Irreducible rational ordered vector spaces

An *ordered field* is a field F together with a total order $\leqslant$ on F such that:

(i) For all $\lambda, \mu \in F$ with $0 < \lambda$ we have $\mu < \lambda + \mu$.
(ii) For all $\lambda, \mu \in F$ with $0 < \lambda$ and $0 < \mu$ we have $0 < \lambda\mu$.

Two examples of ordered fields are the rationals $\mathbb{Q}$ and the reals $\mathbb{R}$. One can show without much effort that any ordered field is an extension of $\mathbb{Q}$.

From now on we let F denote an ordered field.

An *ordered vector space* is an F-vector space V together with a total order $\leqslant$ on V such that:

(i) For all $\lambda \in F$ and $v \in V$ with $0 < \lambda$ and $0 < v$ we have $0 < \lambda v$
(ii) For all $u, v \in V$ with $0 < u$ we have $v < u + v$.

Two ordered vector spaces V and W are considered the "same" if there exists an order-isomorphism between them, that is, a linear isomorphism $f : V \to W$ such that $u \leqslant v$ in V implies $f(u) \leqslant f(v)$ in W.

For any ordered field F and any $n \in \mathbb{Z}_{\geqslant 0}$, the pair $(F^n, \leqslant)$ is an ordered field if we define

$$(\lambda_1, \dots, \lambda_n) < (\mu_1, \dots, \mu_n) \iff \lambda_j < \mu_j$$

where $j = \max\{i : \lambda_i \neq \mu_i\}$ provided the latter is non-empty. This order is called the anti-lexicographic order on F^n.

Another interesting example is the rational vector space $V = \mathbb{Q} + \mathbb{Q}\sqrt{2}$ ordered as a subfield of $\mathbb{R}$. One can show that this space is not order-isomorphic to $\mathbb{Q}^2$ with the anti-lexicographic order. On the other hand, the case $F = \mathbb{R}$ is much simpler: by a theorem of Koshi, [1], any finite-dimensional ordered real vector space is order-isomorphic to $\mathbb{R}^n$ with the anti-lexicographic order.

Let V be an ordered vector space and U be a subspace. We say U is *convex* if for any $u \in U$ and $v \in V$ with $0 < v < u$ we have $v \in U$. Clearly $\{0\}$ and V are convex. One can show that the set of convex subspaces of V, which we denote by $C(V)$, is totally ordered by inclusion.

This notion is fundamental in the study of ordered vector spaces. For example, showing that $V = \mathbb{Q} + \mathbb{Q}\sqrt{2} \subset \mathbb{R}$ is not anti-lexicographically ordered amounts to show that $C(V) = \{\{0\}, V\}$. Here, we call an ordered vector space with this property *irreducible*.

By a theorem in [2], for any finite-dimensional ordered vector space V there is an order isomorphism

$$V \simeq \bigoplus_{W \in C(V) \setminus \{0\}} W/W'$$

where W' denotes the predecessor of W in $C(V)$ (for the definition of the order on the above sum see the reference). Thus, any finite-dimensional ordered vector space is an anti-lexicographic sum of irreducible ordered vector spaces.

The objective of the project is to study the irreducible rational vector spaces. Can we describe the order-automorphisms of such a space? Can we decide when two such spaces are order-isomorphic? Of course, a first step in the project is to study the existing literature.

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[1] Koshi, S. 1979. Vector spaces with linear order. Commentationes Mathematicae. Special Issue 2: 183187.

[2] Torreao Dassen, E L. 2011. Basis reduction for layered lattices. Proefschrift. Universiteit Leiden (December 20, 2011).