1. For (i) $\Rightarrow$ (ii), let a, b, and c be the sides of a Heron triangle with circumradius k. Then the circumdiameter $2k$ is even and by Corollary 5, so are a, b, and c. The triangle with sides $a/2$, $b/2$, and $c/2$ is a Heron triangle by Lemma 3 and has circumdiameter k. For (ii) $\Rightarrow$ (iii), let a, b, and c be the sides of a Heron triangle with circumdiameter k. Set $r = \gcd(a, b, c, k)$. Then the Heron triangle with sides a/r, b/r, and c/r has circumdiameter k/r and we have $\gcd(a/r, b/r, c/r, k/r) = 1$. From Theorem 2 we find $k/r > 1$, so k/r has a prime divisor p, which by the same theorem satisfies $p \equiv 1 \bmod 4$. For (iii) $\Rightarrow$ (i), let p and k be as in (iii). According to Theorem 2 there is a Heron triangle with circumdiameter p. Multiplication by $2k/p$ yields a Heron triangle with circumradius k. $\square$

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2. Kramer, A.-V. and Luca, F., Some remarks on Heron triangles, *Acta Acad. Paedagog. Agriensis Sect. Mat. (N.S.)* **27** (2000) 25–38 (2001).
3. Niven, I. and Zuckerman, H., *An introduction to the Theory of numbers*, third edition, John Wiley & Sons, 1972.