Computing Brauer–Manin obstructions on diagonal quartic surfaces

Martin Bright

University of Bristol

Arithmetic of K3 surfaces, Banff, 2008
Outline

1 Introduction
   - The Hasse principle
   - The Brauer group
   - The Brauer–Manin obstruction

2 Computing the Brauer–Manin obstruction
   - Computing the algebraic Brauer group
   - Finding the Azumaya algebras
   - Magma demo

3 Theoretical results on the evaluation map
   - Smooth models
   - Unramified places
   - Tamely ramified places
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3 Theoretical results on the evaluation map
   - Smooth models
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   - Tamely ramified places
The Hasse principle

Let $X$ be a variety over a number field $k$. Write $\mathbb{A}_k$ for the ring of adèles of $k$. The set of adelic points of $X$ is $X(\mathbb{A}_k)$; the set of rational points $X(k)$ is contained in it, under the diagonal embedding. If $X$ is a complete variety, then

$$X(\mathbb{A}_k) = \prod_v X(k_v)$$

where the product is over all places $v$ of $k$. 

Some classes of varieties satisfy the Hasse principle: that is, $X(\mathbb{A}_k) \neq \emptyset \Rightarrow X(k) \neq \emptyset$. In this case, it is straightforward to decide whether $X$ has rational points, since the condition on the left is a finite computation.
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Failure of the Hasse principle

- Unfortunately, many interesting classes of varieties do not satisfy the Hasse principle. In particular, K3 surfaces do not.

For example, the diagonal quartic surface

\[ X_4^0 + X_4^1 = 6X_4^2 + 12X_4^3 \]

has points in every completion of \( \mathbb{Q} \), but no rational points.

Manin showed that one can use the Brauer group of \( X \) to define a subset of \( X(\mathbb{A}_k) \) which must contain \( X(k) \). If this set is empty, we say that there is a Brauer–Manin obstruction to the Hasse principle for \( X \). This accounted for all counterexamples to the Hasse principle known then.
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We might hope to be able to evaluate an element of $Br k(X)$ at a point of $X$, to obtain an element of $Br k$. 

\[ \text{Martin Bright (University of Bristol)} \] \[ \text{Computing Brauer–Manin obstructions} \] \[ \text{Banff 2008} \] \[ 6 / 26 \]
The Brauer group of the function field

- Every field $K$ has a Brauer group $\text{Br}(K)$, the group of equivalence classes of central simple algebras over $K$. In particular, this is true of the function field $k(X)$.
- We might hope to be able to evaluate an element of $\text{Br} k(X)$ at a point of $X$, to obtain an element of $\text{Br} k$.
- Just as a rational function cannot be evaluated at every point of a variety, so a typical element of $\text{Br} k(X)$ cannot be evaluated everywhere – it is ramified along some divisor.
The Brauer group of a variety

- Let $X$ be a smooth, geometrically irreducible variety over $k$. The Brauer group of $X$, written $\text{Br } X$, can be informally defined as the subgroup of $\text{Br } k(X)$ of those elements which can be evaluated everywhere. These algebras are called Azumaya algebras.
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- We will be interested only in algebraic elements of $\text{Br } X$, that is, those which are split by an extension of $k$. These can be described in Galois cohomology as

$$\text{Br}_1 X = \ker (H^2(k, k(\bar{X})^\times) \to H^2(k, \text{Div } \bar{X})).$$
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$$\text{Br}_1 X = \ker \left( H^2(k, k(\overline{X})^\times) \to H^2(k, \text{Div} \overline{X}) \right).$$

- Equivalently, a class $\alpha$ in $H^2(k, k(\overline{X})^\times)$ lies in $\text{Br}_1 X$ if and only if, for all points $P \in X$, we can represent $\alpha$ by a cocycle taking values in $\mathcal{O}_X^\times, P$. 

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Computing Brauer–Manin obstructions
Banff 2008 7 / 26
Example

Let $l/k$ be a quadratic extension, and suppose that $f$ is a rational function on $X$ whose divisor is a norm from $l$, say $(f) = N_{l/k}D$. Then the quaternion algebra $A = (l/k, f)$ is an Azumaya algebra on $X$. 
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- To see this, let $P$ be any point of $X$. If $f$ is invertible at $P$, then $A$ can be evaluated at $P$ to get $A(P) = (l/k, f(P))$. 
Example

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- To see this, let $P$ be any point of $X$. If $f$ is invertible at $P$, then $A$ can be evaluated at $P$ to get $A(P) = (l/k, f(P))$.
- Otherwise, there is some divisor $D' \sim D$ which avoids $P$; let $(g) = D' - D$. Then the algebra $(l/k, fN_{l/k}g)$ is isomorphic to $A$ and can be evaluated at $P$. 
Let $v$ be a place of $k$. Recall from class field theory that there is a canonical injection $\text{inv}_v : \text{Br} \ k_v \to \mathbb{Q}/\mathbb{Z}$, such that the sequence

$$0 \to \text{Br} \ k \to \bigoplus_v \text{Br} \ k_v \xrightarrow{\sum_v \text{inv}_v} \mathbb{Q}/\mathbb{Z}$$

is exact.
The Brauer–Manin obstruction

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- If \( A \) is an Azumaya algebra on \( X \) and \( P_v \in X(k_v) \), then \( A \) can be evaluated at \( P_v \) to get an element of \( \text{Br } k_v \). So \( A \) gives maps

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X(k_v) \to \mathbb{Q}/\mathbb{Z}, \quad P_v \mapsto \text{inv}_v A(P_v)
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for each \( v \).
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If $\mathcal{A}$ is an Azumaya algebra on $X$ and $P_v \in X(k_v)$, then $\mathcal{A}$ can be evaluated at $P_v$ to get an element of $\text{Br } k_v$. So $\mathcal{A}$ gives maps

$$X(k_v) \rightarrow \mathbb{Q}/\mathbb{Z}, \quad P_v \mapsto \text{inv}_v \mathcal{A}(P_v)$$

for each $v$.

Combining these two facts, we get...
We deduce that, if \((P_v) \in X(\mathbb{A}_k)\) is the diagonal image of a rational point, then

\[
\sum_v \text{inv}_v A(P_v) = 0.
\]

Given a subset \(B\) of \(\text{Br} X\), define

\[
X_A(k) := \{ (P_v) \in X(\mathbb{A}_k) | \sum_v \text{inv}_v A(P_v) = 0 \text{ for all } A \in B \}.
\]

We have shown that \(X(k) \subset X(\mathbb{A}_k)\).

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**The Brauer–Manin obstruction**

\[
\begin{array}{c}
X(k) \longrightarrow X(\mathbb{A}_k) \\
\downarrow \quad \downarrow \quad A \\
\text{Br } k \quad \bigoplus_v \text{Br } k_v
\end{array}
\]
The Brauer–Manin obstruction

\[ X(k) \overset{A}{\rightarrow} X(\mathbb{A}_k) \]

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Given a subset \(B\) of \(\text{Br } X\), define \(X(\mathbb{A}_k)^B\) as

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We have shown that \(X(k) \subset X(\mathbb{A}_k)^{\text{Br } X}\).
The Brauer–Manin obstruction

\[ X(k) \rightarrow X(\mathbb{A}_k) \]

\[ \downarrow \quad \downarrow \]

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\]

We have shown that \(X(k) \subset X(\mathbb{A}_k)^{\text{Br } X}\).
If $X(\mathbb{A}_k)^B$ is empty, we say there is a Brauer–Manin obstruction to the Hasse principle coming from $B$. If $X(\mathbb{A}_k)^B$ is not the whole of $X(\mathbb{A}_k)$, we say there is a Brauer–Manin obstruction to weak approximation.
Comments

- If $X(\mathbb{A}_k)^B$ is empty, we say there is a Brauer–Manin obstruction to the Hasse principle coming from $B$. If $X(\mathbb{A}_k)^B$ is not the whole of $X(\mathbb{A}_k)$, we say there is a Brauer–Manin obstruction to weak approximation.
- Given $A \in \text{Br } X$, it is often possible to compute $X(\mathbb{A}_k)^A$ effectively.
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We have constant Azumaya algebras $\text{Br } k \subset \text{Br } X$, but the condition they impose is vacuous. So the Brauer–Manin obstruction is determined by $\text{Br } X / \text{Br } k$. 
If $X(\mathbb{A}^1_k)^B$ is empty, we say there is a Brauer–Manin obstruction to the Hasse principle coming from $B$. If $X(\mathbb{A}^1_k)^B$ is not the whole of $X(\mathbb{A}^1_k)$, we say there is a Brauer–Manin obstruction to weak approximation.

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We will show how to compute generators for the algebraic part, $\text{Br}_1 X / \text{Br } k$, and the associated obstruction.
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Computing the algebraic Brauer group

Recall that the algebraic part of the Brauer group, $\text{Br}_1 X$, can be described as a Galois cohomology group

$$\text{Br}_1 X = \ker \left( H^2(k, k(\bar{X})^\times) \to H^2(k, \text{Div}\bar{X}) \right).$$

On the face of it this is not very useful, as $H^2(k, k(\bar{X})^\times)$ is not something we want to be computing with.
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- However, we only need to know generators for $\text{Br}_1 X / \text{Br} k$. Write the homomorphism above as a composition

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- The kernel-cokernel exact sequence for this composition of maps is

$$0 \to \ker f \to \text{Br}_1 X \to \ker g \to \text{coker } f$$

and we can identify these groups.
Computing the algebraic Brauer group

\[ 0 \rightarrow \ker f \rightarrow \text{Br}_1 X \rightarrow \ker g \rightarrow \text{coker } f \]
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\[ 0 \to \ker f \to \Br_1 X \to \ker g \to \coker f \]

Using the exact sequence

\[ 0 \to \bar{k}^\times \to k(\bar{X})^\times \to \Princ \bar{X} \to 0 \]

shows that \( \ker f = \text{im} (\Br k) \), and that \( \coker f = H^3(k, \bar{k}^\times) = 0 \).
Computing the algebraic Brauer group

\[ \text{Br } k \rightarrow \text{Br}_1 X \rightarrow \ker g \rightarrow 0 \]

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shows that \( \ker g \) is the image of the boundary map \( \partial : H^1(k, \text{Pic} \, \bar{X}) \rightarrow H^2(k, \text{Princ} \, \bar{X}) \). Since \( \text{Div} \, \bar{X} \) is an induced module, this map is injective.
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- So there is an isomorphism \( \text{Br}_1 X / \text{Br } k \cong H^1(k, \text{Pic } \bar{X}) \).
Computing the algebraic Brauer group

- We have an isomorphism $\text{Br}_1 X / \text{Br} k \cong H^1(k, \text{Pic } \bar{X})$. If $\text{Pic } \bar{X}$ is finitely generated, then we can hope to understand this group. If $\text{Pic } \bar{X}$ is also free, then $\text{Br}_1 X / \text{Br} k$ is finite.

On a diagonal quartic surface, there are 48 straight lines. We can write down their equations, and they generate $\text{Pic } \bar{X}$. The Galois group of the field of definition of the 48 lines is always a subgroup of the "generic" Galois group, which is an extension of $C_2$ by $C_2 \times C_4 \times C_4$. Going through all the possible Galois actions finds all possibilities for $\text{Br}_1 X / \text{Br} k$. It is always killed by 8, and has 2-rank at most 7.
Computing the algebraic Brauer group

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- If we know explicitly a finite, Galois-stable set of generators for $\text{Pic} \bar{X}$, and the Galois action on them, then computing $H^1(k, \text{Pic} \bar{X})$ is straightforward.
Computing the algebraic Brauer group

- We have an isomorphism $\text{Br}_1 X / \text{Br} k \cong H^1(k, \text{Pic} \tilde{X})$. If $\text{Pic} \tilde{X}$ is finitely generated, then we can hope to understand this group. If $\text{Pic} \tilde{X}$ is also free, then $\text{Br}_1 X / \text{Br} k$ is finite.

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Finding the Azumaya algebras

- Getting our hands on explicit generators for $H^1(k, \text{Pic} \, \bar{X})$ is only the first step to computing the algebraic Brauer–Manin obstruction. We now need to turn them into explicit generators for $\text{Br}_1 X / \text{Br} \, k$.
Finding the Azumaya algebras

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- The isomorphism $H^1(k, \text{Pic } \bar{X}) \cong \text{Br}_1 X / \text{Br } k$ arose as a composition of various maps:

$$H^1(k, \text{Pic } \bar{X}) \xrightarrow{\partial} H^2(k, \text{Princ } \bar{X}) \xrightarrow{\bar{g}} H^2(k, k(\bar{X})^\times).$$
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$$H^1(k, \text{Pic} \bar{X}) \xrightarrow{\partial} H^2(k, \text{Princ} \bar{X}) \leftrightarrow H^2(k, k(\bar{X})^\times).$$

- The first of these, $\partial$, is a boundary map in cohomology and is straightforward to compute: lift from $\text{Pic} \bar{X}$ to $\text{Div} \bar{X}$ and take the coboundary. Note that there is a choice of lifts here, giving different but cohomologous images.
Finding the Azumaya algebras

- Getting our hands on explicit generators for $H^1(k, \text{Pic } \tilde{X})$ is only the first step to computing the algebraic Brauer–Manin obstruction. We now need to turn them into explicit generators for $\text{Br}_1 X / \text{Br}_k$.

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- The first of these, $\partial$, is a boundary map in cohomology and is straightforward to compute: lift from $\text{Pic } \tilde{X}$ to $\text{Div } \tilde{X}$ and take the coboundary. Note that there is a choice of lifts here, giving different but cohomologous images.

- Computing $g^{-1}$ involves lifting from $\text{Princ } \tilde{X}$ to $k(\tilde{X})^\times$, a potentially slow operation. Moreover, lifting just anyhow will not give us a cocycle – to do that, we need to make effective the fact that $H^3(k, \bar{k}^\times) = 0$. 
Using a small splitting field

- Some of these problems become easier if the elements of $H^1(k, \text{Pic } \bar{X})$ we're looking at are split by a small extension $l/k$.

\[
H^1(k, \text{Pic } \bar{X}) \xrightarrow{\partial} H^2(k, \text{Princ } \bar{X}) \xleftarrow{g} H^2(k, k(\bar{X})^\times)
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\[ \begin{array}{cccc}
H^1(k, \text{Pic} \bar{X}) & \xrightarrow{\partial} & H^2(k, \text{Princ} \bar{X}) & \xleftarrow{g} & H^2(k, k(\bar{X})^\times) \\
\uparrow \text{inf} & & \uparrow \text{inf} & & \uparrow \text{inf} \\
H^1(l/k, \text{Pic} X_l) & \rightarrow & H^2(l/k, \text{Princ} X_l) & \leftarrow & H^2(l/k, k(X_l)^\times)
\end{array} \]

If $l/k$ is cyclic, things get even more straightforward. But we have introduced a new problem: we probably don't know a set of divisors defined over $l$ which generate Pic $X_l$. 
Using a small splitting field

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\[
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\end{array}
\]

\[
\begin{array}{ccc}
\text{inf} & \|
\end{array}
\]

\[
\begin{array}{ccc}
H^1(l/k, \text{Pic } X_l) & \longrightarrow & H^2(l/k, \text{Princ } X_l)
\end{array}
\]

\[
\begin{array}{ccc}
\sim & \|
\end{array}
\]

\[
\begin{array}{ccc}
N \text{Pic } X_l & \overset{N}{\longrightarrow} & \frac{\text{Princ } X}{N \text{Princ } X_l}
\end{array}
\]

\[
\begin{array}{ccc}
\sim & \|
\end{array}
\]

\[
\begin{array}{ccc}
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\end{array}
\]

\[
\begin{array}{ccc}
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\inf & \uparrow & \inf & \uparrow & \inf & \uparrow \\
H^1(l/k, \text{Pic} \, X_l) & \longrightarrow H^2(l/k, \text{Princ} \, X_l) & \longrightarrow & H^2(l/k, k(X_l)^\times) \\
\sim & \uparrow & \sim & \uparrow & \sim & \uparrow \\
\frac{N \text{Pic} \, X_l}{\langle \sigma - 1 \rangle} & \xrightarrow{N} \frac{\text{Princ} \, X}{N \text{Princ} \, X_l} & \longleftarrow & \frac{k(X)^\times}{Nk(X_l)^\times}
\end{align*}
\]

- If $l/k$ is cyclic, things get even more straightforward.
- But we have introduced a new problem: we probably don’t know a set of divisors defined over $l$ which generate $\text{Pic} \, X_l$.  

Magma demo
Outline

1. Introduction
   - The Hasse principle
   - The Brauer group
   - The Brauer–Manin obstruction

2. Computing the Brauer–Manin obstruction
   - Computing the algebraic Brauer group
   - Finding the Azumaya algebras
   - Magma demo

3. Theoretical results on the evaluation map
   - Smooth models
   - Unramified places
   - Tamely ramified places
Theoretical results on the evaluation map

Let \( \mathcal{A} \) be an Azumaya algebra on \( X \), and fix a finite place \( \nu \). We will apply some geometry to understand the evaluation map

\[
X(k_\nu) \rightarrow \mathbb{Q}/\mathbb{Z} \quad P \mapsto \text{inv}_\nu \mathcal{A}(P).
\]

- We saw in the demonstration that, at primes of good reduction, the invariant was everywhere zero. For each \( P \in X(k_\nu) \), we could always find one of our representative algebras \((-1, f)\) such that \( f(P) \) was a unit in \( k_\nu \).
Theoretical results on the evaluation map

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- We saw in the demonstration that, at primes of good reduction, the invariant was everywhere zero. For each $P \in X(k_\nu)$, we could always find one of our representative algebras $(-1, f)$ such that $f(P)$ was a unit in $k_\nu$.
- Of course, we could spoil this: we could change our algebra by a constant algebra ramified at $\nu$. The invariant would still be constant, but not necessarily zero.
Smooth models

It is much easier to investigate the behaviour of $\mathcal{A}(P)$ when $P$ reduces to a smooth point. What does this mean for diagonal quartic surfaces?

- Consider the diagonal quartic surface

  $$X : a_0X_0^4 + a_1X_1^4 + a_2X_2^4 + a_3X_3^4 = 0$$

where $a_i \in \mathbb{Q}$. We may clearly assume that the $a_i$ are coprime integers, and that none of them is divisible by a fourth power. Reducing the equation modulo $p$ gives a surface over $\mathbb{F}_p$ which may be singular.
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Reducing the equation modulo \( p \) gives a surface over \( \mathbb{F}_p \) which may be singular.

- But this is only one model of \( X \); we can easily produce others.

- Suppose, say, that \( p \) divides \( a_0 \) but none of the other \( a_i \). We can replace \( X_i \) by \( pX_i \) for \( i = 1, 2, 3 \) and then remove the resulting power of \( p \), giving a new surface isomorphic (over \( \mathbb{Q} \)) to \( X \).
In this way we obtain up to four different models. It is not difficult to show that any point in $X(\mathbb{Q}_p)$ reduces to a smooth point modulo $p$ in at least one of these models.
Smooth models

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- Geometrically, we have shown that there exists a model $\mathcal{X}/\mathbb{Z}_p$ for $X$, obtained by blowing up our original one, such that any point of $X(\mathbb{Q}_p)$ extends to a smooth point of $\mathcal{X}(\mathbb{Z}_p)$. The different equations describe the components of this model.
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- In fact, this can be accomplished for any smooth variety over $\mathbb{Q}_p$; such a model is called a weak Néron model.
Theorem

Let $X$ be a smooth, geometrically irreducible variety over $k_v$. Let $A \in \text{Br}_1 X$ be an Azumaya algebra split by an unramified extension of $k_v$. Let $\mathcal{X}/\mathcal{O}_v$ be a smooth model of $X$, with $Z$ an irreducible component of the special fibre. Then $\text{inv}_v A(P)$ is constant on the set of points $P$ reducing to $Z$. 

In particular, this is true at primes where $X$ has good reduction. At a prime of good reduction, the Galois module $\text{Pic} \bar{X}$ is unramified. This is because the inertia group, by definition, acts trivially on the reduction of $X$ modulo $p$. So each of the 48 lines on $X$ must be taken to a line with the same reduction modulo $p$. But the 48 lines all have distinct reductions – after all, the reduction of $X$ is a smooth diagonal quartic surface, so contains 48 straight lines.
Unramified places

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- But the 48 lines all have distinct reductions – after all, the reduction of $X$ is a smooth diagonal quartic surface, so contains 48 straight lines.
Tamely ramified places

- Now suppose that $\mathcal{A}$ is split by a totally, tamely ramified Galois extension $L/k_v$ of degree $n$. There are isomorphisms

$$\text{Br}(L/k_v) \cong k_v^\times / NI^\times \cong \mathcal{O}_v / N\mathcal{O}_I^\times \cong F^\times / (F^\times)^n$$

where $F$ is the residue field of $k_v$. 
Tamely ramified places

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- This tells us that, if we have a 2-cocycle describing an element of \( \text{Br}(l/k_v) \), and if it takes unit values, then its class is determined by its reduction modulo \( v \).
Now suppose that $\mathcal{A}$ is split by a totally, tamely ramified Galois extension $I/k_v$ of degree $n$. There are isomorphisms

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where $\mathbb{F}$ is the residue field of $k_v$.

This tells us that, if we have a 2-cocycle describing an element of $\text{Br}(I/k_v)$, and if it takes unit values, then its class is determined by its reduction modulo $v$.

With a little work, we can deduce that $\text{inv}_v \mathcal{A}(P)$ only depends on the residue class of $P$. In fact, we can say more...
Theorem

Let $X$ be a smooth, geometrically irreducible variety over $k_v$, and let $A \in \text{Br}_1 X$ be an Azumaya algebra split by a tamely ramified Galois extension $l/k_v$ of degree $n$. Let $X/O_v$ be a smooth model of $X$, with $Z$ a geometrically irreducible component of the special fibre. Then, after possibly modifying $A$ by a constant algebra, there is a $Z$-torsor $T$ under $\mu_n$ such that the following diagram commutes.

$$
\begin{array}{ccc}
X(k_v)_Z & \xrightarrow{A} & \text{Br } l/k_v \\
\downarrow & & \downarrow \rho \\
Z(F) & \xrightarrow{T} & F^\times/(F^\times)^n
\end{array}
$$
Consequences for diagonal quartics

- On a diagonal quartic surface $X$, the 48 lines are all defined over some 2-power degree extension of the base field; so this extension is either unramified or tamely ramified except at 2.
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- So, if $p \neq 2$, evaluating the Brauer–Manin obstruction at $p$ comes down to studying some torsors under $\mu_n$ on the reduction of $X$ at $p$. 

If $X$ has good reduction, then the reduction is again a smooth quartic surface, so the only torsors under $\mu_n$ are constant; we see again that the Brauer–Manin obstruction there is constant.

If the reduction of $X$ is a cone, then consider a straight line $L$ in that cone. There are no non-constant torsors under $\mu_n$ on $L$, even after removing the vertex; so we deduce that the Brauer–Manin evaluation map is constant on the set of points of $X(\mathbb{Q}_p)$ reducing to points on $L$. 

Martin Bright (University of Bristol)
Computing Brauer–Manin obstructions
Banff 2008 26 / 26
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- If the reduction of \( X \) is a cone, then consider a straight line \( L \) in that cone. There are no non-constant torsors under \( \mu_n \) on \( L \), even after removing the vertex; so we deduce that the Brauer–Manin evaluation map is constant on the set of points of \( X(\mathbb{Q}_p) \) reducing to points on \( L \).