EXPLICIT ARAKELOV GEOMETRY

R.S. de Jong

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Promotor:
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Prof. dr E.M. Opdam
Dr. B.J.J. Moonen

Faculty of Natural Sciences, Mathematics and Computer Science
University of Amsterdam

To Désirée

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Contents

Introduction

(i) Arakelov geometry is a technique for studying diophantine problems from a geometrical point of view. In short, given a diophantine problem, one considers an arithmetic scheme associated with that problem, and adds in the complex points of that scheme by way of "compactification". Next, one endows all arithmetic bundles on the scheme with an additional structure over the complex numbers, meaning one endows them with certain hermitian metrics. It is well-known from traditional topology or geometry that compactifying a space often introduces a convenient structure to it, which makes a study of it easier generally. The same holds in our case: by introducing an additional Arakelov structure to a given arithmetic situation one ends up with a convenient set-up to formulate, study and even prove diophantine properties of the original situation. For instance one could think of questions dealing with the size of the solutions to a given diophantine problem. Fermat's method of descent can perhaps be viewed as a prototype of Arakelov geometry on arithmetic schemes.

(ii) Probably the best way to start an introduction to Arakelov geometry is to consider the simplest type of arithmetic scheme possible, namely the spectrum of a ring of integers in a number field, for instance $\mathrm{Spec}(\mathbb{Z})$. In the nineteenth century, some authors, like Kummer, Kronecker, Dedekind and Weber, drew attention to the remarkable analogy that one has between the properties of rings of integers in a number field, on the one hand, and the properties of coordinate rings of affine non-singular curves on the other. In particular, they started the parallel development of a theory of "places" or "prime divisors" on both sides of the analogy. Most important, morally speaking, was however that the success of this theory allowed mathematicians to see that number theory on the one hand, and geometry on the other, are unified by a bigger picture. This way of thinking continued to be stressed in the twentieth century, most notably by Weil, and it is fair to say that the later development of the concept of a scheme by Grothendieck is directly related to these early ideas.

The idea of "compactifying" the spectrum of a ring of integers can be motivated as follows. We start at the geometric side. Let C be an affine non-singular curve over an algebraically closed field. The first thing we do is to "compactify" it: by making an appropriate embedding of C into projective space and taking the Zariski closure, one gets a complete non-singular curve $\overline{C}$. This curve is essentially unique. Now we consider divisors on $\overline{C}$: a divisor is a finite formal integral linear combination $D = \sum_P n_P P$ of points on $\overline{C}$. The divisors form in a natural way a group $\mathrm{Div}(\overline{C})$. We obtain a natural group homomorphism $\mathrm{Div}(\overline{C}) \to \mathbb{Z}$ by taking the degree $\deg D = \sum_P n_P$. In order to obtain an interesting theory from this, one associates to any non-zero rational function f on $\overline{C}$ a divisor $(f) = \sum_P v_P(f) P$, where $v_P(f)$ denotes the multiplicity of f at P. By factoring out the divisors of rational functions one obtains the so-called Picard group $\mathrm{Pic}(\overline{C})$ of $\overline{C}$. Now a fundamental result is that the degree of the divisor of a rational function is 0, and hence the degree factors through a homomorphism $\mathrm{Pic}(\overline{C}) \to \mathbb{Z}$. It turns out that the kernel $\mathrm{Pic}^0(\overline{C})$ of this homomorphism can be given a natural structure of projective algebraic variety. This variety is a fundamental invariant attached to $\overline{C}$ and is studied extensively in algebraic geometry. The fundamental property that the degree of a divisor of a rational function is 0 is not true in general when we consider only affine curves. This makes the step of compactifying C so important.

Turning next to the arithmetic side, given the success of compactifying a curve at the geometric side, one wants to define analogues of divisor, degree and compactification, in such a way that the degree of a divisor of a rational function is 0. This leads us to an arithmetic analogue of the degree 0 part of the Picard group. The compactification step is as follows: let $B = \mathrm{Spec}(O_K)$ be the spectrum of the ring of integers O_K in a number field K. We formally add to B the set of embeddings $\sigma : K \hookrightarrow \mathbb{C}$ of K into $\mathbb{C}$. By algebraic number theory this set is finite of cardinality $[K : \mathbb{Q}]$. Now we consider Arakelov divisors on this enlarged B: an Arakelov divisor on B is a finite formal linear combination $D = \sum_P n_P P + \sum_\sigma \alpha_\sigma \cdot \sigma$, with the first sum running over the non-zero prime ideals of O_K, with $n_P \in \mathbb{Z}$, and with the second sum running over the complex embeddings of K, with $\alpha_\sigma \in \mathbb{R}$. Note that the non-zero prime ideals of O_K correspond to the closed points of B. The set of Arakelov divisors forms in a natural way a group $\widehat{\mathrm{Div}}(B)$. On it we have an Arakelov degree $\widehat{\deg}\, D = \sum_P n_P \log \#(O_K/P) + \sum_\sigma \alpha_\sigma$ which takes values in $\mathbb{R}$. The Arakelov divisor associated to a non-zero rational function $f \in K$ is given as $(f) = \sum_P v_P(f) \log \#(O_K/P) + \sum_\sigma v_\sigma(f)\sigma$ with $v_P(f)$ the multiplicity of f at P, *i.e.*, the multiplicity of P in the prime ideal decomposition of f, and with $v_\sigma(f) = -\log |f|_\sigma$. The crucial idea is now that the product formula accounts for the fact that $\widehat{\deg}(f) = 0$ for any non-zero $f \in K$. So indeed, by factoring out the divisors of rational functions, we obtain a Picard group $\widehat{\mathrm{Pic}}(B)$ with a degree $\widehat{\mathrm{Pic}}(B) \to \mathbb{R}$. To illustrate the use of these constructions, we refer to Tate's thesis: there Tate showed that the degree 0 part $\widehat{\mathrm{Pic}}^0(B)$, the analogue of the $\mathrm{Pic}^0(\overline{C})$ from geometry, can be seen as a natural starting point to prove finiteness theorems in algebraic number theory, such as Dirichlet's unit theorem, or the finiteness of the class group. In fact, Tate uses a slight variant of our $\widehat{\mathrm{Pic}}^0(B)$, but we shall ignore this fact.

(iii) Shafarevich asked for an extension of the above idea to varieties defined over a number field. In particular he asked for this extension in the context of the Mordell conjecture. Let C be a curve over a field k. The statement that the set $C(k)$ of rational points of C is finite, is called the Mordell conjecture for C/k. Now for curves over a function field in characteristic 0, the Mordell conjecture (under certain trivial conditions on C) was proven to be true in the 1960s by Manin and Grauert. However, the Mordell conjecture for curves over a number field was by then still unknown, and the technique of proof could not be straightforwardly generalised. A different approach to the Mordell conjecture for function fields was given by Parshin and Arakelov. The main feature of their approach is that it leads to an effective version of the conjecture: they define a function h, called a height function, on the set of rational points, with the property that for all A, the set of P with $h(P) \leq A$ is finite, and can in principle be explicitly enumerated. Now what they prove is that the height of a rational point can be bounded *a priori*. Hence, it is possible in principle to construct an exhaustive list of the rational points of a given curve.

In order to prove this result, the essential step is to associate to the curve C/k a model $p : \mathcal{X} \to B$ with $\mathcal{X}$ a complete algebraic surface, and with B a non-singular projective curve with function field k, such that the generic fiber of $\mathcal{X}$ is isomorphic to C. The rational points of C/k correspond then to the sections $P : B \to \mathcal{X}$ of p. The essential tool, then, is classical intersection theory on $\mathcal{X}$. It turns out that certain inequalities between the canonical classes of this surface can be derived, and these inequalities make it possible to bound the height of a section.

The obvious question, in the light of the Mordell conjecture for number fields, is whether this set-up can be carried over to the case of curves defined over a number field. As was said before, Shafarevich asked for such an analogue, but eventually it was Arakelov who, building on ideas of Shafarevich and Parshin, came up with a promising solution. His results are written down in the important paper *An intersection theory for divisors on an arithmetic surface*, published in 1974.

Let us describe the idea of that paper. Let C/K be a curve over a number field K. To it there is associated a scheme $p : \mathcal{X} \to B = \mathrm{Spec}(O_K)$, called an arithmetic surface, which is a fibration in curves over B, just as in the classical context of function fields mentioned above. The generic fiber of $p : \mathcal{X} \to B$ is isomorphic to C, and for almost all non-zero primes P of O_K, the fiber at the corresponding closed point is equal to the reduction of C modulo P. Again, the set of rational points of C/K corresponds to the set of sections $P : B \to \mathcal{X}$. In order to attack the Mordell conjecture for C, one wants to have an intersection theory for divisors on $\mathcal{X}$. The first idea, as always, is to compactify the scheme $\mathcal{X}$. We do this by formally adding in, for each complex embedding σ of K, the complex points of C, base changed along σ to $\mathbb{C}$. These complex points come with the natural structure of a Riemann surface, and yield the so-called "fibers at infinity" F_σ of $\mathcal{X}$. Now, an Arakelov divisor on $\mathcal{X}$ is a sum $D = D_{\mathrm{fin}} + D_{\mathrm{inf}}$ with D_{fin} a traditional Weil divisor on $\mathcal{X}$, and with $D_{\mathrm{inf}} = \sum_\sigma \alpha_\sigma F_\sigma$ an "infinite" contribution with $\alpha_\sigma \in \mathbb{R}$. The set of such divisors forms in a natural way a group $\widehat{\mathrm{Div}}(\mathcal{X})$. The main result of Arakelov is that one has a natural symmetric and bilinear intersection pairing on this group, and that this pairing factors through the Arakelov divisors of rational functions of $\mathcal{X}$. The crucial case to consider is the intersection of two distinct sections P, Q of $p : \mathcal{X} \to B$, viewed as divisors on $\mathcal{X}$. We have a finite contribution $(P,Q)_{\mathrm{fin}}$ which is given using the traditional intersection numbers on $\mathcal{X}$, but we also have an "infinite" contribution $(P,Q)_{\mathrm{inf}}$, which is defined to be a sum $-\sum_\sigma \log G(P_\sigma, Q_\sigma)$ over the complex embeddings σ. Here G is a kind of "distance" function on X_σ, the Riemann surface corresponding to σ. Arakelov defines G by writing down the axioms that it is supposed to satisfy, and by observing that these axioms allow a unique solution. The function G, called the Arakelov-Green function, is a very important invariant attached to each (compact and connected) Riemann surface. One of the properties of Arakelov's intersection theory is that an adjunction formula holds true, as in the classical function field case.

Given Arakelov's intersection theory on arithmetic surfaces, the set-up appears to be present to try to attack the Mordell conjecture. Unfortunately, no proof exists yet which translates the original ideas of Parshin and Arakelov into the number field setting. The major problem is that as yet there seem to exist no good arithmetic analogues of the classical canonical class inequalities. However, we do have an ineffective proof of the Mordell conjecture for number fields, due to Faltings. He was inspired by Szpiro to work on this conjecture using Arakelov theory, but ultimately he found a proof which runs, strictly speaking, along different lines. Nevertheless, Faltings obtained many interesting results in Arakelov intersection theory, and he wrote down these results in his 1984 landmark paper *Calculus on arithmetic surfaces*. Here Faltings shows that, besides the adjunction formula, also other theorems from classical intersection theory on algebraic surfaces have a true analogue for arithmetic surfaces, such as the Riemann-Roch theorem, the Hodge index theorem, and the Noether formula. The formulation of the Noether formula requires the introduction of a new fundamental invariant δ of Riemann surfaces, and in his paper Faltings asks for a further study of the properties of this invariant.

(iv) As we said above, the major difficulty in translating the classical techniques for effective Mordell into the number field setting is the lack of good canonical class inequalities. For example, one would like to formulate and prove a convenient analogue of the classical Bogomolov-Miyaoka-Yau-inequality for algebraic surfaces, and attempts to do this have been made by for example Parshin and Moret-Bailly in the 1980s. It was shown by Bost, Mestre and Moret-Bailly, however, that a certain naive analogue of the classical inequality is false. But parallel to this it also became clear that besides effective Mordell, also other major diophantine conjectures, such as Szpiro's conjecture and the *abc*-conjecture, would follow if one had good canonical class inequalities for arithmetic surfaces. No doubt it is very worthwhile to look further and better for such inequalities.

Unfortunately, during the last decades not much progress seems to have been made on this problem. The difficulties generally arise because of the difficult complex differential geometry that one encounters while dealing with the contributions at infinity. Also, we have no good idea how the canonical classes of an arithmetic surface can be calculated, and neither do we have any good idea how to relate them to other, perhaps easier, invariants. Many authors therefore continue to stress the importance of finding ways to calculate canonical classes of arithmetic surfaces, and of making up an inventory of the possible values that may occur. It is clear that a better understanding of the invariants associated to "infinity" is much needed.

Several authors have done Arakelov intersection theory from this point of view. A first important step was taken by Bost, Mestre and Moret-Bailly, who studied the explicit and calculational aspects of the first non-trivial case, namely of curves of genus 2 (the Arakelov theory of elliptic curves is well-understood, see for instance Faltings' paper). After that, several other isolated examples have been considered: for example Ullmo *et al.* studied the Arakelov theory of the modular curves $X_0(N)$, and Guàrdia in his thesis covered a certain class of plane quartic curves admitting many automorphisms.

In the present thesis we wish to contribute to the problem of doing explicit Arakelov geometry by trying to find a description of the main numerical invariants of arithmetic surfaces that makes it possible to calculate them efficiently. We give explicit formulas for the Arakelov-Green function as well as for the Faltings delta-invariant, where it should be remarked that these invariants are defined only in a very implicit way. We show how we can make things even more explicit in the case of elliptic and hyperelliptic curves. Finally, we indicate how efficient calculations are to be done, and in fact we include some explicit numerical examples.

(v) We now turn to a more specialised description of the main results of this thesis. For an explanation of the notation we refer to the main text.

Chapter 1 is an introduction to Arakelov theory. We introduce the main characters, such as the Arakelov-Green function, the delta-invariant, the Faltings height and the relative dualising sheaf, and we prove some fundamental properties about them. The results described in this chapter are certainly not new, although our proofs sometimes differ from the standard ones.

In Chapter 2 we state and prove our explicit formulas for the Arakelov-Green function and Faltings' delta-invariant. Let X be a compact and connected Riemann surface of genus $g > 0$, and let G be the Arakelov-Green function of X. Let μ be the fundamental (1,1)-form of X and let $\|\vartheta\|$ be the normalised theta function on $\mathrm{Pic}_{g-1}(X)$. Let $S(X)$ be the invariant defined by

$$\log S(X) := -\int_X \log \|\vartheta\|(gP - Q) \cdot \mu(P)\,,$$

with Q an arbitrary point on X. It can be checked that the integral is well-defined and does not depend on the choice of Q. Let $\mathcal{W}$ be the classical divisor of Weierstrass points on X. We have then the following explicit formula for the Arakelov-Green function.

Theorem. *For P, Q points on X, with P not a Weierstrass point, we have*

$$G(P,Q)^g = S(X)^{1/g^2} \cdot \frac{\|\vartheta\|(gP - Q)}{\prod_{W \in \mathcal{W}} \|\vartheta\|(gP - W)^{1/g^3}}\,.$$

Here the product runs over the Weierstrass points of X, counted with their weights. The formula is valid also for Weierstrass points P, provided that we take the leading coefficients of a power series expansion about P in both numerator and denominator.

As to Faltings' delta-invariant $\delta(X)$ of X, we prove the following result. Let $\Phi : X \times X \to \mathrm{Pic}_{g-1}(X)$ be the map sending (P, Q) to the class of $(gP - Q)$. For a fixed $Q \in X$, let $i_Q : X \to X \times X$ be the map sending P to (P, Q), and put $\phi_Q = \Phi \cdot i_Q$.

Theorem. *Define the line bundle L_X by*

$$L_X := \left(\bigotimes_{W \in \mathcal{W}} \phi_W^* \left(O(\Theta) \right) \right)^{\otimes (g-1)/g^3} \otimes_{O_X} \left(\Phi^*(O(\Theta))|_{\Delta_X} \otimes_{O_X} \Omega_X^{\otimes g} \right)^{\otimes -(g+1)} \otimes_{O_X} \otimes \left(\Omega_X^{\otimes g(g+1)/2} \otimes_{O_X} \left(\wedge^g H^0(X, \Omega_X^1) \otimes_{\mathbb{C}} O_X \right)^{\vee} \right)^{\otimes 2} .$$

Then the line bundle L_X is canonically trivial. If $T(X)$ is the norm of the canonical trivialising section of L_X, the formula

$$\exp(\delta(X)/4) = S(X)^{-(g-1)/g^2} \cdot T(X)$$

holds.

We have the following explicit formula for $T(X)$. For P on X, not a Weierstrass point, and z a local coordinate about P, we put

$$\|F_z\|(P) := \lim_{Q \to P} \frac{\|\vartheta\|(gP - Q)}{|z(P) - z(Q)|^g} .$$

Further we let $W_z(\omega)(P)$ be the Wronskian at P in z of an orthonormal basis $\{\omega_1, \ldots, \omega_g\}$ of the differentials $H^0(X, \Omega_X^1)$ with respect to the hermitian inner product $(\omega, \eta) \mapsto \frac{i}{2} \int_X \omega \wedge \overline{\eta}$.

Theorem. *The invariant $T(X)$ satisfies the formula*

$$T(X) = \|F_z\|(P)^{-(g+1)} \cdot \prod_{W \in \mathcal{W}} \|\vartheta\|(gP - W)^{(g-1)/g^3} \cdot |W_z(\omega)(P)|^2 ,$$

where again the product runs over the Weierstrass points of X, counted with their weights, and where P can be any point of X that is not a Weierstrass point.

It follows that the invariant $T(X)$ can be given in purely classical terms.

Chapters 3 and 4 are devoted to the proof of the following result, specialising to hyperelliptic Riemann surfaces.

Theorem. *Let X be a hyperelliptic Riemann surface of genus $g \geq 2$, and let $\|\Delta_g\|(X)$ be its modified modular discriminant. Then for the invariant $T(X)$ of X, the formula*

$$T(X) = (2\pi)^{-2g} \cdot \|\Delta_g\|(X)^{-\frac{3g-1}{8ng}}$$

holds.

The proof of this theorem follows by combining two results relating the Arakelov-Green function to the invariants $T(X)$ and $\|\Delta_g\|(X)$. Although these results look quite similar, the proofs that we give of these results use very different techniques. For the first result, which we prove in Chapter 3, we only use function theory on hyperelliptic Riemann surfaces. For the second result, which we prove in Chapter 4, we broaden our perspective and consider hyperelliptic curves over an arbitrary base scheme. The result follows then from a consideration of a certain isomorphism of line bundles over the moduli stack of hyperelliptic curves. Special care is needed to deal with its specialisation to characteristic 2, where the locus of Weierstrass points behaves in an atypical way.

In Chapter 5 we focus on the Arakelov theory of elliptic curves. Mainly because the fundamental (1,1)-form μ behaves well under isogenies, a fruitful theory emerges in this case. We give a reasonably self-contained and fairly elementary exposition of this theory. We recover some well-known results, due to Faltings, Szpiro and Autissier, but with alternative proofs. In particular, we base our discussion on a complex projection formula for isogenies, which seems new. The main new results that we derive from this formula are as follows.

Theorem. *Let X and X' be Riemann surfaces of genus 1. Let $\|\eta\|(X)$ and $\|\eta\|(X')$ be the values* *of the normalised eta-function associated to X and X', respectively. Suppose we have an isogeny $f : X \to X'$. Then we have*

$$\prod_{P \in \mathrm{Ker} f, P \neq 0} G_X(0,P) = \frac{\sqrt{N} \cdot \|\eta\|(X')^2}{\|\eta\|(X)^2} \,,$$

where N is the degree of f.

The above theorem answers a question posed by Szpiro.

Theorem. *Let E and E' be elliptic curves over a number field K, related by an isogeny $f : E \to E'$. Let $p : \mathcal{E} \to B$ and $p' : \mathcal{E}' \to B$ be arithmetic surfaces over the ring of integers of K with generic fibers isomorphic to E and E', respectively. Suppose that the isogeny f extends to a B-morphism $f : \mathcal{E} \to \mathcal{E}'$; for example, this is guaranteed if $\mathcal{E}'$ is a minimal arithmetic surface. Let D be an Arakelov divisor on $\mathcal{E}$ and let D' be an Arakelov divisor on $\mathcal{E}'$. Then the equality of intersection products $(f^*D', D) = (D', f_*D)$ holds.*

In the final Chapter 6 we explain how our explicit formulas can be used to effectively calculate examples of canonical classes. It turns out that the major difficulty is always the calculation of the invariant $S(X)$.

Theorem. *Consider the hyperelliptic curve X of genus 3 and defined over $\mathbb{Q}$, with hyperelliptic equation*

$$y^2 = x(x-1)(4x^5 + 24x^4 + 16x^3 - 23x^2 - 21x - 4) \,.$$

Then X has semi-stable reduction over $\mathbb{Q}$ with bad reduction only at the primes $p = 37, p = 701$ and $p = 14717$. For the corresponding Riemann surface (also denoted by X) we have

$$\begin{aligned} \log T(X) &= -4.44361200473681284... \\ \log S(X) &= 17.57... \\ \delta(X) &= -33.40... \end{aligned}$$

and for the curve $X/\mathbb{Q}$ we have

$$\begin{aligned} h_F(X) &= -1.280295247656532068... \\ e(X) &= 20.32... \end{aligned}$$

for the Faltings height and the self-intersection of the relative dualising sheaf, respectively.

The main results of this thesis are also described in the following papers.

R. de Jong, *Arakelov invariants of Riemann surfaces.* Submitted to Documenta Mathematica.

R. de Jong, *On the Arakelov theory of elliptic curves.* Submitted to l'Enseignement Mathématique.

R. de Jong, *Faltings' delta-invariant of a hyperelliptic Riemann surface.* Submitted to the Proceedings of the Texel Conference "The analogy between number fields and function fields".

R. de Jong, *Jacobian Nullwerte associated to hyperelliptic Riemann surfaces.* In preparation.

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Chapter 1

Review of Arakelov geometry

In this chapter we review the fundamental notions of Arakelov geometry, as developed in Arakelov's paper [Ar2] and Faltings' paper [Fa2]. These papers will serve as the basic references throughout the whole chapter.

In Section 1.1 we discuss the complex differential geometric notions that are needed to provide the "contributions at infinity" in Arakelov intersection theory. In Section 1.2 we turn then to this intersection theory itself, and discuss its formal properties. In Section 1.3 we recall the defining properties of the determinant of cohomology and the Deligne bracket, and show how they are metrised over the complex numbers. These metrisations allow us to give an arithmetic version of the Riemann-Roch theorem. In Section 1.4 we introduce Faltings' delta-invariant, and give two fundamental formulas in which this invariant occurs. In Section 1.5 we recall the definition and basic properties of semi-stable curves and show how they are used to define Arakelov invariants for curves over number fields. Finally in Section 1.6 we discuss the arithmetic significance of the delta-invariant by stating and sketching a proof of the arithmetic Noether formula, due to Faltings and Moret-Bailly.

1.1 Analytic part

Let X be a compact and connected Riemann surface of genus $g > 0$, and let Ω^1_X be its holomorphic cotangent bundle. On the space of holomorphic differential forms $H^0(X, \Omega^1_X)$ we have a natural hermitian inner product given by

$$(\omega, \eta) = \frac{i}{2} \int_X \omega \wedge \overline{\eta} \,.$$

Here we use the notation $i = \sqrt{-1}$. We use this inner product[1] to form an orthonormal basis $\{\omega_1, \ldots, \omega_g\}$ of $H^0(X, \Omega^1_X)$. Then we define a canonical (1,1)-form μ on X by setting

$$\mu := \frac{i}{2g} \sum_{k=1}^{g} \omega_k \wedge \overline{\omega}_k \,.$$

Clearly the form μ does not depend on the choice of orthonormal basis, and we have $\int_X \mu = 1$.

Definition 1.1.1. The canonical Arakelov-Green function G is the unique function $X \times X \to \mathbb{R}_{\geq 0}$ such that the following properties hold:

(i) $G(P,Q)^2$ is C^∞ on $X \times X$ and $G(P,Q)$ vanishes only at the diagonal Δ_X. For a fixed $P \in X$, an open neighbourhood U of P and a local coordinate z on U we can write $\log G(P,Q) =$

[1] We warn the reader that some authors use the normalisation $\frac{i}{2\pi}$ instead of $\frac{i}{2}$.

$\log |z(Q)| + f(Q)$ for $P \neq Q \in U$, with f a C^∞-function;

(ii) for all $P \in X$ we have $\partial_Q \overline{\partial}_Q \log G(P,Q)^2 = 2\pi i \mu(Q)$ for $Q \neq P$;

(iii) for all $P \in X$ we have $\int_X \log G(P,Q)\mu(Q) = 0$.

Of course, the existence and uniqueness of such a function require proof. Such a proof is given in [Ar2]. However, that proof relies on methods from the theory of partial differential equations, and is ineffective in the sense that it does not give a way to construct G. One of the results in this thesis is an explicit formula for G which is well-suited for concrete calculations (see Theorem 2.1.2).

The defining properties of G imply the symmetry relation $G(P,Q) = G(Q,P)$ for all $P, Q \in X$. This follows by an easy application of Green's formula, which we state at the end of this section. The symmetry of G will be crucial for obtaining the symmetry of the Arakelov intersection product that we shall define in Section 1.2.

We now describe how the Arakelov-Green function gives rise to certain canonical metrics on the line bundles $O_X(D)$, where D is a divisor on X. It suffices to consider the case of a point $P \in X$, for the general case follows from this by taking tensor products. Let s be the canonical generating section of the line bundle $O_X(P)$. We then define a smooth hermitian metric $\|\cdot\|_{O_X(P)}$ on $O_X(P)$ by putting $\|s\|_{O_X(P)}(Q) = G(P,Q)$ for any $Q \in X$. By property (ii) of the Arakelov-Green function, the curvature form (*cf.* [GH], p. 148) of $O_X(P)$ is equal to μ, and in general, the curvature form of $O_X(D)$ is $\deg(D) \cdot \mu$, with $\deg(D)$ the degree of D.

Definition 1.1.2. A line bundle L with a smooth hermitian metric $\|\cdot\|$ is called admissible if its curvature form is a multiple of μ. We also call the metric $\|\cdot\|$ itself admissible in this case.

We will frequently make use of the following observation.

Proposition 1.1.3. *Let $\|\cdot\|$ and $\|\cdot\|'$ be admissible metrics on a line bundle L. Then the quotient $\|\cdot\|/\|\cdot\|'$ is a constant function on X.*

Proof. The logarithm of the quotient is a smooth harmonic function on X, hence it is constant. □

It follows that any admissible line bundle L is, up to a constant scaling factor, isomorphic to the admissible line bundle $O_X(D)$ for a certain divisor D. In Section 1.2 we will generalise the notion of admissible line bundle to arithmetic surfaces, and define an intersection product for admissible line bundles.

An important example of an admissible line bundle is the holomorphic cotangent bundle Ω^1_X. We define a metric on it as follows. Consider the line bundle $O_{X\times X}(\Delta_X)$ on $X \times X$. By the adjunction formula, we have a canonical residue isomorphism $O_{X \times X}(-\Delta_X)|_{\Delta_X} \xrightarrow{\sim} \Omega^1_X$. We obtain a smooth hermitian metric $\|\cdot\|$ on $O_{X\times X}(\Delta_X)$ by putting $\|s\|(P,Q) = G(P,Q)$, where s is the canonical generating section.

Definition 1.1.4. We define the metric $\|\cdot\|_{\mathrm{Ar}}$ on Ω^1_X by requiring that the residue isomorphism be an isometry.

Theorem 1.1.5. *(Arakelov [Ar2]) The metric $\|\cdot\|_{\mathrm{Ar}}$ is admissible.*

It remains to state Green's formula. We will use this formula once more in Section 3.8. It can be proved in a straightforward way using Stokes' formula.

Lemma 1.1.6. *(Green's formula) Let ϕ, ψ be functions on X such that for any $P \in X$, any small enough open neighbourhood U of P and any local coordinate z on U we can write $\log \phi(Q) = v_P(\phi) \log |z(Q)| + f(Q)$ and $\log \psi(Q) = v_P(\psi) \log |z(Q)| + g(Q)$ for all $P \neq Q \in U$ with $v_P(\phi), v_P(\psi)$ integers and f, g two C^∞-functions on U. Then the formula*

$$\frac{i}{\pi} \int_X \left(\log \phi \cdot \partial\overline{\partial} \log \psi - \log \psi \cdot \partial\overline{\partial} \log \phi \right) = \sum_{P \in X} \left(v_P(\phi) \log \psi(P) - v_P(\psi) \log \phi(P) \right)$$

holds.

1.2 Intersection theory

In this section we describe the intersection theory on an arithmetic surface in the original style of Arakelov [Ar2]. For the general facts that we use on arithmetic surfaces we refer to [Li].

Definition 1.2.1. An arithmetic surface is a proper flat morphism $p : \mathcal{X} \to B$ of schemes with $\mathcal{X}$ regular and with B the spectrum of the ring of integers in a number field K, such that the generic fiber $\mathcal{X}_K$ is a geometrically connected curve. If $\mathcal{X}_K$ has genus g, we also say that $\mathcal{X}$ is of genus g.

The arithmetic genus is constant in the fibers of an arithmetic surface, and all geometric fibers except finitely many are non-singular. Further we have $p_* O_{\mathcal{X}} = O_B$ for an arithmetic surface $p : \mathcal{X} \to B$, and hence, by the Zariski connectedness theorem, all fibers of p are connected.

Definition 1.2.2. An arithmetic surface $p : \mathcal{X} \to B$ of positive genus is called minimal if every proper birational B-morphism $\mathcal{X} \to \mathcal{X}'$ with $p' : \mathcal{X}' \to B$ an arithmetic surface, is an isomorphism.

For any geometrically connected, non-singular proper curve C of positive genus defined over a number field K there exists a minimal arithmetic surface $p : \mathcal{X} \to B$ together with an isomorphism $\mathcal{X}_\eta \xrightarrow{\sim} C$. This minimal arithmetic surface is unique up to isomorphism.

We now proceed to discuss the Arakelov divisors on an arithmetic surface $p : \mathcal{X} \to B$.

Definition 1.2.3. (*Cf.* [Ar2]) An Arakelov divisor on $\mathcal{X}$ is a finite formal integral linear combination of irreducible closed subschemes on $\mathcal{X}$ (*i.e.,* a Weil divisor), plus a contribution $\sum_\sigma \alpha_\sigma \cdot F_\sigma$ running over the embeddings $\sigma : K \hookrightarrow \mathbb{C}$ of K into the complex numbers. Here the $\alpha_\sigma \in \mathbb{R}$, and the F_σ are formal symbols, called the "fibers at infinity", corresponding to the Riemann surfaces $X_\sigma = (\mathcal{X} \otimes_{\sigma,B} \mathbb{C})(\mathbb{C})$. We have a natural group structure on the set of such divisors, denoted by $\widehat{\mathrm{Div}}(\mathcal{X})$.

Given an Arakelov divisor D, we write $D = D_{\mathrm{fin}} + D_{\mathrm{inf}}$ with D_{fin} its finite part, *i.e.,* the underlying Weil divisor, and with $D_{\mathrm{inf}} = \sum_\sigma \alpha_\sigma \cdot F_\sigma$ its infinite part. To a non-zero rational function f on $\mathcal{X}$ we associate an Arakelov divisor $(f) = (f)_{\mathrm{fin}} + (f)_{\mathrm{inf}}$ with $(f)_{\mathrm{fin}}$ the usual divisor of f on $\mathcal{X}$, and $(f)_{\mathrm{inf}} = \sum_\sigma v_\sigma(f) \cdot F_\sigma$ with $v_\sigma(f) = -\int_{X_\sigma} \log |f|_\sigma \cdot \mu_\sigma$. Here μ_σ is the fundamental (1,1)-form on X_σ given in Section 1.1. The infinite contribution $v_\sigma(f) \cdot F_\sigma$ is supposed to be an analogue of the contribution to (f) in the fiber above a closed point $b \in B$, which is given by $\sum_C v_C(f) \cdot C$ where C runs through the irreducible components of the fiber above b, and where v_C denotes the normalised discrete valuation on the function field of $\mathcal{X}$ defined by C. The "fiber at infinity" F_σ should be seen as "infinitely degenerate", with each point P of X_σ corresponding to an irreducible component, such that the valuation v_P of f along this component is given by $v_P(f) = -\log |f|_\sigma(P)$.

Definition 1.2.4. We say that two Arakelov divisors D_1,D_2 are linearly equivalent if their difference is of the form (f) for some non-zero rational function f. We denote by $\widehat{\mathrm{Cl}}(\mathcal{X})$ the group of Arakelov divisors on $\mathcal{X}$ modulo linear equivalence.

Next we discuss the intersection theory of Arakelov divisors, and show that this intersection theory respects linear equivalence. A vertical divisor on $\mathcal{X}$ is a divisor which consists only of irreducible components of the fibers of p. A horizontal divisor on $\mathcal{X}$ is a divisor which is flat over B. For typical cases D_1, D_2 of Arakelov divisors, the intersection product (D_1, D_2) is then defined as follows: (i) if D_1 is a vertical divisor, and D_2 is a Weil divisor, without any components in common with D_1, then the intersection (D_1, D_2) is defined as $(D_1, D_2) = \sum_b (D_1, D_2)_b \log \# k(b)$ where b runs through the closed points of B and where $(D_1, D_2)_b$ denotes the usual intersection multiplicity (*cf.* [Li], Section 9.1) of D_1, D_2 above b. (ii) if D_1 is a horizontal divisor, and D_2 is a "fiber at infinity" F_σ, then $(D_1, D_2) = \deg(D_1)$ with $\deg(D_1)$ the generic degree of D_1. (iii) if D_1 and D_2 are distinct sections of p, then (D_1, D_2) is defined as $(D_1, D_2) = (D_1, D_2)_{\mathrm{fin}} + (D_1, D_2)_{\mathrm{inf}}$ with $(D_1, D_2)_{\mathrm{fin}} = \sum_b (D_1, D_2)_b \log \#k(b)$ as in (i) and with $(D_1, D_2)_{\mathrm{inf}} = -\sum_\sigma \log G_\sigma(D_1^\sigma, D_2^\sigma)$ with G_σ the Arakelov-Green function (*cf.* Section 1.1) on X_σ. Note that $-\log G(P, Q)$ becomes a kind of intersection multiplicity "at infinity". The intersection numbers defined in this way extend by linearity to a pairing on $\widehat{\mathrm{Div}}(\mathcal{X})$.

Theorem 1.2.5. *(Arakelov [Ar2]) There exists a natural bilinear symmetric intersection pairing* $\widehat{\mathrm{Div}}(\mathcal{X}) \times \widehat{\mathrm{Div}}(\mathcal{X}) \to \mathbb{R}$. *This pairing factors through linear equivalence, giving an intersection pairing* $\widehat{\mathrm{Cl}}(\mathcal{X}) \times \widehat{\mathrm{Cl}}(\mathcal{X}) \to \mathbb{R}$.

Morally speaking, by "compactifiying" the arithmetic surface by adding in the "fibers at infinity", and by "compactifiying" the horizontal divisors on the arithmetic surface by allowing also for their complex points, we have created a framework that allows us to define a natural intersection theory respecting linear equivalence. This makes for a formal analogy with the classical intersection theory that we have on smooth proper surfaces defined over an algebraically closed field.

Let us sketch a proof of the second statement of Theorem 1.2.5 by showing that for a section D of p, and a non-zero rational function f on $\mathcal{X}$, we have $(D, (f)) = 0$. First let us determine, in general, the Arakelov-Green function $G(\mathrm{div}(f), P)$ for a non-zero meromorphic function f on a compact and connected Riemann surface X of positive genus. We note that $\partial_P \overline{\partial}_P \log G(\mathrm{div}(f), P)^2 = 0$ outside $\mathrm{div}(f)$, since the degree of $\mathrm{div}(f)$ is 0. But we also have $\partial\overline{\partial} \log |f|^2 = 0$ outside $\mathrm{div}(f)$, since f is holomorphic outside $\mathrm{div}(f)$. This implies that $G(\mathrm{div}(f), P) = e^\alpha \cdot |f|(P)$ for some constant α, and after taking logarithms and integrating against μ we find, by property (iii) of Definition 1.1.1, that $\alpha = -\int_X \log |f| \cdot \mu = v(f)$. We compute then

$$\begin{aligned}(D, (f)) &= (D, (f)_{\mathrm{fin}} + \sum_\sigma v_\sigma(f) \cdot F_\sigma) \\ &= (D, (f)_{\mathrm{fin}})_{\mathrm{fin}} + (D, (f)_{\mathrm{fin}})_{\mathrm{inf}} + \sum_\sigma v_\sigma(f) \\ &= \sum_b v_b(f|_D) \log \#k(b) - \sum_\sigma \log \left(e^{v_\sigma(f)} \cdot |f|_\sigma(D^\sigma) \right) + \sum_\sigma v_\sigma(f) \\ &= \sum_b v_b(f|_D) \log \#k(b) - \sum_\sigma \log |f|_\sigma(D^\sigma) ,\end{aligned}$$

which is zero by the product formula for K.

Finally, we connect the notion of Arakelov divisor with the notion of admissible line bundle.

Definition 1.2.6. An admissible line bundle L on $\mathcal{X}$ is the datum of a line bundle L on $\mathcal{X}$, together with smooth hermitian metrics on the restrictions of L to the X_σ, such that these restrictions are all admissible in the sense of Section 1.1. The group of isomorphism classes of admissible line bundles on $\mathcal{X}$ is denoted by $\widehat{\mathrm{Pic}}(\mathcal{X})$.

To each Arakelov divisor $D = D_{\mathrm{fin}} + D_{\mathrm{inf}}$ with $D_{\mathrm{inf}} = \sum_\sigma \alpha_\sigma \cdot F_\sigma$ we can associate an admissible line bundle $O_\mathcal{X}(D)$, as follows. For the underlying line bundle, we take $O_\mathcal{X}(D_{\mathrm{fin}})$. For the metric on $O_\mathcal{X}(D_{\mathrm{fin}})|_{X_\sigma}$ we take the canonical metric on $O_\mathcal{X}(D_{\mathrm{fin}})|_{X_\sigma}$ as in Section 1.1, multiplied by $e^{-\alpha_\sigma}$. Clearly, for two Arakelov divisors D_1 and D_2 which are linearly equivalent, the corresponding admissible line bundles $O_\mathcal{X}(D_1)$ and $O_\mathcal{X}(D_2)$ are isomorphic. The proof of the following theorem is then a rather formal exercise.

Theorem 1.2.7. *(Arakelov [Ar2]) There exists a canonical isomorphism of groups* $\widehat{\mathrm{Cl}}(\mathcal{X}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(\mathcal{X})$.

Theorem 1.2.7, together with Theorem 1.2.5, allows us to speak of the intersection product of two admissible line bundles, and we will often do this.

1.3 Determinant of cohomology

The determinant of cohomology for an arithmetic surface $p : \mathcal{X} \to B$ is a gadget on the base B which allows us to formulate an arithmetic Riemann-Roch theorem for p (Theorem 1.3.8). In the present section we will describe the determinant of cohomology in full generality. Our Riemann-Roch theorem will be a formal analogue of the Riemann-Roch that one obtains by taking the determinant of cohomology on a proper morphism $p : \mathcal{X} \to B$ with $\mathcal{X}$ a smooth proper surface and B a smooth proper curve, both defined over an algebraically closed field. With the help of arithmetic Riemann-Roch, we will be able to formulate and prove an arithmetic analogue of the Noether formula (see Section 1.6). References for this section are [De2] and [Mo1].

The determinant of cohomology is determined by a set of uniquely defining properties.

Definition 1.3.1. (*Cf.* [Mo1], §1) Let $p : \mathcal{X} \to B$ be a proper morphism of Noetherian schemes. To each coherent $O_{\mathcal{X}}$-module F on X, flat over O_B, we associate a line bundle $\det Rp_*F$ on B, called the determinant of cohomology of F, satisfying the following properties:

(i) The association $F \mapsto \det Rp_*F$ is functorial for isomorphisms $F \xrightarrow{\sim} F'$ of coherent $O_{\mathcal{X}}$-modules.

(ii) The construction of $\det Rp_*F$ commutes with base change, *i.e.*, each cartesian diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "u'"] \arrow[d, "p'"] & \mathcal{X} \arrow[d, "p"] \\ B' \arrow[r, "u"] & B \end{tikzcd}$$

gives rise to a canonical isomorphism $u^*(\det Rp_*F) \xrightarrow{\sim} \det Rp'_*(u'^*F)$.

(iii) Each exact sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

of flat coherent $O_{\mathcal{X}}$-modules gives rise to an isomorphism

$$\det Rp_*F \xrightarrow{\sim} \det Rp_*F' \otimes \det Rp_*F''$$

compatible with base change and with isomorphisms of exact sequences.

(iv) Let $E^{\cdot} = (0 \to E^0 \to E^1 \to \cdots \to E^n \to 0)$ be a finite complex of O_B-modules which are locally free of finite rank, and suppose there is given a quasi-isomorphism $E^{\cdot} \to Rp_*F$. Then one has a canonical isomorphism

$$\det Rp_*F \xrightarrow{\sim} \bigotimes_{k=0}^{n} \left(\det E^k\right)^{\otimes(-1)^k} ,$$

compatible with base change. Here $\det E$ denotes the maximal exterior power of a locally free O_B-module E of finite rank.

(v) In particular, when the O_B-modules R^kp_*F are locally free, one has a canonical isomorphism

$$\det Rp_*F \xrightarrow{\sim} \bigotimes_{k=0}^{n} \left(\det R^kp_*F\right)^{\otimes(-1)^k} ,$$

compatible with base change.

(vi) Let $\chi_{\mathcal{X}/B}(F)$ be the locally constant function $x \mapsto \chi(F_x)$ on B. Let $u \in \Gamma(B, O_B^*)$ be multiplication by u in F. By (i), this gives an automorphism of $\det Rp_*F$; this automorphism is multiplication by $u^{\chi_{\mathcal{X}/B}(F)}$.

(vii) If M is a line bundle on B then one has a canonical isomorphism

$$\det Rp_*(F \otimes p^*M) \xrightarrow{\sim} (\det Rp_*F) \otimes M^{\otimes \chi_{\mathcal{X}/B}(F)}$$

of line bundles on B.

In the case $B = \mathrm{Spec}(\mathbb{C})$, we will often use the shorthand notation $\lambda(F)$ for the determinant of cohomology of F. Explicitly, we have $\lambda(F) = \otimes_{k=0}^{n} (\det H^k(\mathcal{X}, F))^{\otimes(-1)^k}$, where n is the dimension of $\mathcal{X}$.

An important canonical coherent sheaf in the situation where $p : \mathcal{X} \to B$ is proper, flat and locally a complete intersection, is the relative dualising sheaf $\omega_{\mathcal{X}/B}$, *cf.* [Li], Section 6.4. In fact, the sheaf $\omega_{\mathcal{X}/B}$ is invertible, and satisfies the following important duality relation (Serre duality): let F be any coherent sheaf on $\mathcal{X}$, flat over O_B. Then we have a canonical isomorphism

$$\det Rp_*F \xrightarrow{\sim} \det Rp_*(\Omega^1_{\mathcal{X}/B} \otimes F^\vee)$$

of line bundles on B. The relative dualising sheaf behaves well with respect to base change: let $u : B' \to B$ be a morphism, let $\mathcal{X}' = \mathcal{X} \times_B B'$ and let $u' : \mathcal{X}' \to \mathcal{X}$ be the projection onto the first factor. Then we have a canonical isomorphism $u'^*\omega_{\mathcal{X}/B} \xrightarrow{\sim} \omega_{\mathcal{X}'/B'}$. As a consequence, by property (ii) in Definition 1.3.1 we have a canonical isomorphism $u^*(\det p_*\omega_{\mathcal{X}/B}) \xrightarrow{\sim} \det p'_*(\omega_{\mathcal{X}'/B'})$ on B'. Here $p' : \mathcal{X}' \to B'$ is the projection on the second factor. If $p : \mathcal{X} \to B$ is a smooth curve, the relative dualising sheaf $\omega_{\mathcal{X}/B}$ can be identified with the sheaf $\Omega^1_{\mathcal{X}/B}$ of relative differentials. A convenient description is also possible if the fibers of p are nodal curves, see [DM], §1.

For our Riemann-Roch theorem we need a metric on the determinant of cohomology $\det Rp_*L$, where L is an admissible line bundle on an arithmetic surface $p : \mathcal{X} \to B$. So, let us restrict for the moment to the case that $B = \mathrm{Spec}(\mathbb{C})$, and consider the determinant of cohomology $\lambda(L)$, where L is an admissible line bundle on a compact and connected Riemann surface X of positive genus g. The following theorem gives a satisfactory answer to our question.

Theorem 1.3.2. *(Faltings [Fa2]) For every admissible line bundle L there exists a unique metric on $\lambda(L)$ such that the following axioms hold:*

(i) any isomorphism $L_1 \xrightarrow{\sim} L_2$ of admissible line bundles induces an isometry $\lambda(L_1) \xrightarrow{\sim} \lambda(L_2)$;

(ii) if we scale the metric on L by a factor α, the metric on $\lambda(L)$ is scaled by a factor $\alpha^{\chi(L)}$, where $\chi(L) = \deg L - g + 1$;

(iii) for any admissible line bundle L and any point P, the exact sequence

$$0 \to L \to L(P) \to P_*P^*L(P) \to 0$$

induces an isometry

$$\lambda(L(P)) \xrightarrow{\sim} \lambda(L) \otimes P^*L(P)\,;$$

here $L(P)$ carries the metric coming from the canonical isomorphism $L(P) \xrightarrow{\sim} L \otimes_{O_X} O_X(P)$;

(iv) for $L = \Omega^1_X$, the metric on $\lambda(L) = \wedge^g H^0(X, \Omega^1_X)$ is defined by the hermitian inner product $(\omega, \eta) \mapsto \frac{i}{2}\int_X \omega \wedge \overline{\eta}$ on $H^0(X, \Omega^1_X)$.

We will refer to the metric in the theorem as the Faltings metric on the determinant of cohomology. For the proof of Theorem 1.3.2 we shall use the so-called Deligne bracket. Since we will make essential use of this tool later on, we define it here in detail.

Definition 1.3.3. (*Cf.* [De2]) Let $p : \mathcal{X} \to B$ be a proper, flat curve which is locally a complete intersection. Let L, M be two line bundles on $\mathcal{X}$. Then $\langle L, M \rangle$ is to be the O_B-module which is generated, locally for the étale topology on B, by the symbols $\langle l, m \rangle$ for local sections l, m of L, M, with relations

$$\langle l, fm \rangle = f(\mathrm{div}(l)) \cdot \langle l, m \rangle \quad , \quad \langle fl, m \rangle = f(\mathrm{div}(m)) \cdot \langle l, m \rangle \, .$$

Here $f(\mathrm{div}(l))$ should be interpreted as a norm: for an effective relative Cartier divisor D on $\mathcal{X}$ we set $f(D) = N_{D/B}(f)$, and then for $\mathrm{div}(l) = D_1 - D_2$ with D_1, D_2 effective we set $f(\mathrm{div}(l)) = f(D_1) \cdot f(D_2)^{-1}$. One checks that this is independent of the choices of D_1, D_2. Furthermore, it can be shown that the O_B-module $\langle L, M \rangle$ is actually a line bundle on B.

We have the following properties for the Deligne bracket.

(i) For given line bundles L_1, L_2, M_1, M_2, L, M on $\mathcal{X}$ we have canonical isomorphisms $\langle L_1 \otimes L_2, M \rangle \xrightarrow{\sim} \langle L_1, M \rangle \otimes \langle L_2, M \rangle$, $\langle L, M_1 \otimes M_2 \rangle \xrightarrow{\sim} \langle L, M_1 \rangle \otimes \langle L, M_2 \rangle$, and $\langle L, M \rangle \xrightarrow{\sim} \langle M, L \rangle$;

(ii) The formation of the Deligne bracket commutes with base change, *i.e.*, each cartesian diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "u'"] \arrow[d, "p'"] & \mathcal{X} \arrow[d, "p"] \\ B' \arrow[r, "u"] & B \end{tikzcd}$$

gives rise to a canonical isomorphism $u^* \langle L, M \rangle \xrightarrow{\sim} \langle u'^* L, u'^* M \rangle$;

(iii) For $P : B \to \mathcal{X}$ a section of p we have a canonical isomorphism $P^* L \xrightarrow{\sim} \langle O_{\mathcal{X}}(P), L \rangle$;

(iv) If the B-morphism $q : \mathcal{X}' \to \mathcal{X}$ is the blowing-up of a singular point on $\mathcal{X}$, then we have a canonical isomorphism $\langle q^* L, q^* M \rangle \xrightarrow{\sim} \langle L, M \rangle$;

(v) For the relative dualising sheaf $\omega_{\mathcal{X}/B}$ of p and any section $P : B \to \mathcal{X}$ of p we have a canonical adjunction isomorphism $\langle P, P \rangle^{\otimes -1} \xrightarrow{\sim} \langle P, \omega_{\mathcal{X}/B} \rangle$.

The relation with the determinant of cohomology is given by the following formula: let L, M be line bundles on $\mathcal{X}$, then we have a canonical isomorphism

$$\langle L, M \rangle \xrightarrow{\sim} \det Rp_*(L \otimes M) \otimes (\det Rp_* L)^{\otimes -1} \otimes (\det Rp_* M)^{\otimes -1} \otimes \det p_* \omega_{\mathcal{X}/B} \, .$$

This formula gives us new information on the determinant of cohomology, namely, it follows from the formula that we have a canonical isomorphism

$$(*) \qquad (\det Rp_* L)^{\otimes 2} \xrightarrow{\sim} \langle L, L \otimes \omega_{\mathcal{X}/B}^{-1} \rangle \otimes (\det p_* \omega_{\mathcal{X}/B})^{\otimes 2}$$

of line bundles on B. This isomorphism can be interpreted as Riemann-Roch for the morphism $p : \mathcal{X} \to B$. We will use Riemann-Roch to put metrics on the $\lambda(L)$. First of all we show how the Deligne bracket can be metrised in a natural way.

Definition 1.3.4. (*Cf.* [De2]) Let L, M be admissible line bundles on a Riemann surface X. Then for local sections l, m of L and M we put

$$\log \| \langle l, m \rangle \| = (\log \| m \|) \, [\mathrm{div}(l)] \, .$$

It can be checked that this gives a well-defined metric on $\langle L, M\rangle$, and in fact the isomorphisms from (i), (iii) and (v) above are isometries for this metric.

Proof of Theorem 1.3.2. We will construct a metric on $\lambda(L)$ such that axioms (i)–(iv) are satisfied. First of all we use property (iv) from Theorem 1.3.2 to put a metric on $\lambda(\omega)$. Next we use Definition 1.3.4 above to put a metric on the brackets $\langle L, L\otimes\omega^{-1}\rangle$. Then by Riemann-Roch (*) we obtain a metric on $\lambda(L)$. From this construction, the axioms (i) and (ii) are clear; it only remains to see that property (iii) is satisfied. But this we can see by the following argument due to Mazur: we have isometries

$$\lambda(L)^{\otimes 2} \xrightarrow{\sim} \langle L, L\otimes\omega^{-1}\rangle \otimes \lambda(\omega)^{\otimes 2} \quad \text{and} \quad \lambda(L(P)))^{\otimes 2} \xrightarrow{\sim} \langle L(P), L(P)\otimes\omega^{-1}\rangle \otimes \lambda(\omega)^{\otimes 2}\,.$$

Combining, we obtain an isometry

$$\lambda(L(P))^{\otimes 2} \otimes \lambda(L)^{\otimes -2} \xrightarrow{\sim} \langle L(P), L(P)\otimes\omega^{-1}\rangle \otimes \langle L, L\otimes\omega^{-1}\rangle^{\otimes -1}\,.$$

By expanding the brackets, we see that the latter is isometric to $P^*L(P) \otimes P^*(L\otimes\omega^{-1})$. By the adjunction formula, this is isometric with $(P^*L(P))^{\otimes 2}$. Hence property (iii) also holds, and Theorem 1.3.2 is proven. $\square$

Note that the Riemann-Roch isomorphism (*), which is by now an isometry given the various metrisations, gives us that the canonical Serre duality isomorphism $\lambda(L) \xrightarrow{\sim} \lambda(\Omega^1_X \otimes L^{-1})$ is an isometry.

To conclude this section, we explain what all this means for admissible line bundles on arithmetic surfaces. Using the metrisation of the determinant of cohomology, one obtains, for any arithmetic surface $p : \mathcal{X} \to B = \mathrm{Spec}(R)$ and any admissible line bundle L on $\mathcal{X}$, the determinant of cohomology $\det Rp_*L$ as a metrised line bundle (or metrised projective R-module) on B.

Definition 1.3.5. For a metrised projective R-module M we define a degree as follows: choose a non-zero element s of M, then

$$\widehat{\deg}\, M = \log \#(M/R\cdot s) - \sum_\sigma \log \|s\|_\sigma\,.$$

One can check using the product formula that this definition is independent of the choice of s.

It follows directly from Definitions 1.3.4 and 1.3.5 that for two admissible line bundles L, M on $\mathcal{X}$ we have $\widehat{\deg}\,\langle L, M\rangle = (L, M)$, the intersection product from Section 1.2.

We are now ready to reap the fruits of our work. Let $\omega_{\mathcal{X}/B}$ be the admissible line bundle on $\mathcal{X}$ whose underlying line bundle is the relative dualising sheaf of p, and where the metrics at infinity are the canonical ones as in Section 1.1.

Proposition 1.3.6. *(Adjunction formula, Arakelov [Ar2]) For any section $P : B \to \mathcal{X}$ we have an equality $-(P, P) = (P, \omega_{\mathcal{X}/B})$.*

Proof. This follows immediately from property (iii) of the Deligne bracket and the definition of the admissible metric on Ω^1_X for a compact and connected Riemann surface X, given in Section 1.1. $\square$

Proposition 1.3.7. *Let $q : B' \to B$ be a finite morphism with B' the spectrum of the ring of integers in a finite extension F of the quotient field K of R. Let $\mathcal{X}' \to \mathcal{X} \times_B B'$ be the minimal desingularisation of $\mathcal{X} \times_B B'$, and let $r : \mathcal{X}' \to \mathcal{X}$ be the induced morphism. Then we have, for any two admissible line bundles L, M on $\mathcal{X}$, an equality $(r^*L, r^*M) = [F : K](L, M)$.*

Proof. This follows from properties (ii) and (iv) of the Deligne bracket. $\square$

Proposition 1.3.8. *(Riemann-Roch theorem, Faltings [Fa2]) Let L be an admissible line bundle on $\mathcal{X}$. Then the formula*

$$\widehat{\deg}\det Rp_*L = \frac{1}{2}(L, L\otimes\omega_{\mathcal{X}/B}^{-1}) + \widehat{\deg}\det p_*\omega_{\mathcal{X}/B}$$

holds.

Proof. This follows directly from the fact that Riemann-Roch (*) is an isometry. □

1.4 Faltings' delta-invariant

The definition of the Faltings metric on the determinant of cohomology (see Theorem 1.3.2) is rather implicit, since it is given as the unique metric satisfying a certain set of axioms. In this section we want to make the Faltings metric more explicit. It turns out that there is a close relationship with theta functions, which we briefly review first. The connection is provided by Faltings' delta-invariant, which is defined in Theorem 1.4.6. We end this section by giving two fundamental formulas in which the delta-invariant occurs.

Let again X be a compact and connected Riemann surface of genus $g>0$. Let $\mathrm{Pic}_{g-1}(X)$ be the degree $g-1$ part in the Picard variety of isomorphism classes of line bundles on X. Choose a symplectic basis for the homology $H_1(X,\mathbb{Z})$ of X and choose a basis $\{\omega_1,\ldots,\omega_g\}$ of the holomorphic differentials $H^0(X,\Omega_X^1)$. Let $\Omega=(\Omega_1|\Omega_2)$ be the period matrix given by these data. By Riemann's first bilinear relations, the matrix Ω_1 is invertible and the matrix $\tau=\Omega_1^{-1}\Omega_2$ lies in $\mathcal{H}_g$, the Siegel upper half-space of complex symmetric $g\times g$-matrices with positive definite imaginary part.

Lemma 1.4.1. *(Riemann's second bilinear relations) The matrix identity*

$$\left(\frac{i}{2}\int_X \omega_k\wedge\overline{\omega}_l\right)_{1\le k,l\le g} = \frac{i}{2}(\overline{\Omega}_2{}^t\Omega_1 - \overline{\Omega}_1{}^t\Omega_2) = \overline{\Omega}_1(\mathrm{Im}\tau)^t\Omega_1$$

holds.

Proof. For the first equality, see for instance [GH], pp. 231–232. The second follows from the first by the fact that τ is symmetric. □

Choose a point $P_0\in X$, and let $\{\eta_1,\ldots,\eta_g\}=\{\omega_1,\ldots,\omega_g\}\cdot{}^t\Omega_1^{-1}$. Then by a classical theorem of Abel and Jacobi, the map

$$\mathrm{Div}_{g-1}(X)\longrightarrow\mathbb{C}^g/\mathbb{Z}^g+\tau\mathbb{Z}^g \quad , \quad \sum n_kP_k\mapsto\sum n_k\int_{P_0}^{P_k}(\eta_1,\ldots,\eta_g)$$

descends to well-defined bijective map

$$u:\mathrm{Pic}_{g-1}(X)\xrightarrow{\sim}\mathbb{C}^g/\mathbb{Z}^g+\tau\mathbb{Z}^g\,.$$

Let $\vartheta(z;\tau)$ be Riemann's theta function given by

$$\vartheta(z;\tau):=\sum_{n\in\mathbb{Z}^g}\exp(\pi i^tn\tau n+2\pi i^tnz)\,.$$

Due to its transformation properties under translation of z by an element of the lattice $\mathbb{Z}^g+\tau\mathbb{Z}^g$, the function ϑ can be viewed as a global section of a line bundle on $\mathbb{C}^g/\mathbb{Z}^g+\tau\mathbb{Z}^g$. We denote by Θ_0 the divisor of this section. Let $\Theta\subset\mathrm{Pic}_{g-1}(X)$ be the divisor given by the classes of line bundles admitting a global section. Riemann has shown that there is a close relationship between these two "theta-divisors".

Theorem 1.4.2. *(Riemann) There is an element $\kappa = \kappa(P_0)$ in $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ such that the following holds. Let t_κ denote translation by κ in $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$. Then the equality of divisors $(t_\kappa \cdot u)^*\Theta_0 = \Theta$ holds. In particular we have a canonical isomorphism of line bundles $(t_\kappa \cdot u)^* O(\Theta_0) \xrightarrow{\sim} O(\Theta)$ on $\mathrm{Pic}_{g-1}(X)$. Furthermore, for a divisor D of degree $g-1$ on X we have $(t_\kappa \cdot u)(K-D) = -(t_\kappa \cdot u)(D)$, where K is a canonical divisor on X. In particular, the map $t_\kappa \cdot u$ identifies the set of classes of semi-canonical divisors (i.e., divisors D with $2D$ linearly equivalent to K) with the set of 2-division points on $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$.*

We want to put a metric on the line bundle $O(\Theta)$. By Riemann's theorem, it suffices to put a metric on the line bundle $O(\Theta_0)$ on $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$. Let s be the canonical section of $O(\Theta_0)$, and let ν be the canonical translation-invariant (1,1)-form on $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ given by

$$\nu := \frac{i}{2} \sum_{1 \le k,l \le g} (\mathrm{Im}\tau)^{-1}_{k,l} dz_k \wedge d\overline{z}_l \,.$$

The $2g$-form $\frac{1}{g!}\nu^g$ gives the Haar measure on $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$. We let $\|\cdot\|_{\Theta_0}$ be the metric on $O(\Theta_0)$ uniquely defined by the following properties:

(i) the curvature form of $\|\cdot\|_{\Theta_0}$ is equal to ν;

(ii) $\frac{1}{g!}\int_{\mathbb{C}^g/\mathbb{Z}^g+\tau\mathbb{Z}^g} \|s\|^2_{\Theta_0} \nu^g = 2^{-g/2}$.

Definition 1.4.3. We denote by $\|\cdot\|_\Theta$ the metric on $O(\Theta)$ induced by $\|\cdot\|_{\Theta_0}$ via Riemann's theorem, and we write $\|\vartheta\|$ as a shorthand for $\|(t_\kappa \cdot u)^* s\|_\Theta$, or, by abuse of notation, for $\|s\|_{\Theta_0}$.

Note that $\|\vartheta\|(K-D) = \|\vartheta\|(D)$ for any divisor D of degree $g-1$, and that $\|\vartheta\|(D)$ vanishes if and only if D is linearly equivalent to an effective divisor.

By checking the properties (i) and (ii) one finds the following explicit formula for $\|\vartheta\|$.

Proposition 1.4.4. *Let $z \in \mathbb{C}^g$ and $\tau \in \mathcal{H}_g$, the Siegel upper half-space of degree g. Then the formula*

$$\|\vartheta\|(z;\tau) = (\det \mathrm{Im}\tau)^{1/4} \exp(-\pi {}^t y \cdot (\mathrm{Im}\tau)^{-1} \cdot y) \cdot |\vartheta(z;\tau)|$$

holds. Here $y = \mathrm{Im}\, z$.

It is not difficult to check using Lemma 1.4.1 that if we embed X into $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ by integration $j : P \mapsto \int_{P_0}^P (\eta_1, \ldots, \eta_g)$, we have $j^*\nu = g \cdot \mu$. One can view this as an alternative definition of the form μ.

Proposition 1.4.5. *Let D be a divisor on X, and consider the map $\phi_D : X \to \mathrm{Pic}_{g-1}(X)$ given by $P \mapsto [D - \chi(D) \cdot P]$, where $\chi(D) = \deg D - g + 1$. Then the line bundle $\phi_D^*(O(\Theta))$ on X is admissible and has degree $g \cdot \chi(D)^2$.*

Proof. A computation using the formula in Proposition 1.4.4 shows that outside $\phi_D^*(\Theta)$ we have $\partial_P \overline{\partial}_P \log \|\vartheta\|(D - \chi(D) \cdot P)^2 = 2\pi i \chi(D)^2 \cdot j^*\nu = 2\pi i g \chi(D)^2 \cdot \mu$. Thus, the curvature form of $\phi_D^*(O(\Theta))$ is a multiple of μ, and the degree of $\phi_D^*(O(\Theta))$ is $g \cdot \chi(D)^2$. □

The following theorem introduces Faltings' delta-invariant, connecting Faltings' metric on the determinant of cohomology with the metric on $O(\Theta)$ defined in Definition 1.4.3. It follows from axiom (ii) in Theorem 1.3.2 that for an admissible line bundle L of degree $g-1$, the metric on $\lambda(L)$ is in fact independent of the metric on L.

Theorem 1.4.6. *(Faltings [Fa2]) There is a constant $\delta = \delta(X)$ such that the following holds. Let L be an admissible line bundle of degree $g-1$. Then there is a canonical isomorphism $\lambda(L) \xrightarrow{\sim} O(-\Theta)[L]$, and the norm of this isomorphism is equal to $\exp(\delta/8)$.*

For the proof we need the following lemma. For the general definition of the scheme $\underline{\mathrm{Pic}}_{g-1}(\mathcal{X}/B)$, its theta divisor Θ, and for the existence of the universal bundle, we refer the reader to [Mo1], Section 2.

Lemma 1.4.7. *Let B be a noetherian scheme and let $p : \mathcal{X} \to B$ be a smooth proper curve admitting a section. There is, up to a unique isomorphism, a unique universal line bundle $\mathcal{U}$ on the product $\mathcal{X} \times \underline{\mathrm{Pic}}_{g-1}(\mathcal{X}/B)$. Let $q : \mathcal{X} \times \underline{\mathrm{Pic}}_{g-1}(\mathcal{X}/B) \to \underline{\mathrm{Pic}}_{g-1}(\mathcal{X}/B)$ be the projection onto the second factor. Then there is a canonical isomorphism $\det Rq_*\mathcal{U} \xrightarrow{\sim} O(-\Theta)$ of line bundles on $\underline{\mathrm{Pic}}_{g-1}(\mathcal{X}/B)$, compatible with base change.*

Proof. This is in [Mo1], Section 2.4. □

Sketch of the proof of Theorem 1.4.6. Let r be a non-negative integer, let E be a divisor of degree $r + g - 1$ on X, and consider the map $\varphi_E : X^r \to \mathrm{Pic}_{g-1}(X)$ given by $(P_1, \ldots, P_r) \mapsto O_X(E - (P_1 + \cdots + P_r))$. Let $\mathcal{U}$ be the universal line bundle on $X \times \mathrm{Pic}_{g-1}(X)$, and consider the pullback diagram

$$\begin{tikzcd} X^r \times X \arrow[r, "\tilde{\varphi}_E"] \arrow[d, "p"] & \mathrm{Pic}_{g-1}(X) \times X \arrow[d, "q"] \\ X^r \arrow[r, "\varphi_E"] & \mathrm{Pic}_{g-1}(X) \end{tikzcd}$$

with p, q the projections on the first factor and with $\tilde{\varphi}_E = (\varphi_E, \mathrm{id}_X)$. By Lemma 1.4.7 and Definition 1.3.1 we have a canonical isomorphism $\det Rp_*(\tilde{\varphi}_E^*\mathcal{U}) \xrightarrow{\sim} \varphi_E^*(O(-\Theta))$ of line bundles on X^r. It clearly suffices for our purposes to prove that the norm of this isomorphism is constant. But this follows from a calculation as performed in [Fa2], p. 397, showing that the curvature forms of the line bundles at both sides of the isomorphism are equal. □

In order to perform the calculation referred to at the end of the above proof, Faltings makes use of the following lemma. We, in turn, will use this lemma to derive an explicit formula from Theorem 1.4.6.

Lemma 1.4.8. *Let L be an admissible line bundle on X and let $P_1, \ldots, P_r$ be r points on X. Then we have a canonical isomorphism*

$$\lambda(L \otimes O_X(P_1 + \ldots + P_r)^\vee) \xrightarrow{\sim} \lambda(L) \otimes \bigotimes_{k=1}^{r} P_k^* L^\vee \otimes \bigotimes_{k<l} P_l^* O_X(P_k) \, ,$$

and this isomorphism is an isometry.

Proof. This follows just by iteration of axiom (iii) from Theorem 1.3.2. □

A fundamental theorem of Riemann states that if $D = P_1 + \cdots + P_g$ is an effective divisor of degree g such that $\phi_D(X)$ is not contained in Θ, we have an equality of divisors $\phi_D^*(\Theta) = D$ on X. By Propositions 1.1.3 and 1.4.5, the canonical isomorphism $\phi_D^*(O(\Theta)) \xrightarrow{\sim} O_X(P_1 + \cdots + P_g)$ has constant norm on X. In other words, there is a constant $c = c(P_1, \ldots, P_g)$ depending only on $P_1, \ldots, P_g$ such that $\|\vartheta\|(P_1 + \cdots + P_g - Q) = c \cdot \prod_{k=1}^g G(P_k, Q)$ for all $Q \in X$. We will now compute this constant. Let $\{\omega_1, \ldots, \omega_g\}$ be a basis of the differentials $H^0(X, \Omega_X^1)$ and let $P_1, \ldots, P_g$ be g points on X. Let $z_1, \ldots, z_g$ be local coordinates about $P_1, \ldots, P_g$ and write $\omega_k = f_{kl} \cdot dz_l$ locally at P_l. Then we write $\| \det \omega_k(P_l)\|_{\mathrm{Ar}} = |\det(f_{kl}(0))| \cdot \prod_{k=1}^g \|dz_k\|_{\mathrm{Ar}}(P_k)$. This definition does not depend on the choices of the local coordinates $z_1, \ldots, z_g$.

Theorem 1.4.9. *(Faltings [Fa2]) Let $\{\omega_1, \ldots, \omega_g\}$ be an orthonormal basis of $H^0(X, \Omega_X^1)$ provided with the hermitian inner product $(\omega, \eta) \mapsto \frac{i}{2} \int_X \omega \wedge \overline{\eta}$. Let $P_1, \ldots, P_g, Q$ be generic points on X.*

Then the formula

$$\|\vartheta\|(P_1 + \cdots + P_g - Q) = \exp(-\delta(X)/8) \cdot \frac{\|\det \omega_k(P_l)\|_{\mathrm{Ar}}}{\prod_{k<l} G(P_k, P_l)} \cdot \prod_{k=1}^{g} G(P_k, Q)$$

holds.

Proof. We apply Lemma 1.4.8 to the admissible line bundle $L = \Omega_X^1(Q) = \Omega_X^1 \otimes_{O_X} O_X(Q)$ and the points $P_1, \ldots, P_g$. We obtain the required formula by computing the norm of a canonical section on the left and the right hand side of the isomorphism in Lemma 1.4.8. By Serre duality we have $\lambda(\Omega_X^1(Q) \otimes O_X(P_1 + \cdots + P_g)^\vee) \cong \lambda(O_X(P_1 + \cdots + P_g - Q))$. For generic points $P_1, \ldots, P_g, Q$, the line bundle $O_X(P_1 + \cdots + P_g - Q)$ has no global sections. In this case, the determinant $\lambda(O_X(P_1 + \cdots + P_g - Q))$ is canonically isomorphic to $\mathbb{C}$ and hence has a canonical section 1. By Theorem 1.4.6, it has norm $\exp(-\delta(X)/8) \cdot \|\vartheta\|(P_1 + \cdots + P_g - Q)^{-1}$. Now let's look at the right hand side of the isomorphism in Lemma 1.4.8. We have a canonical isomorphism

$$\lambda(\Omega_X^1(Q)) \xrightarrow{\sim} \bigotimes_{k=1}^{g} P_k^* \Omega_X^1(Q)$$

given by taking the determinant of the evaluation map

$$H^0(X, \Omega_X^1(Q)) = H^0(X, \Omega_X^1) \xrightarrow{\sim} \bigoplus_{k=1}^{g} P_k^* \Omega_X^1(Q) \,.$$

The norm of this isomorphism is $\|\det \omega_k(P_l)\|_{\mathrm{Ar}} \cdot \prod_{k=1}^{g} G(P_k, Q)$, and hence we have a canonical element in $\lambda(\Omega_X^1(Q))^\vee \otimes \bigotimes_{k=1}^{g} P_k^* \Omega_X^1(Q)$ of that same norm. We end up with a canonical element in

$$\lambda(\Omega_X^1(Q)) \otimes \left(\bigotimes_{k=1}^{g} P_k^* \Omega_X^1(Q) \right)^\vee \otimes \bigotimes_{k<l} P_l^* O_X(P_k)$$

of norm

$$\|\det \omega_k(P_l)\|_{\mathrm{Ar}}^{-1} \cdot \prod_{k=1}^{g} G(P_k, Q)^{-1} \cdot \prod_{k<l} G(P_k, P_l) \,.$$

The theorem follows by equating the two norms. □

An important counterpart to Faltings' formula has been proved by Guàrdia [Gu1]. We will make essential use of this formula in Section 4.5 where, as an appendix to our work involved in determining a certain auxiliary Arakelov invariant for hyperelliptic Riemann surfaces, we prove a relation between products of certain Jacobian Nullwerte and products of certain Thetanullwerte. The new ingredient in Guàrdia's formula is a function $\|J\|$ on $\mathrm{Sym}^g X$, which we shall introduce first.

Recall that we have fixed for our Riemann surface X a symplectic basis of its homology and a basis $\{\omega_1, \ldots, \omega_g\}$ of $H^0(X, \Omega_X^1)$, giving rise to a period matrix $\Omega = (\Omega_1|\Omega_2)$. We have put $\tau = \Omega_1^{-1}\Omega_2$ and $\{\eta_1, \ldots, \eta_g\} = \{\omega_1, \ldots, \omega_g\} \cdot {}^t\Omega_1^{-1}$.

Lemma 1.4.10. *Consider $\wedge^g H^0(X, \Omega_X^1)$ with its metric derived from the hermitian inner product $(\omega, \eta) \mapsto \frac{i}{2}\int_X \omega \wedge \overline{\eta}$ on $H^0(X, \Omega_X^1)$. Then the formula $\|\omega_1 \wedge \ldots \wedge \omega_g\|^2 = (\det \mathrm{Im}\tau) \cdot |\det \Omega_1|^2$ holds.*

Proof. Note that $\|\omega_1 \wedge \ldots \wedge \omega_g\|^2 = \det\left((\omega_k, \omega_l)\right)_{k,l}$. The formula follows then from Lemma 1.4.1. □

Definition 1.4.11. For $w_1, \ldots, w_g \in \mathbb{C}^g$ we put

$$\begin{aligned} J(w_1, \ldots, w_g) &:= \det\left(\tfrac{\partial\vartheta}{\partial z_k}(w_l)\right), \\ \|J\|(w_1, \ldots, w_g) &:= (\det \mathrm{Im}\tau)^{\frac{g+2}{4}} \cdot \exp(-\pi \textstyle\sum_{k=1}^{g} {}^t y_k \cdot (\mathrm{Im}\tau)^{-1} \cdot y_k) \cdot |J(w_1, \ldots, w_g)|, \end{aligned}$$

where $y_k = \mathrm{Im}\, w_k$ for $k = 1, \ldots, g$. The latter definition depends only on the classes of the vectors w_k in $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$. Next, fix g points $P_1, \ldots, P_g$ on X and choose g vectors $w_1, \ldots, w_g$ in $\mathbb{C}^g$ by requiring that for each $k = 1, \ldots, g$, the divisor $\sum_{\substack{l=1 \\ l \neq k}}^{g} P_l$ corresponds by Riemann's theorem 1.4.2 to the class $[w_k] \in \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$. We then define $\|J\|(P_1, \ldots, P_g) := \|J\|(w_1, \ldots, w_g)$. One may check that this definition does not depend on the choice of the matrix τ.

We have $\|J\|(P_1, \ldots, P_g) = 0$ if and only if the points $P_1, \ldots, P_g$ are linearly dependent on the image of X under the canonical map $X \to \mathbb{P}(H^0(X, \Omega_X^1)^\vee)$.

The following theorem is Corollary 2.6 in [Gu1].

Theorem 1.4.12. *(Guàrdia [Gu1]) Let $P_1, \ldots, P_g, Q$ be generic points on X. Then the formula*

$$\|\vartheta\|(P_1 + \cdots + P_g - Q)^{g-1} = \exp(\delta(X)/8) \cdot \|J\|(P_1, \ldots, P_g) \cdot \frac{\prod_{k=1}^{g} G(P_k, Q)^{g-1}}{\prod_{k<l} G(P_k, P_l)}$$

holds.

Proof. If P is a point on X and t is a local coordinate about P, then by definition of the Arakelov metric on Ω_X^1 we have $\lim_{Q \to P} |t(P) - t(Q)|/G(P,Q) = \|dt\|_{\mathrm{Ar}}$. By a slight abuse of notation we write $\eta_k = \eta_k(P)dt$ and $\omega_k = \omega_k(P)dt$ for $k = 1, \ldots, g$. In this notation we have, for any divisor D of degree $g-1$,

$$\lim_{Q \to P} \|\vartheta\|(D + P - Q)/|t(P) - t(Q)| = (\det \mathrm{Im}\tau)^{1/4} \cdot \exp(-\pi {}^t y \cdot (\mathrm{Im}\tau)^{-1} \cdot y) \cdot \left|\sum_{k=1}^{g} \frac{\partial\vartheta}{\partial z_k}(w) \cdot \eta_k(P)\right|,$$

by the formula in Proposition 1.4.4. Here $y = \mathrm{Im}\, w$ and $w \in \mathbb{C}^g$ lifts a class that corresponds to D in $\mathrm{Pic}_{g-1}(X)$. Let us assume that $\{\omega_1, \ldots, \omega_g\}$ was an orthonormal basis of $H^0(X, \Omega_X^1)$. We are going to apply the above to the equation

$$\|\vartheta\|(P_1 + \cdots + P_g - Q) = \exp(-\delta(X)/8) \cdot \frac{\|\det \omega_k(P_l)\|_{\mathrm{Ar}}}{\prod_{k<l} G(P_k, P_l)} \cdot \prod_{k=1}^{g} G(P_k, Q)$$

which is Faltings' fundamental formula from Theorem 1.4.9. Let $t_1, \ldots, t_g$ be local coordinates about the points $P_1, \ldots, P_g$, and let w_k for each $k = 1, \ldots, g$ correspond to the divisor $\sum_{\substack{l=1 \\ l \neq k}}^{g} P_l$. Dividing through $|t_k(P_k) - t_k(Q)|$ and taking the limit $Q \to P_k$ we obtain

$$\begin{aligned} &(\det \mathrm{Im}\tau)^{1/4} \cdot \exp(-\pi {}^t y_k \cdot (\mathrm{Im}\tau)^{-1} \cdot y_k) \cdot \left|\sum_{l=1}^{g} \frac{\partial\vartheta}{\partial z_l}(w_k) \cdot \eta_l(P_k)\right| \\ &\qquad = \exp(-\delta(X)/8) \cdot \frac{\|\det \omega_k(P_l)\|_{\mathrm{Ar}}}{\prod_{k<l} G(P_k, P_l)} \cdot \prod_{l \neq k} G(P_k, P_l) \cdot \frac{1}{\|dt_k\|_{\mathrm{Ar}}(P_k)}. \end{aligned}$$

Multiplying over $k = 1, \ldots, g$ we obtain

$$
\begin{aligned}
(\det \mathrm{Im}\tau)^{g/4} \cdot \exp(-\pi \sum_{k=1}^{g} {}^t y_k \cdot (\mathrm{Im}\tau)^{-1} \cdot y_k) \cdot | \prod_{k=1}^{g} \sum_{l=1}^{g} \frac{\partial \vartheta}{\partial z_l}(w_k) \cdot \eta_l(P_k)| \\
= \exp(-g\delta(X)/8) \cdot \left(\frac{\| \det \omega_k(P_l) \|_{\mathrm{Ar}}}{\prod_{k<l} G(P_k, P_l)} \right)^g \cdot \prod_{k<l} G(P_k, P_l)^2 \cdot \prod_{k=1}^{g} \frac{1}{\|dt_k\|_{\mathrm{Ar}}(P_k)} .
\end{aligned}
$$

Riemann's singularity theorem (see [GH], pp. 341–342) says that for any effective divisor D on X, the projectivised tangent space $\mathbb{P}T_{\Theta,D}$ at the class of D in $\Theta \subset \mathrm{Pic}_{g-1}(X)$ contains the image of the divisor D on X under the canonical map $X \to \mathbb{P}T_{\mathrm{Pic}_{g-1}(X),D} \cong \mathbb{P}(H^0(X, \Omega_X^1)^\vee)$. For us this means that $\sum_{m=1}^{g} \frac{\partial \vartheta}{\partial z_m}(w_k) \cdot \eta_m(P_l) = 0$ whenever $k \neq l$. As a consequence, we can write

$$
\prod_{k=1}^{g} \sum_{l=1}^{g} \frac{\partial \vartheta}{\partial z_l}(w_k) \cdot \eta_l(P_k) = J(w_1, \ldots, w_g) \cdot \det \eta_k(P_l) .
$$

Plugging this in we obtain

$$
\begin{aligned}
(\det \mathrm{Im}\tau)^{-1/2} \cdot \|J\|(P_1, \ldots, P_g) \cdot | \det \eta_k(P_l)| \\
= \exp(-g\delta(X)/8) \cdot \left(\frac{\| \det \omega_k(P_l) \|_{\mathrm{Ar}}}{\prod_{k<l} G(P_k, P_l)} \right)^g \cdot \prod_{k<l} G(P_k, P_l)^2 \cdot \frac{| \det \omega_k(P_l)|}{\| \det \omega_k(P_l) \|_{\mathrm{Ar}}} .
\end{aligned}
$$

It follows from Lemma 1.4.10 that $| \det \eta_k(P_l)| = (\det \mathrm{Im}\tau)^{1/2} | \det \omega_k(P_l)|$. Hence we arrive at

$$
\|J\|(P_1, \ldots, P_g) = \exp(-g\delta(X)/8) \cdot \left(\frac{\| \det \omega_k(P_l) \|_{\mathrm{Ar}}}{\prod_{k<l} G(P_k, P_l)} \right)^{g-1} .
$$

The required formula is obtained by eliminating the factor $\| \det \omega_k(P_l) \|_{\mathrm{Ar}}$ using Faltings' fundamental formula again. $\square$

It is not so clear either from Theorem 1.4.6 or from the formulas of Faltings and Guàrdia derived above, how one can compute the delta-invariant for a given Riemann surface X. In fact, in the introduction to his paper [Fa2], Faltings says that he cannot give an explicit formula for it, except in the case of elliptic curves. However, as will become apparent in Section 1.6, the delta-invariant plays a very fundamental role in the function theory of the moduli space of curves, and therefore it deserves to be studied further. In Chapter 2 we will answer Faltings' question by giving a simple closed formula for the delta-invariant which holds in arbitrary genus.

1.5 Semi-stability

In this section and the next we formulate results that hold only in general for semi-stable arithmetic surfaces. We start by recalling the definition of a semi-stable curve.

Definition 1.5.1. Let B be a locally Noetherian scheme. A proper flat curve $p : \mathcal{X} \to B$ is called semi-stable if all geometric fibers of p are reduced, connected and have only ordinary double points as singularities, the arithmetic genus of the fibers is positive, and each non-singular rational component of a geometric fiber meets the other components in at least 2 points. For a semi-stable curve $p : \mathcal{X} \to B$ and a closed point $b \in B$ we denote by δ_b the number of singular points in the fiber at b. If $p : \mathcal{X} \to B$ is a semi-stable arithmetic surface, we denote by $\Delta_{\mathcal{X}/B}$ the divisor $\sum_b \delta_b \cdot b$ on B, where b runs through the closed points of B.

We will need the following result in Section 2.5. The proof uses the celebrated Hodge index theorem for arithmetic surfaces (*cf.* [Fa2], §5). This is well-documented and we will not discuss this further.

Proposition 1.5.2. *(Faltings [Fa2]) Let $p : \mathcal{X} \to B$ be a semi-stable arithmetic surface of genus $g > 0$, and let D be an effective Arakelov divisor on $\mathcal{X}$. Then*

(i) $(\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) \geq 0$,

(ii) $4g(g-1) \cdot (\omega_{X/B}, D) \geq (\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) \cdot \deg D$.

Proof. This is Theorem 5 in [Fa2]. □

We next consider the properties of semi-stable arithmetic surfaces with respect to base change.

Proposition 1.5.3. *Let $q : B' \to B$ be a finite morphism with B' the spectrum of the ring of integers in a finite extension L of the quotient field K of R. Let $\mathcal{X}' \to \mathcal{X} \times_B B'$ be the minimal desingularisation of $\mathcal{X} \times_B B'$, and let $r : \mathcal{X}' \to \mathcal{X}$ be the induced morphism.*

(i) The arithmetic surface $\mathcal{X}' \to B'$ is again semi-stable.

(ii) We have an equality of divisors $\Delta_{\mathcal{X}'/B'} = q^* \Delta_{\mathcal{X}/B}$ on B'.

(iii) There exists a canonical isomorphism $r^* \omega_{\mathcal{X}/B} \xrightarrow{\sim} \omega_{\mathcal{X}'/B'}$ on $\mathcal{X}'$.

(iv) There exists a canonical isomorphism $\det p'_* \omega_{\mathcal{X}'/B'} \xrightarrow{\sim} q^* \det p_* \omega_{\mathcal{X}/B}$ on B'.

Proof. As to (i) and (ii), these follow from the fact (*cf.* [La], Theorem V.5.1) that a double point in the fiber of $\mathcal{X} \times_B B'$ at a closed point b' is resolved by a chain of $e-1$ irreducible components isomorphic to $\mathbb{P}^1$ and having geometric self-intersection -2. Here e is the ramification index of $q : B' \to B$ at b'. Statement (iii) is in [La], Proposition V.5.5. Finally (iv) follows from (iii) and the defining properties of the determinant of cohomology. □

Proposition 1.5.3 makes it possible to define invariants of curves defined over a number field.

Theorem 1.5.4. *(Stable reduction theorem, Grothendieck, Deligne-Mumford et al. [DM]) For any geometrically connected, non-singular proper curve C of positive genus over a number field K there exists a finite extension L of K and a semi-stable arithmetic surface $p : \mathcal{X} \to B$ over the ring of integers of L such that the generic fiber of p is isomorphic to $X \otimes_K L$.*

We note that a semi-stable arithmetic surface is a minimal model of its generic fiber.

Proposition 1.5.5. *Let C/K be a curve of positive genus, and let L be a finite extension of K over which C acquires semi-stable reduction. Let $p : \mathcal{X} \to B$ be a semi-stable arithmetic surface over the ring of integers of L. Then the quantities $\widehat{\deg} \det p_* \omega_{\mathcal{X}/B} / [L : \mathbb{Q}]$ and $(\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) / [L : \mathbb{Q}]$ do not depend on the choice of L, hence they define invariants of C.*

Proof. This follows from Propositions 1.5.3, 1.3.7 and 1.3.8. □

Definition 1.5.6. We denote by $h_F(C)$ the quantity $\widehat{\deg} \det p_* \omega_{\mathcal{X}/B} / [L : \mathbb{Q}]$ from the above proposition. It is often referred to as the Faltings height of C. We denote by $e(C)$ the quantity $(\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) / [L : \mathbb{Q}]$ from the above proposition.

In [Fa1] it is proved that for a fixed number field K, the set of isomorphism classes of C/K of fixed positive genus and of bounded Faltings height, is finite.

1.6 Noether's formula

In this section we demonstrate the importance of the delta-invariant by showing that it can be seen as the norm of the so-called Mumford-isomorphism on the moduli space of curves (Theorem 1.6.1). This fundamental isomorphism was first obtained in [Mu1] by an application of the Grothendieck-Riemann-Roch theorem. In [Fa2] and [Mo2] we find an explicit construction of this isomorphism. We briefly discuss this construction in the proof of Theorem 1.6.1, leaving it to the reader to check the details in the aforementioned papers. As a consequence of the calculation of the norm of the Mumford-isomorphism we obtain the celebrated Noether formula in Arakelov theory (Corollary 1.6.3). Throughout this section we will freely use the language of stacks as in [Fa2] and [Mo2].

Let $g > 0$ be an integer. Let $\mathcal{M}_g$ be the moduli stack of smooth curves of genus g, and let $p : \mathcal{U}_g \to \mathcal{M}_g$ be the universal curve. For line bundles on $\mathcal{U}_g$ we have as in Section 1.3 the notion of Deligne bracket and determinant of cohomology on $\mathcal{M}_g$. In particular, if ω is the relative dualising sheaf of $p : \mathcal{U}_g \to \mathcal{M}_g$, then we have the line bundles $\det p_*\omega$ and $\langle \omega, \omega \rangle$ on $\mathcal{M}_g$.

Theorem 1.6.1. *(Mumford [Mu1], Faltings [Fa2], Moret-Bailly [Mo2]) There exists an isomorphism*

$$\mu : (\det p_*\omega)^{\otimes 12} \xrightarrow{\sim} \langle \omega, \omega \rangle$$

of line bundles on $\mathcal{M}_g$. This isomorphism is unique up to a sign. Its norm on $\mathcal{M}_g(\mathbb{C})$ is equal to $(2\pi)^{-4g} \exp(\delta)$.

Sketch of the proof. In order to prove existence it suffices, roughly speaking, to construct for each smooth proper curve $p : \mathcal{X} \to B$ of genus g an isomorphism $(\det p_*\omega_{\mathcal{X}/B})^{\otimes 12} \xrightarrow{\sim} \langle \omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B} \rangle$ which is compatible with base change. We will sketch such a construction only for $p : \mathcal{X} \to B$ which come equipped with a theta-characteristic L, *i.e.* a line bundle with an isomorphism $L^{\otimes 2} \xrightarrow{\sim} \omega_{\mathcal{X}/B}$. The general case requires a more subtle argument. Using L, we make $J := \underline{\mathrm{Pic}}_{g-1}(\mathcal{X}/B)$ into an abelian scheme over B. For this we refer to [Mo1], Section 2. Let $e : B \to J$ be its zero-section, and let $\Omega^1_{J/B}$ be the sheaf of relative 1-forms. In the case $B = \mathrm{Spec}(\mathbb{C})$, the global sections $H^0(J, \Omega^g_J)$ come equipped with a hermitian inner product $(\alpha, \beta) \mapsto (i/2)^g (-1)^{g(g-1)/2} \int_{J(\mathbb{C})} \alpha \wedge \overline{\beta}$. The next four steps give then the required isomorphism. (i) Let Θ be the theta divisor of $\underline{\mathrm{Pic}}_{g-1}(\mathcal{X}/B)$, see once more [Mo1], Section 2. Then there is a canonical isomorphism $e^*(\Omega^g_{J/B}) \xrightarrow{\sim} e^*(O(\Theta))^{\otimes 2}$, compatible with base change. This is Moret-Bailly's *formule clé*, see [Mo2] and [Mo3]. (ii) Let $j : \mathcal{X} \to J$ be the usual embedding, unique up to translation, which exists locally for the étale topology on B. Then there is a canonical isomorphism $e^*(\Omega^g_{J/B}) \xrightarrow{\sim} \det p_*\omega_{\mathcal{X}/B}$, compatible with base change. (iii) There is a canonical isomorphism $\det Rp_*L \xrightarrow{\sim} e^*(O(-\Theta))$, compatible with base change. This follows directly from Lemma 1.4.7. (iv) There are canonical isomorphisms

$$\det Rp_*(\omega^{\otimes 2}) \xrightarrow{\sim} \langle \omega, \omega \rangle \otimes \det p_*\omega \quad \text{and} \quad (\det Rp_*L)^{\otimes 8} \xrightarrow{\sim} \langle \omega, \omega \rangle^{\otimes -1} \otimes (\det p_*\omega)^{\otimes 8} ,$$

compatible with base change. These isomorphisms follow from the Riemann-Roch theorem for $p : \mathcal{X} \to B$, discussed in Section 1.3. The uniqueness up to sign of the isomorphism μ follows from the fact (see [Mo2], Lemme 2.2.3) that $H^0(\mathcal{M}_g, \mathbb{G}_m) = \{+1, -1\}$. The statement on the norm of μ follows from the fact that the isomorphism in (i) has norm $(2\pi)^{-4g}$ (this is the main result of [Mo3]), the isomorphism in (iii) has norm $\exp(\delta/8)$ by definition of the delta-invariant, and the other isomorphisms are isometries. $\square$

As was shown in [DM], for any $g \geq 1$ we have a moduli stack $\overline{\mathcal{M}}_g$ classifying stable curves of genus g. It contains the moduli stack $\mathcal{M}_g$ of smooth proper curves of genus g as an open substack. It is customary to denote by Δ the closed subset $\overline{\mathcal{M}}_g - \mathcal{M}_g$, provided with its reduced structure; this is a normal crossings divisor in $\overline{\mathcal{M}}_g$ (see [DM]). The divisor Δ is the union of different components

$$\Delta = \Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_r \,, \quad r = \lfloor g/2 \rfloor \,,$$

where Δ_0 denotes the closure of the locus corresponding to irreducible curves with a single node, and where Δ_k for $k > 0$ denotes the closure of the locus corresponding to reducible curves with components of genus k and genus $g-k$. Mumford [Mu1] has shown that the isomorphism μ extends over $\overline{\mathcal{M}}_g$.

Theorem 1.6.2. *(Mumford [Mu1]) There exists an isomorphism*

$$\mu : (\det p_*\omega)^{\otimes 12} \xrightarrow{\sim} \langle \omega, \omega \rangle \otimes O_{\overline{\mathcal{M}}_g}(\Delta)$$

of line bundles on $\overline{\mathcal{M}}_g$. This isomorphism is unique up to sign.

By considering the Mumford-isomorphism on the base of a semi-stable arithmetic surface $p : \mathcal{X} \to B$ and taking degrees on left and right we obtain the arithmetic Noether formula.

Corollary 1.6.3. *(Noether's formula, Faltings [Fa2], Moret-Bailly [Mo2]) Let $p : \mathcal{X} \to B$ be a semi-stable arithmetic surface of genus $g > 0$, with B the spectrum of the ring of integers in a number field K. Then the formula*

$$12 \,\widehat{\deg} \det p_* \omega_{\mathcal{X}/B} = (\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) + \sum_b \delta_b \log \# k(b) + \sum_{\sigma : K \hookrightarrow \mathbb{C}} \delta(X_\sigma) - 4g[K:\mathbb{Q}] \log(2\pi)$$

holds. Here b runs through the closed points in B, and σ runs through the complex embeddings of K.

A detailed investigation as in [Jo] and [We] shows that when viewed as a function on the moduli space $\mathcal{M}_g(\mathbb{C})$, the delta-invariant acquires logarithmic singularities along the components of the boundary divisor Δ. We will come back to this in Section 2.4. As was remarked by Faltings in his introduction to [Fa2], the delta-invariant can be viewed as the minus logarithm of a "distance" to Δ. This interpretation is supported by the Noether formula.

Chapter 2

Analytic invariants

The purpose of this chapter is to give explicit formulas for the Arakelov-Green function and the delta-invariant, introduced in Chapter 1. In order to do this, we introduce two new invariants S and T of Riemann surfaces. These invariants are reasonably explicit and can be efficiently calculated. In Section 2.1 we state our results. After giving the proofs in Section 2.2, we specialise to the case of elliptic curves in Section 2.3. In particular we obtain Faltings' formula for the delta-invariant for elliptic curves, given in [Fa2]. The asymptotic behavior of the invariants S and T is considered in Section 2.4. In Section 2.5 we give some applications of our formulas in intersection theory. Among other things we prove a lower bound for the self-intersection of the relative dualising sheaf. Finally we comment upon the use of Arakelov geometry in a recent bound for the complexity of an algorithm, due to Edixhoven, for computing certain Galois representations.

2.1 Results

Let X be a compact and connected Riemann surface of genus $g > 0$. Our first result deals with the Arakelov-Green function G of X. Let P be a generic point on X. By the remarks after the proof of Lemma 1.4.8, there is a constant $c = c(P)$ depending only on P such that for all $Q \in X$ we have $G(P,Q)^g = c(P) \cdot \|\vartheta\|(gP - Q)$. This has been observed by some authors before, see for instance the remarks in [Jo], p. 229. Our contribution is that we make the dependence on P of the constant $c(P)$ clear. Our result involves the divisor $\mathcal{W}$ of Weierstrass points on X. This is a divisor of degree $g^3 - g$ on X, given as the divisor of a Wronskian differential formed out of a basis of the holomorphic differentials $H^0(X, \Omega^1_X)$. For each point $P \in X$, the multiplicity of P in $\mathcal{W}$ is given by a weight $w(P)$, which can also be calculated by means of the classical gap sequence at P (see Remark 2.2.9).

Definition 2.1.1. We define the invariant $S(X)$ of X by means of the formula

$$\log S(X) := -\int_X \log \|\vartheta\|(gP - Q) \cdot \mu(P)\,,$$

where Q can be any point in X.

We will see later (Proposition 2.2.3) that the integrand has logarithmic singularities only at the Weierstrass points of X, which are integrable. Hence the integral is well-defined. That the definition does not depend on the choice of Q follows from the translation-invariance of the form ν on $\mathrm{Pic}_{g-1}(X)$.

The invariant $S(X)$ appears as a normalisation constant in the formula that we propose for the Arakelov-Green function.

Theorem 2.1.2. *Let $P, Q \in X$ with P not a Weierstrass point. Then the formula*

$$G(P,Q)^g = S(X)^{1/g^2} \cdot \frac{\|\vartheta\|(gP-Q)}{\prod_{W \in \mathcal{W}} \|\vartheta\|(gP-W)^{1/g^3}}$$

holds. Here the Weierstrass points are counted with their weights.

For P a Weierstrass point, and $Q \neq P$, both numerator and denominator in the formula of Theorem 2.1.2 vanish with order $w(P)$, the weight of P. The formula remains true also in this case, provided that we take the leading coefficients of the appropriate power series expansions about P in both numerator and denominator. Note that apart from the normalisation term involving $S(X)$, the Arakelov-Green function can be expressed in terms of certain values of the $\|\vartheta\|$-function. These values are very easy to calculate numerically. The (real) 2-dimensional integral involved in computing $S(X)$ is harder to carry out in general, but it is still not difficult.

Other ways of expressing the Arakelov-Green function in terms of quantities associated to X and μ have been given, for instance one might use the eigenvalues and eigenfunctions of the Laplacian (see [Fa2], Section 3), or one might use abelian differentials of the second and third kind (see [La], Chapter II). There is also a closed formula due to Bost [Bo]

$$\log G(P,Q) = \frac{1}{g!} \int_{\Theta + P - Q} \log \|\vartheta\| \cdot \nu^{g-1} + A(X) \,,$$

expressing the Arakelov-Green function in terms of an integral over the translated theta divisor. Here ν is the canonical translation-invariant (1,1)-form on $\mathrm{Pic}_{g-1}(X)$, and the quantity $A(X)$ is a certain normalisation constant, perhaps comparable to our $S(X)$.

One of our motives for finding a new explicit formula was the need to have a formula that makes the efficient calculation of the Arakelov-Green function possible. The other approaches that we mentioned are perhaps less suitable for this objective. For instance, the formula given by Bost involves a (real) $2g-2$-dimensional integral over a region which seems not easy to parametrise. Also, for each new pair of points (P,Q) one has to calculate such an integral again, whereas in our approach one only has to calculate a certain integral once.

Our second result deals with Faltings' delta-invariant $\delta(X)$. Let $\Phi : X \times X \to \mathrm{Pic}_{g-1}(X)$ be the map sending (P,Q) to the class of $(gP-Q)$. For a fixed $Q \in X$, let $i_Q : X \to X \times X$ be the map sending P to (P,Q), and put $\phi_{-Q} = \Phi \cdot i_Q$. This coincides with the definition of ϕ_D in Proposition 1.4.5 for divisors D, where we take $D = -Q$. Define the line bundle L_X by

$$L_X := \left(\bigotimes_{W \in \mathcal{W}} \phi_{-W}^* \left(O(\Theta) \right) \right)^{\otimes (g-1)/g^3} \otimes_{O_X} \left(\Phi^*(O(\Theta))|_{\Delta_X} \otimes_{O_X} \Omega_X^{\otimes g} \right)^{\otimes -(g+1)} \otimes_{O_X} \otimes \left(\Omega_X^{\otimes g(g+1)/2} \otimes_{O_X} \left(\wedge^g H^0(X, \Omega_X^1) \otimes_{\mathbb{C}} O_X \right) \right)^{\vee} \right)^{\otimes 2} .$$

We have then the following theorem.

Theorem 2.1.3. *The line bundle L_X is canonically trivial. Let $T(X)$ be the norm of the canonical trivialising section of L_X. Then the formula*

$$\exp(\delta(X)/4) = S(X)^{-(g-1)/g^2} \cdot T(X)$$

holds.

Despite appearances to the contrary, the invariant $T(X)$ admits a very concrete description, see Propositions 2.2.7 and 2.2.8. In fact, the computation of $T(X)$ involves only elementary operations on special values of the functions $\|\vartheta\|$ and Guàrdia's $\|J\|$. Thus, we have now a very simple closed formula for the delta-invariant, reducing its calculation to the calculation of the invariants $S(X)$ and $T(X)$, the former involving a (real) 2-dimensional integral, and the latter being elementary to calculate. We shall demonstrate the practical significance of our formulas for calculating Arakelov invariants in Chapter 6.

It seems an important problem to relate the invariants $S(X)$ and $T(X)$ to more classical invariants. In Chapters 3 and 4 we prove a result that does this for $T(X)$ with X a hyperelliptic Riemann surface. This is already quite involved.

Next, it seems worthwhile to study whether our invariants $S(X)$ and $T(X)$ give rise to proper, strongly $(g-2)$-pseudoconvex functions on $\mathcal{M}_g(\mathbb{C})$. This notion arises in the context of Morse theory on manifolds. The importance of finding such functions is stressed by Hain and Looijenga (private communication); indeed, if such functions would be seen to exist, numerous interesting results (both known and still conjectural) on the geometry of $\mathcal{M}_g(\mathbb{C})$ would be implied. Perhaps the explicit nature of our invariants opens a way to constructing such functions.

Our inspiration to study Weierstrass points in order to obtain results in Arakelov theory stems from the papers [Ar1], [Bu] and [Jo]. Especially the latter paper has been useful. For example, our formula for the delta-invariant in Theorem 2.1.3 is closely related to the formula from Theorem 2.6 of that paper. Our improvement on that formula is perhaps that we give an explicit splitting of the delta-invariant in a new invariant $S(X)$ involving an integral, and a new invariant $T(X)$ which is purely "classical". These invariants seem to be of interest in their own right.

2.2 Proofs

In this section we prove Theorems 2.1.2 and 2.1.3. The major idea will be to give Arakelov-theoretic versions of classical results on Weierstrass points.

First we recall the Wronskian differential that defines the divisor of Weierstrass points on X. An alternative approach is sketched in Remark 2.2.9 below. Let $\{\psi_1, \ldots, \psi_g\}$ be a basis of $H^0(X, \Omega_X^1)$. Let P be a point on X and let z be a local coordinate about P. Write $\psi_k = f_k \cdot dz$ for $k = 1, \ldots, g$. The Wronskian determinant about P is then the holomorphic function

$$W_z(\psi) := \det \left(\frac{1}{(l-1)!} \frac{d^{l-1} f_k}{dz^{l-1}} \right)_{1 \leq k,l \leq g} .$$

Let $\tilde{\psi}$ be the $g(g+1)/2$-fold holomorphic differential

$$\tilde{\psi} := W_z(\psi) \cdot (dz)^{\otimes g(g+1)/2} .$$

Then $\tilde{\psi}$ is independent of the choice of the local coordinate z, and extends to a non-zero global section of $\Omega_X^{g(g+1)/2}$. A change of basis changes the Wronskian differential by a non-zero scalar factor, so that the divisor of a Wronskian differential $\tilde{\psi}$ on X is unique: we denote this divisor by $\mathcal{W}$, the divisor of Weierstrass points.

The Wronskian differential leads to a canonical sheaf morphism

$$\left(\wedge^g H^0(X, \Omega_X^1) \otimes_{\mathbb{C}} O_X \right) \longrightarrow \Omega_X^{g(g+1)/2}$$

given by

$$\xi_1 \wedge \ldots \wedge \xi_g \mapsto \frac{\xi_1 \wedge \ldots \wedge \xi_g}{\psi_1 \wedge \ldots \wedge \psi_g} \cdot \tilde{\psi} .$$

This gives a canonical section in $\Omega_X^{\otimes g(g+1)/2} \otimes_{O_X} \left(\wedge^g H^0(X, \Omega_X^1) \otimes_{\mathbb{C}} O_X \right)^\vee$ whose divisor is $\mathcal{W}$.

Proposition 2.2.1. *The canonical isomorphism*

$$\Omega_X^{\otimes g(g+1)/2} \otimes_{O_X} \left(\wedge^g H^0(X,\Omega_X^1) \otimes_{\mathbb{C}} O_X\right)^\vee \xrightarrow{\sim} O_X(\mathcal{W})$$

has a constant norm on X.

Proof. This follows since both sides have the same curvature form, and the divisors of the canonical sections are equal. □

Definition 2.2.2. We shall denote by $R(X)$ the norm of the isomorphism from Proposition 2.2.1. In more concrete terms we have $\prod_{W\in\mathcal{W}} G(P,W) = R(X)\cdot\|\tilde{\omega}\|_{\mathrm{Ar}}(P)$ for any $P \in X$, where $\{\omega_1,\ldots,\omega_g\}$ is an orthonormal basis of $H^0(X,\Omega_X^1)$, and where the norm of the Wronskian differential $\tilde{\omega}$ is taken in the line bundle $\Omega_X^{\otimes g(g+1)/2}$ with its canonical metric induced from the canonical metric on Ω_X^1.

Taking logarithms and integrating against $\mu(P)$ gives, by property (iii) of the Arakelov-Green function, the formula $\log R(X) = -\int_X \log\|\tilde{\omega}\|_{\mathrm{Ar}}(P)\cdot\mu(P)$.

Recall from Section 2.1 the map $\Phi : X\times X \to \mathrm{Pic}_{g-1}(X)$ sending (P,Q) to the class of $(gP-Q)$. A classical result on the divisor of Weierstrass points is that the equality of divisors

$$\Phi^*(\Theta) = \mathcal{W}\times X + g\cdot\Delta_X$$

holds on $X\times X$, see for example [Fay], p. 31. Denote by $p_1 : X\times X \to X$ the projection on the first factor. Using Proposition 2.2.1, the above equality of divisors yields a canonical isomorphism of line bundles

$$\Phi^*(O(\Theta)) \xrightarrow{\sim} p_1^*\left(\Omega_X^{\otimes g(g+1)/2} \otimes \left(\wedge^g H^0(X,\Omega_X^1)\otimes_{\mathbb{C}} O_X\right)^\vee\right) \otimes O_{X\times X}(\Delta_X)^{\otimes g}$$

on $X\times X$. We will reprove this isomorphism in the next proposition, and show that its norm is constant on $X\times X$. After Corollary 2.2.5 to this proposition, the proofs of Theorems 2.1.2 and 2.1.3 are just a few lines.

Proposition 2.2.3. *On $X\times X$, there exists a canonical isomorphism of line bundles*

$$\Phi^*(O(\Theta)) \xrightarrow{\sim} p_1^*\left(\Omega_X^{\otimes g(g+1)/2} \otimes \left(\wedge^g H^0(X,\Omega_X^1)\otimes_{\mathbb{C}} O_X\right)^\vee\right) \otimes O_{X\times X}(\Delta_X)^{\otimes g}\,.$$

The norm of this isomorphism is everywhere equal to $\exp(\delta(X)/8)$.

Proof. We are done if we can prove that

$$\exp(\delta(X)/8)\cdot\|\vartheta\|(gP-Q) = \|\tilde{\omega}\|_{\mathrm{Ar}}(P)\cdot G(P,Q)^g$$

for all $P,Q\in X$, where $\{\omega_1,\ldots,\omega_g\}$ is an orthonormal basis of $H^0(X,\Omega_X^1)$. But this follows from the formula in Theorem 1.4.9, by a computation which is performed in [Jo], p. 233. Let P be a point on X, and choose a local coordinate z about P. By definition of the canonical metric on Ω_X^1 we have then that $\lim_{Q\to P}|z(P)-z(Q)|/G(P,Q) = \|dz\|_{\mathrm{Ar}}(P)$. Letting $P_1,\ldots,P_g$ approach P in Theorem 1.4.9 we get

$$\begin{aligned}
\lim_{P_l\to P}\frac{\|\det\omega_k(P_l)\|_{\mathrm{Ar}}}{\prod_{k<l}G(P_k,P_l)} &= \lim_{P_l\to P}\left\{\frac{\|\det\omega_k(P_l)\|_{\mathrm{Ar}}}{\prod_{k<l}|z(P_k)-z(P_l)|}\cdot\frac{\prod_{k<l}|z(P_k)-z(P_l)|}{\prod_{k<l}G(P_k,P_l)}\right\}\\
&= \left\{\lim_{P_l\to P}\frac{|\det\omega_k(P_l)|}{\prod_{k<l}|z(P_k)-z(P_l)|}\right\}\cdot\|dz\|_{\mathrm{Ar}}^{g+g(g-1)/2}(P)\\
&= |W_z(\omega)(P)|\cdot\|dz\|_{\mathrm{Ar}}^{g(g+1)/2}(P)\\
&= \|\tilde{\omega}\|_{\mathrm{Ar}}(P)\,.
\end{aligned}$$

The required formula is therefore just a limiting case of Theorem 1.4.9 where all P_k approach P. $\square$

Corollary 2.2.4. *The formula* $S(X) = R(X) \cdot \exp(\delta(X)/8)$ *holds.*

Proof. This follows easily by taking logarithms in the formula

$$\exp(\delta(X)/8) \cdot \|\vartheta\|(gP - Q) = \|\tilde{\omega}\|_{\mathrm{Ar}}(P) \cdot G(P,Q)^g$$

and integrating against $\mu(P)$. Here we use again property (iii) of the Arakelov-Green function and the formula $\log R(X) = -\int_X \log \|\tilde{\omega}\|_{\mathrm{Ar}}(P) \cdot \mu(P)$, which was noted above. $\square$

Corollary 2.2.5. *(i) Let* $Q \in X$*. Then we have a canonical isomorphism*

$$\phi_{-Q}^*(O(\Theta)) \xrightarrow{\sim} O_X(\mathcal{W} + g \cdot Q)$$

of constant norm $S(X)$ *on* X*.*
(ii) We have a canonical isomorphism

$$(\Phi^*(O(\Theta))|_{\Delta_X}) \otimes_{O_X} \Omega_X^{\otimes g} \xrightarrow{\sim} O_X(\mathcal{W})$$

of constant norm $S(X)$ *on* X*.*

Proof. We obtain the isomorphism in (i) by restricting the isomorphism from Proposition 2.2.3 to a slice $X \times \{Q\}$, and using Proposition 2.2.1. Its norm is then equal to $R(X) \cdot \exp(\delta(X)/8)$, which is $S(X)$ by Corollary 2.2.4. For the isomorphism in (ii) we restrict the isomorphism from Proposition 2.2.3 to the diagonal, and apply the canonical adjunction isomorphism $O_{X \times X}(-\Delta_X)|_{\Delta_X} \xrightarrow{\sim} \Omega_X^1$. Again we get norm $R(X) \cdot \exp(\delta(X)/8)$, since the adjunction isomorphism is an isometry. $\square$

Note that Corollary 2.2.5 gives an alternative interpretation to the invariant $S(X)$.

Proof of Theorem 2.1.2. By taking norms of canonical sections on left and right in the isomorphism from Corollary 2.2.5 (i) we obtain

$$G(P,Q)^g \cdot \prod_{W \in \mathcal{W}} G(P,W) = S(X) \cdot \|\vartheta\|(gP - Q)$$

for any $P, Q \in X$. Now take the (weighted) product over $Q \in \mathcal{W}$. This gives

$$\prod_{W \in \mathcal{W}} G(P,W)^{g^3} = S(X)^{g^3 - g} \cdot \prod_{W \in \mathcal{W}} \|\vartheta\|(gP - W)\,.$$

Plugging this in in the first formula gives

$$G(P,Q)^g \cdot S(X)^{\frac{g^3-g}{g^3}} \cdot \prod_{W \in \mathcal{W}} \|\vartheta\|(gP - W)^{1/g^3} = S(X) \cdot \|\vartheta\|(gP - Q)\,,$$

from which the theorem follows. $\square$

Proof of Theorem 2.1.3. From Corollary 2.2.5 (i) we obtain, again by taking the (weighted) product over $Q \in \mathcal{W}$, a canonical isomorphism

$$\left(\bigotimes_{W \in \mathcal{W}} \phi_{-W}^* O(\Theta) \right) \xrightarrow{\sim} O_X(g^3 \cdot \mathcal{W})$$

of norm $S(X)^{g^3-g}$. It follows that we have a canonical isomorphism

$$\left(\bigotimes_{W \in \mathcal{W}} \phi_{-W}^* O(\Theta) \right)^{\otimes (g-1)/g^3} \xrightarrow{\sim} O_X((g-1) \cdot \mathcal{W})$$

of norm $S(X)^{(g-1)(g^3-g)/g^3}$. From Corollary 2.2.5 (ii) we obtain a canonical isomorphism

$$\left((\Phi^*(O(\Theta))|_{\Delta_X}) \otimes_{O_X} \Omega_X^{\otimes g} \right)^{\otimes -(g+1)} \xrightarrow{\sim} O_X(-(g+1)\mathcal{W})$$

of norm $S(X)^{-(g+1)}$. Finally from Proposition 2.2.1 and Corollary 2.2.4 we have a canonical isomorphism

$$\left(\Omega_X^{\otimes g(g+1)/2} \otimes_{O_X} \left(\wedge^g H^0(X, \Omega_X^1) \otimes_{\mathbb{C}} O_X \right)^\vee \right)^{\otimes 2} \xrightarrow{\sim} O_X(2\mathcal{W})$$

of norm $S(X)^2 \exp(-\delta(X)/4)$. It follows that indeed the line bundle L_X is canonically trivial, and that its canonical trivialising section has norm

$$S(X)^{-(g-1)(g^3-g)/g^3} \cdot S(X)^{g+1} \cdot S(X)^{-2} \cdot \exp(\delta(X)/4) = S(X)^{(g-1)/g^2} \cdot \exp(\delta(X)/4) \, .$$

By definition this is $T(X)$, so the theorem follows. $\square$

Theorem 2.1.2 leads to an alternative formula for $S(X)$.

Proposition 2.2.6. *Let P be a point on X, not a Weierstrass point. Then the formula*

$$\log S(X) = -g^2 \cdot \int_X \log \|\vartheta\|(gP - Q) \cdot \mu(Q) + \frac{1}{g} \cdot \sum_{W \in \mathcal{W}} \log \|\vartheta\|(gP - W)$$

holds. Here the sum is over the Weierstrass points of X, counted with their weights.

Proof. Take logarithms in Theorem 2.1.2 and integrate against $\mu(Q)$. $\square$

It remains for us to give an explicit formula for the invariant $T(X)$. Let $P \in X$ not a Weierstrass point and let z be a local coordinate about P. Define $\|F_z\|(P)$ as

$$\|F_z\|(P) := \lim_{Q \to P} \frac{\|\vartheta\|(gP - Q)}{|z(P) - z(Q)|^g} \, .$$

Let $\{\omega_1, \ldots, \omega_g\}$ be an orthonormal basis of $H^0(X, \Omega_X^1)$.

Proposition 2.2.7. *The formula*

$$T(X) = \|F_z\|(P)^{-(g+1)} \cdot \prod_{W \in \mathcal{W}} \|\vartheta\|(gP - W)^{(g-1)/g^3} \cdot |W_z(\omega)(P)|^2$$

holds.

Proof. Let F be the canonical section of $(\Phi^*(O(\Theta))|_{\Delta_X}) \otimes \Omega_X^{\otimes g}$ given by the canonical isomorphism in Corollary 2.2.5 (ii). For its norm we have $\|F\| = \|F_z\| \cdot \|dz\|_{\mathrm{Ar}}^g$ in the local coordinate z. The canonical section of $\bigotimes_{W \in \mathcal{W}} \phi_{-W}^* O(\Theta)$ has norm $\prod_{W \in \mathcal{W}} \|\vartheta\|(gP - W)$ at P. Finally, the canonical section of $\Omega_X^{\otimes g(g+1)/2} \otimes_{O_X} \left(\wedge^g H^0(X, \Omega_X^1) \otimes_{\mathbb{C}} O_X \right)^\vee$ has norm $\|\tilde{\omega}\|_{\mathrm{Ar}} = |W_z(\omega)| \cdot \|dz\|_{\mathrm{Ar}}^{g(g+1)/2}$. The proposition follows then from the definition of $T(X)$. $\square$

We next give a formula for $T(X)$ in which only first order partial derivatives of the theta function occur.

Proposition 2.2.8. *Let $P_1, \ldots, P_g, Q$ be generic points on X. Then the formula*

$$T(X) = \left(\frac{\|\vartheta\|(P_1 + \cdots + P_g - Q)}{\prod_{k=1}^g \|\vartheta\|(gP_k - Q)^{1/g}} \right)^{2g-2} \cdot \\ \cdot \left(\frac{\prod_{k \neq l} \|\vartheta\|(gP_k - P_l)^{1/g}}{\|J\|(P_1, \ldots, P_g)^2} \right) \cdot \prod_{W \in \mathcal{W}} \prod_{k=1}^{g} \|\vartheta\|(gP_k - W)^{(g-1)/g^4}$$

holds.

Proof. The formula follows from Theorem 1.4.12, using Theorem 2.1.2 to eliminate the occurring values of the Arakelov-Green function G, and using Theorem 2.1.3 to eliminate the factor $\exp(\delta(X)/8)$. The factors involving $S(X)$ that are introduced in this way cancel out. □

Remark 2.2.9. An alternative way to obtain the divisor of Weierstrass points $\mathcal{W}$ on X is to use gap-sequences. Let $P \in X$ be a point.

Definition 2.2.10. The gap-sequence $\Gamma(P)$ at P is the set

$$\begin{aligned} \Gamma(P) &= \{a \geq 1 \,|\, \text{there is no meromorphic function} \, f \, \text{with} \, (f)_\infty = a \cdot P\} \\ &= \{a \geq 1 \,|\, \text{there exists a holomorphic 1-form} \, \omega \, \text{with a zero of exact order } a-1 \text{ at } P\}\,. \end{aligned}$$

Here $(f)_\infty$ denotes the polar part of a meromorphic function f. The equality implied by the definition follows from the Riemann-Roch theorem.

The following facts are then not difficult to see:

(i) $\mathbb{N} \setminus \Gamma(P)$ is a semi-group;

(ii) $1 \in \Gamma(P)$;

(iii) For $a \in \Gamma(P)$ we have $a \leq 2g - 1$;

(iv) The set $\Gamma(P)$ has cardinality g.

Let $\Gamma(P) = \{a_1, \ldots, a_g\}$ with $a_1 < \ldots < a_g$. We then define the weight of P to be the deviation of the gap-sequence from the sequence $\{1, \ldots, g\}$:

Definition 2.2.11. The weight $w(P)$ of P is the number $w(P) = \sum_{k=1}^g (a_k - k)$.

It follows that always $w(P) \leq g(g-1)/2$.

Definition 2.2.12. We call P a Weierstrass point if $\Gamma(P)$ differs from $\{1, \ldots, g\}$. Equivalently, we call P a Weierstrass point if $w(P) > 0$ or if $h^0(gP) > 1$.

In [Gun], pp. 123–125 we find a proof of the following proposition.

Proposition 2.2.13. *Let $\tilde{\psi} = W_z(\psi) \cdot (dz)^{\otimes g(g+1)/2}$ be a Wronskian differential in $H^0(X, \Omega_X^{g(g+1)/2})$. Then we have an equality of divisors* $\operatorname{div} \psi = \sum_{P \in X} w(P) \cdot P$.

As an example, consider a hyperelliptic Riemann surface X of genus $g \geq 2$. A hyperelliptic map $X \to \mathbb{P}^1$ has $2g+2$ ramification points, and for each ramification point P, the gap-sequence $\Gamma(P)$ at P equals $\Gamma(P) = \{1, 3, \ldots, 2g-1\}$. Hence, each P has weight $g(g-1)/2$, and the ramification points are exactly the Weierstrass points of X.

2.3 Elliptic curves

In this section we make the invariants $S(X)$ and $T(X)$ explicit for a Riemann surface X of genus 1. We can write $X = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ where τ is an element in the complex upper half plane. It is determined up to a transformation with an element of $\mathrm{SL}(2,\mathbb{Z})$. Since

$$\frac{i}{2}\int_X dz \wedge d\bar{z} = \mathrm{Im}\tau \,,$$

the holomorphic differential $dz/\sqrt{\mathrm{Im}\tau}$ is an orthonormal basis of $H^0(X,\Omega^1_X)$.

As usual we write $q = \exp(2\pi i\tau)$ and then we have the eta-function $\eta(\tau) = q^{1/24}\prod_{k=1}^{\infty}(1-q^k)$ and the modular discriminant $\Delta(\tau) = \eta(\tau)^{24} = q\prod_{k=1}^{\infty}(1-q^k)^{24}$. The latter is a modular form on $\mathrm{SL}(2,\mathbb{Z})$ of weight 12. We put $\|\eta\|(X) = (\mathrm{Im}\tau)^{1/4}\cdot|\eta(\tau)|$ and $\|\Delta\|(X) = \|\eta\|(X)^{24} = (\mathrm{Im}\tau)^6\cdot|\Delta(\tau)|$. These definitions do not depend on the choice of τ.

Theorem 2.3.1. *The formula*

$$S(X) = \frac{1}{\|\eta\|(X)}$$

holds.

As an immediate consequence we find a formula for the Arakelov-Green function, given already in [Fa2], Section 7.

Corollary 2.3.2. *The formula*

$$G(P,Q) = \frac{\|\vartheta\|(P-Q)}{\|\eta\|(X)}$$

holds.

Proof. Apply the previous result to the formula in Theorem 2.1.2. □

Proof of Theorem 2.3.1. We follow an analogous computation in [La], Chapter II, §5. The fundamental (1,1)-form μ is given by $\mu = \frac{i}{2}(dz\wedge d\bar{z})/\mathrm{Im}\tau$. We will perform our integrals over the fundamental domain A for X given by $z = \alpha\tau+\beta$ with $\alpha \in [-\frac{1}{2},\frac{1}{2}]$ and $\beta\in[0,1]$. Write $y = \mathrm{Im}z$. We find

$$\int_A -\pi y^2\cdot(\mathrm{Im}\tau)^{-1}\cdot\mu = \int_{\alpha=-\frac{1}{2}}^{\frac{1}{2}}\int_{\beta=0}^{1} -\pi\alpha^2\cdot\mathrm{Im}\tau\cdot d\alpha d\beta = -\frac{\pi}{12}\cdot\mathrm{Im}\tau$$

and we shall prove

$$\int_A \log|\vartheta(z;\tau)|\cdot\mu(z) = \log|\prod_{k=1}^{\infty}(1-\exp(2\pi ik\tau))| \,.$$

Together this gives

$$\log S(X) = -\int_X \log\|\vartheta\|\cdot\mu = -\log\|\eta\|(X)$$

as required. Let us prove the integral formula. We will make use of the product expansion (*cf.* [Mu2], p. 68)

$$\vartheta(z;\tau) = \prod_{k=1}^{\infty}(1-\exp(2\pi ik\tau))\cdot\prod_{k=0}^{\infty}\{(1+\exp(\pi i(2k+1)\tau-2\pi iz))\,(1+\exp(\pi i(2k+1)\tau+2\pi iz))\}\,.$$

Fix an index $k \geq 0$. In order to compute

$$\int_A \log|(1+\exp(\pi i(2k+1)\tau - 2\pi i z))| \cdot \mu(z)$$
$$= \int_{\alpha=-1/2}^{1/2} \int_{\beta=0}^{1} \log|(1+\exp(\pi i(2k+1)\tau - 2\pi i(\alpha\tau+\beta)))| d\alpha d\beta$$

we observe the following: for $\alpha < 1/2$ we have

$$|\exp(\pi i(2k+1)\tau - 2\pi i(\alpha\tau+\beta))| < 1\,,$$

and next for $w \in \mathbb{C}$ with $|w| < 1$ we have

$$-\log(1-w) = \sum_{m=1}^{\infty} \frac{w^m}{m}\,,$$

where the convergence is uniform on compact subsets. This gives

$$\int_{\alpha=-1/2}^{1/2} \int_{\beta=0}^{1} \log|(1+\exp(\pi i(2k+1)\tau - 2\pi i(\alpha\tau+\beta)))| d\alpha d\beta$$
$$= \int_{\alpha=-1/2}^{1/2} \int_{\beta=0}^{1} \mathrm{Re} \left\{ \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \cdot \exp(\pi i(2k+1)m\tau - 2m\pi i(\alpha\tau+\beta)) \right\} d\alpha d\beta$$
$$= \mathrm{Re} \sum_{m=1}^{\infty} \int_{\alpha=-1/2}^{1/2} \int_{\beta=0}^{1} \frac{(-1)^{m+1}}{m} \cdot \exp(\pi i(2k+1)m\tau - 2m\pi i(\alpha\tau+\beta)) d\alpha d\beta = 0\,,$$

where the latter equality holds since for any m,

$$\int_{\beta=0}^{1} \exp(\pi i(2k+1)m\tau - 2m\pi i(\alpha\tau+\beta)) d\beta = 0$$

as one sees directly. In a similar vein one proves that

$$\int_A \log|(1+\exp(\pi i(2k+1)\tau + 2\pi i z))| \cdot \mu(z) = 0$$

for any fixed $k \geq 0$. Together this gives the required integral formula. □

In Chapter 5, where we study the Arakelov theory of elliptic curves more closely, we give an alternative proof of Corollary 2.3.2. This proof relies on special properties of the Arakelov-Green function of X, which we discover later on.

Next we turn to the invariant $T(X)$.

Theorem 2.3.3. *The formula*

$$T(X) = (2\pi)^{-2} \cdot \|\Delta\|(X)^{-1/4}$$

holds.

Using Theorem 2.1.3 we find the following corollary, which is also in Section 7 of [Fa2].

Corollary 2.3.4. *(Faltings [Fa2]) For Faltings' delta-invariant $\delta(X)$ of X, the formula*

$$\delta(X) = -\log\|\Delta\|(X) - 8\log(2\pi)$$

holds.

Proof of Theorem 2.3.3. We make use of the explicit formula for $T(X)$ in Proposition 2.2.7. Take the euclidean coordinate z as a local coordinate on $X = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$, and choose $\omega = dz/\sqrt{\mathrm{Im}\tau}$ as an orthonormal basis of $H^0(X, \Omega^1_X)$. Choose an arbitrary point $P \in X$. The Riemann vector is given by $\kappa = \frac{1+\tau}{2}$. An explicit computation yields

$$T(X) = \|F_z\|(P)^{-2} \cdot |W_z(\omega)|(P)^2 = (\mathrm{Im}\tau)^{-3/2} \cdot \exp(\pi \cdot \mathrm{Im}\tau/2) \cdot |\frac{\partial\vartheta}{\partial z}(\frac{1+\tau}{2};\tau)|^{-2} .$$

The proposition follows by the formula

$$\left(\exp(\pi i\tau/4) \cdot \frac{\partial\vartheta}{\partial z}\left(\frac{1+\tau}{2};\tau\right)\right)^8 = (2\pi)^8 \cdot \Delta(\tau)$$

which is a consequence of Jacobi's derivative formula (*cf.* [Mu2], Chapter I, §13). □

We could circumvent the computation in the above proof and apply Proposition 2.2.8 directly. However, the idea of using the explicit formula from Proposition 2.2.7 will be applied again in the next chapter, where we compute $T(X)$ for hyperelliptic Riemann surfaces (see especially the proof of Theorem 3.1.2 in Section 3.7). In fact the above proof is a special case of the arguments developed in Chapter 3.

We will give a proof of Jacobi's derivative formula in Section 4.6, using Arakelov theory.

2.4 Asymptotics

In [Fa2], Faltings asked for the behavior of the delta-invariant in a family of Riemann surfaces degenerating to a surface with a single node. An answer to this problem has been formulated by, among others, Jorgenson [Jo] and Wentworth [We]. Given our splitting of the delta-invariant in the invariants $S(X)$ and $T(X)$ (Theorem 2.1.3), it seems natural to ask for the asymptotic behavior of these new invariants. One expects that the question is more subtle than for the delta-invariant as a whole, and indeed this turns out to be the case. In fact, the asymptotic behavior of these invariants depends on the structure of the limit divisor of Weierstrass points. In the present section we give an asymptotic formula only in a generic case (Theorem 2.4.2). We start however by recalling the result of Jorgenson and Wentworth.

Theorem 2.4.1. *(Jorgenson [Jo], Wentworth [We]) Let X_t be a holomorphic family of compact and connected Riemann surfaces of genus $g > 0$ over the punctured disc as in [Fay], Chapter 3, degenerating as $t \to 0$ to a surface with a single node. If the degenerate surface is the union of two compact and connected Riemann surfaces of positive genera g_1, g_2 meeting at a single point, the formula*

$$\delta(X_t) = -\frac{4g_1g_2}{g}\log|t| + O(1)$$

holds. If the degenerate surface remains connected upon removing the node, the formula

$$\delta(X_t) = -\frac{4g-1}{3g}\log|t| - 6\log(-\log|t|) + O(1)$$

holds.

In particular, the asymptotic behavior of the delta-invariant is the same no matter what specific degenerate surface we choose (of the type mentioned in the theorem). This also accounts for the fact (*cf.* [Jo], Theorem 6.2) that the delta-invariant is, up to a log log-term associated to the locus of degenerate surfaces with a non-separating node, a Weil function on $\overline{\mathcal{M}}_g(\mathbb{C})$. As we will see in a minute, this is not true for the invariants $\log S(X)$ and $\log T(X)$. However, we have the following result for a "generic" degenerate surface of separate type.

Theorem 2.4.2. *Suppose that the degenerate surface is the union of two Riemann surfaces of positive genera g_1, g_2 with two points identified, and suppose furthermore that neither of these two points was a Weierstrass point on each of the two separate Riemann surfaces. Then the formulas*

$$\log S(X_t) = -\frac{g_1 g_2}{g} \log |t| + O(1)$$

and

$$\log T(X_t) = -\frac{g_1 g_2 (g^2 + g - 1)}{g^3} \log |t| + O(1)$$

hold.

Proof. We review from [Fay], Chapter 3 the description of the holomorphic family X_t in the separating case. We fix two compact and connected Riemann surfaces X_1 and X_2 of positive genera g_1, g_2, respectively. Further we fix coordinate neighbourhoods U_k about P_k and local coordinates $z_k : U_k \to D$, where D is the unit disk. We let $W_k^t = \{x_k \in X_k \,|\, x_k \in X_k \setminus U_k \,\mathrm{or}\, |z_k(x_k)| > |t|\}$ for $t \in D$ and $\mathcal{C}_t = \{(X, Y) \in D \times D \,|\, XY = t\}$. The family X_t of genus $g = g_1 + g_2$ is then built from these data by putting $X_t = W_1^t \cup \mathcal{C}_t \cup W_2^t$ with the following identifications: $x_1 \in W_1^t \cap U_1$ is identified with $(z_1(x_1), t/z_1(x_1)) \in \mathcal{C}_t$ and $x_2 \in W_2^t \cap U_2$ is identified with $(z_2(x_2), t/z_2(x_2)) \in \mathcal{C}_t$. For $t = 0$, we obtain a singular surface X_0 which is just $X_1 \cup X_2$ with the points P_1, P_2 identified.

From Section 3 of [Jo] we deduce the formulas

$$\begin{aligned}
\log \|\vartheta\|(gP - Q) &= \left\{ \begin{array}{rl} g_2 \log |t|\,, & P, Q \in X_1 \setminus \{P_1\} \\ g_1 \log |t|\,, & P, Q \in X_2 \setminus \{P_2\} \\ 0\,, & \text{otherwise} \end{array} \right\} + O(1)\,, \\
\log \|F_z\|(P) &= \left\{ \begin{array}{ll} g_2 \log |t|\,, & P \in X_1 \setminus \{P_1\} \\ g_1 \log |t|\,, & P \in X_2 \setminus \{P_2\} \end{array} \right\} + O(1)\,, \\
\log |W_z(\omega)(P)| &= \left\{ \begin{array}{ll} \frac{1}{2} g_2 (g_2 + 1) \log |t|\,, & P \in X_1 \setminus \{P_1\} \\ \frac{1}{2} g_1 (g_1 + 1) \log |t|\,, & P \in X_2 \setminus \{P_2\} \end{array} \right\} + O(1)\,, \\
g \textstyle\int_X \log \|\vartheta\|(gP - Q) \cdot \mu(Q) &= g_1 g_2 \log |t| + O(1)\,.
\end{aligned}$$

By Theorem 3.1 in [EH], under the condition stated in the theorem the limit Weierstrass divisor $\mathcal{W}_0$ on X_0, *i.e.*, the intersection of the closure of the Weierstrass divisor on the generic fiber with X_0, is equal to the union of a part $\mathcal{W}_1$ consisting of the ramification points outside P_1 of the linear system $|K_{X_1}((g_2 + 1)P_1)|$ on X_1, and a part $\mathcal{W}_2$ consisting of the ramification points outside P_2 of the linear system $|K_{X_2}((g_1 + 1)P_2)|$ on X_2. Here K_{X_1} and K_{X_2} denote canonical divisors on X_1 and X_2, respectively. In particular, by the Plücker formulas we have $\deg(\mathcal{W}_1) = g_1(g^2 - 1)$ and $\deg(\mathcal{W}_2) = g_2(g^2 - 1)$. Using the first formula above we obtain from this that

$$\sum_{W \in \mathcal{W}} \log \|\vartheta\|(gP - W) = (g^2 - 1) g_1 g_2 \log |t| + O(1)\,.$$

We obtain the limit formula for $\log S(X_t)$ by applying Corollary 2.2.6, and the limit formula for $\log T(X_t)$ by applying Proposition 2.2.7. $\square$

If, contrary to the conditions in the theorem, one of the identified points *is* a Weierstrass point, the limit Weierstrass divisor is in general different from the divisor described in the above proof. It seems interesting to investigate the asymptotic behavior of the invariants $T(X)$ and $S(X)$ in various cases that can occur. For example, using Theorem 3.1.4 below and a result of Cornalba and Harris [CH] it is easy to compute the asymptotic behavior of $T(X)$ in a holomorphic family of hyperelliptic Riemann surfaces degenerating to the union of two hyperelliptic Riemann surfaces meeting in a single point. In this case, the two identified points must be Weierstrass points since they are fixed by the hyperelliptic involution. We are outside the scope of Theorem 2.4.2, and indeed we find a different asymptotic behavior.

It also seems interesting to study the degeneration of the Weierstrass points further in the case that the degenerate surface has a non-separating node. This problem was posed already by Eisenbud and Harris in [EH].

2.5 Applications

In this section we use Proposition 2.2.1 to give a formula for the relative dualising sheaf on a semi-stable arithmetic surface (Proposition 2.5.2). As consequences we derive, among other things, a lower bound for the self-intersection of the relative dualising sheaf (Proposition 2.5.4) and a formula for the self-intersection of a point (Proposition 2.5.8).

Let $p : \mathcal{X} \to B$ be a semi-stable arithmetic surface over the spectrum B of the ring of integers in a number field K. We assume that the generic fiber $\mathcal{X}_K$ is a geometrically connected, smooth proper curve of genus $g > 0$. Denote by $\mathcal{W}$ the Zariski closure in $\mathcal{X}$ of the divisor of Weierstrass points on $\mathcal{X}_K$, and denote by $\omega_{\mathcal{X}/B}$ the relative dualising sheaf of p. We will first deduce some properties of $\mathcal{W}$ on $\mathcal{X}$.

Lemma 2.5.1. *There exists an effective vertical divisor V on $\mathcal{X}$ such that we have a canonical isomorphism*

$$\omega_{\mathcal{X}/B}^{\otimes g(g+1)/2} \otimes_{O_{\mathcal{X}}} \left(p^*(\det p_*\omega_{\mathcal{X}/B})\right)^\vee \xrightarrow{\sim} O_{\mathcal{X}}(V + \mathcal{W})$$

of line bundles on $\mathcal{X}$.

Proof. We have on $\mathcal{X}$ a canonical sheaf morphism $p^*(\det p_*\omega_{\mathcal{X}/B}) \longrightarrow \omega_{\mathcal{X}/B}^{\otimes g(g+1)/2}$ given locally by

$$\xi_1 \wedge \ldots \wedge \xi_g \mapsto \frac{\xi_1 \wedge \ldots \wedge \xi_g}{\psi_1 \wedge \ldots \wedge \psi_g} \cdot \tilde{\psi}$$

for a basis $\{\psi_1, \ldots, \psi_g\}$ of differentials on the generic fiber of $\mathcal{X}$. Multiplying by $(p^*(\det p_*\omega_{\mathcal{X}/B}))^\vee$ we obtain a morphism

$$O_{\mathcal{X}} \longrightarrow \omega_{\mathcal{X}/B}^{\otimes g(g+1)/2} \otimes_{O_{\mathcal{X}}} \left(p^*(\det p_*\omega_{\mathcal{X}/B})\right)^\vee .$$

The image of 1 is a section whose divisor is an effective divisor $V + \mathcal{W}$ where V is vertical. This gives the required isomorphism. □

We will now turn to the Arakelov intersection theory on $\mathcal{X}$. For a complex embedding $\sigma : K \hookrightarrow \mathbb{C}$ we denote by F_σ the "fiber at infinity" associated to σ. The corresponding compact and connected Riemann surface is denoted by X_σ. The next proposition is an analogue of Lemma 3.3 in [Ar1].

Proposition 2.5.2. *Let V be the effective vertical divisor from Lemma 2.5.1. Then we have*

$$\frac{1}{2}g(g+1)\omega_{\mathcal{X}/B} = V + \mathcal{W} + \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log R(X_\sigma) \cdot F_\sigma + p^*(\det p_*\omega_{\mathcal{X}/B})$$

as Arakelov divisors on $\mathcal{X}$. Here the sum runs over the embeddings of K in $\mathbb{C}$.

Proof. Consider the canonical isomorphism from Lemma 2.5.1. The restriction of this isomorphism to X_σ is the isomorphism of Proposition 2.2.1. In particular it has norm $R(X_\sigma)$. The proposition follows. □

Remark 2.5.3. Lemma 2.5.1 shows how the canonical isomorphism from Proposition 2.2.1 over the "fibers at infinity" of $\mathcal{X}$ extends over $\mathcal{X}$ itself. The "difference" between the left and right hand side is measured by the divisor V, which can therefore be seen as a finite analogue of the numbers $\log R(X_\sigma)$ associated with infinity.

We shall deduce three consequences from Proposition 2.5.2. We assume for the moment that $g \geq 2$. We define R_b for a closed point $b \in B$ by the equation $(2g-2) \cdot \log R_b = (V_b, \omega_{\mathcal{X}/B})$, where the intersection is taken in the sense of Arakelov. The assumption that $p : \mathcal{X} \to B$ is semi-stable implies that the quantity $\log R_b$ is always non-negative.

Proposition 2.5.4. *Assume that $g \geq 2$. Then the lower bound*

$$(\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) \geq \frac{8(g-1)}{(2g-1)(g+1)} \left(\sum_b \log R_b + \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log R(X_\sigma) + \widehat{\deg} \det p_* \omega_{\mathcal{X}/B} \right)$$

holds. Here the first sum runs over the closed points $b \in B$, and the second sum runs over the embeddings of K in $\mathbb{C}$.

Proof. Intersecting the equality from Proposition 2.5.2 with $\omega_{\mathcal{X}/B}$ we obtain

$$\begin{aligned} &\frac{1}{2} g(g+1)(\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) = \\ &\qquad = (\mathcal{W}, \omega_{\mathcal{X}/B}) + (2g-2) \left(\sum_b \log R_b + \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log R(X_\sigma) + \widehat{\deg} \det p_* \omega_{\mathcal{X}/B} \right) . \end{aligned}$$

Now since the generic degree of $\mathcal{W}$ is $g^3 - g$ we obtain by Proposition 1.5.2 the lower bound

$$(\mathcal{W}, \omega_{\mathcal{X}/B}) \geq \frac{g^3 - g}{2g(2g-2)} (\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) .$$

Using this in the first equality gives the result. □

We remark that lower bounds of a similar type have been given by Burnol, *cf.* [Bu], Section 3.3. He defines, for a compact and connected Riemann surface X of genus $g \geq 2$, the constants

$$A_k(X) := - \int_X \log \|\vartheta\|(k\Omega^1_X - (2k-1)(g-1)P) \cdot \mu(P)$$

for $k \geq 2$. The integrands have only a finite number of logarithmic singularities, and hence the integrals are well-defined. Burnol arrives then, for a semi-stable arithmetic surface $p : \mathcal{X} \to B$ of genus $g \geq 2$, at the lower bound

$$\begin{aligned} \left(\frac{k^2 - k}{2} + \frac{6g-5}{48(g-1)} \right) (\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) \geq &\frac{1}{2g-2} \sum_\sigma \left(A_k(X_\sigma) - \frac{1}{24} \delta(X_\sigma) - \frac{g \log(2\pi)}{3} \right) \\ &+ \frac{1}{12(2g-2)} \sum_b \delta_b \log \# k(b) \end{aligned}$$

for any $k \geq 2$. He remarks with respect to this lower bound that it only becomes non-trivial (*i.e.* better than the classical lower bound from Proposition 1.5.2) if for all complex embeddings σ we would have $A_k(X_\sigma) \geq \frac{1}{24}\delta(X_\sigma) + g\log(2\pi)/3$. In order to get an idea of how often this may occur, one might start by making a study of the asymptotic behavior of the analytic invariants A_k. This was not carried out in [Bu]. However, with respect to the analytic terms in our lower bound for $(\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B})$ we have by Theorem 2.4.2 and Corollary 2.2.4 the following result.

Proposition 2.5.5. *Let X_t be a holomorphic family of compact and connected Riemann surfaces of genus $g \geq 2$ over the punctured disk, degenerating to the union of two Riemann surfaces of positive genera g_1, g_2 with two points identified. Suppose that neither of these two points was a Weierstrass point on each of the two separate Riemann surfaces. Then the formula*

$$\log R(X_t) = -\frac{g_1 g_2}{2g} \log |t| + O(1)$$

holds.

In particular, the value $\log R(X_t)$ goes to plus infinity under the conditions described in the theorem. It would be interesting to have a more precise, quantitative version of Proposition 2.5.5.

Our second result deals with an upper bound for $\sum_\sigma \log S(X_\sigma)$ for a semi-stable arithmetic surface $p : \mathcal{X} \to B$ of genus $g \geq 2$. Edixhoven has recently found an application of Arakelov theory in a study of the complexity of a certain algorithm that computes Galois representations associated to modular forms. In order to obtain a bound for this complexity, it turned out to be necessary to know how to bound the Arakelov-Green function $\sum_\sigma \log G(P_\sigma, Q_\sigma)$ from above for a semi-stable arithmetic surface $p : \mathcal{X} \to B$. This bound should depend on as few parameters as possible, and should be polynomial in the parameters that measure the length of the input of the algorithm. The present author has tried to attack this problem by looking at the explicit formula in Theorem 2.1.2. He expected that the classical part involving the values of the theta function would not be too difficult to bound from above, and that instead the normalisation constant $S(X)$ could be difficult. Indeed, Edixhoven informed him that Zagier had had these experiences on a similar problem. Things turned out to be otherwise: we can prove a bound for $\sum_\sigma \log S(X_\sigma)$ that meets Edixhoven's demands, but as yet we cannot deal with the classical term. Fortunately, at Edixhoven's request, other authors have searched for bounds on the Arakelov-Green function; we now have satisfactory answers due to Merkl (private communication) and Jorgenson-Kramer [JK2], [JK3], [JK4].

Proposition 2.5.6. *Let $p : \mathcal{X} \to B$ be a semi-stable arithmetic surface of genus $g \geq 2$. Then the upper bound*

$$\sum_\sigma \log S(X_\sigma) \leq \frac{1}{2} \widehat{\deg} \det p_* \omega_{\mathcal{X}/B} + \frac{g^2}{4(g-1)} (\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) + \frac{g}{2} [K : \mathbb{Q}] \log(2\pi)$$

holds.

Proof. In the Noether formula Corollary 1.6.3 we eliminate the terms involving δ by using the formula $\delta(X)/8 = \log S(X) - \log R(X)$, which is Corollary 2.2.4. We eliminate then the term involving $\log R$ by using the formula from Proposition 2.5.4. $\square$

In Section 2.6 we describe, by way of appendix, Edixhoven's algorithm.

The final result of this section deals with the self-intersection of a point P. This self-intersection gives, upon dividing by the degree of the field of definiton of P, the height of P with respect to the relative dualising sheaf. A major problem in diophantine geometry is to obtain certain bounds for this height. We want to contribute to this problem by giving an explicit expression for the self-intersection of a point. Perhaps it turns out to be possible to give bounds of the required shape for each of the summands in the expression.

We can assume that $g \geq 1$ again. We first state a lemma.

Lemma 2.5.7. *Let P be a section of p, not a Weierstrass point on the generic fiber. Then we have a canonical isomorphism*

$$P^*(O_{\mathcal{X}}(\mathcal{V} + \mathcal{W}))^{\otimes 2} \xrightarrow{\sim} (\det Rp_* O_{\mathcal{X}}(gP))^{\otimes 2}$$

of line bundles on B.

Proof. Applying Riemann-Roch to the line bundle $O_{\mathcal{X}}(gP)$ we obtain a canonical isomorphism

$$(\det Rp_* O_{\mathcal{X}}(gP))^{\otimes 2} \xrightarrow{\sim} \langle O_{\mathcal{X}}(gP), O_{\mathcal{X}}(gP) \otimes \omega_{\mathcal{X}/B}^{-1} \rangle \otimes (\det p_* \omega_{\mathcal{X}/B})^{\otimes 2}$$

of line bundles on B. The line bundle at the right hand side is, by the adjunction formula, canonically isomorphic to the line bundle $\langle P, P \rangle^{\otimes g(g+1)} \otimes (\det p_* \omega_{\mathcal{X}/B})^{\otimes 2}$. On the other hand, pulling back the isomorphism from Lemma 2.5.1 along P and using once more the adjunction formula gives a canonical isomorphism

$$\langle P, P \rangle^{\otimes - g(g+1)/2} \xrightarrow{\sim} \langle V + \mathcal{W}, P \rangle \otimes \det p_* \omega_{\mathcal{X}/B} \,.$$

The lemma follows by a combination of these observations. □

Proposition 2.5.8. *Let P be a section of p, not a Weierstrass point on the generic fiber. Then $-\frac{1}{2}g(g+1)(P,P)$ is given by the expression*

$$- \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log G(P_\sigma, \mathcal{W}_\sigma) + \log \# R^1 p_* O_{\mathcal{X}}(gP) + \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log R(X_\sigma) + \widehat{\deg} \det p_* \omega_{\mathcal{X}/B} \,,$$

where σ runs through the complex embeddings of K.

Proof. Intersecting the equality from Proposition 2.5.2 with P, and using the adjunction formula $(\omega, P) = -(P,P)$, we obtain the equality

$$-\frac{1}{2} g(g+1)(P,P) = (V + \mathcal{W}, P) + \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log R(X_\sigma) + \widehat{\deg} \det p_* \omega_{\mathcal{X}/B} \,.$$

It remains therefore to see that $(V + \mathcal{W}, P)_{\mathrm{fin}} = \log \# R^1 p_* O_{\mathcal{X}}(gP)$. For this we consider the isomorphism in Lemma 2.5.7. Note that $p_* O_{\mathcal{X}}(gP)$ is canonically trivialised by the function 1. This gives a canonical section at the right hand side with norm the square of $\# R^1 p_* O_{\mathcal{X}}(gP)$. Under the isomorphism, it is identified with the canonical section on the left-hand side, which has norm the square of $\exp((V + \mathcal{W}, P)_{\mathrm{fin}})$. The required equality follows. □

We see that minus the self-intersection of a point P is large if P is close to a Weierstrass point, either in the p-adic or in the complex topology.

2.6 Edixhoven's algorithm

To conclude this chapter we describe, in a few words, the essentials of Edixhoven's algorithm to compute Galois representations efficiently. We thank Edixhoven for explaining to us these ideas.

Consider for example the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation on the motive M_Δ associated to the discriminant modular form $\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^k)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$. For a prime number p, the integer $\tau(p)$ is the trace of the Frobenius at p acting on M_Δ. Our goal is an algorithm that, given a prime number p, computes the integer $\tau(p)$, and we want that algorithm to run in time polynomial in $\log p$. Earlier algorithms to compute $\tau(p)$ are exponential in $\log p$.

By a famous argument due to Schoof, since the integers $\tau(p)$ can be bounded as $|\tau(p)| \leq 2p^{11/2}$, it suffices to give an algorithm that, given a prime number p and a prime number ℓ, computes the trace of Frobenius at p modulo ℓ in time polynomial in ℓ. Now it can be shown that the mod ℓ étale realisation of M_Δ is the dual of a certain 2-dimensional $\mathbb{F}_\ell$-vector space V_ℓ contained in the ℓ-torsion $J_1(\ell)(\overline{\mathbb{Q}})[\ell]$ of the jacobian $J_1(\ell)$ of the modular curve $X_1(\ell)$. In fact, this V_ℓ is the intersection of the kernels of the endomorphisms $T_q - \tau(q)$, with q running over the primes up to about $\ell^2/24$, acting on $J_1(\ell)(\overline{\mathbb{Q}})[\ell]$. Here T_q is the q-th Hecke operator. We are basically through if, given a prime ℓ, we can compute, in a time polynomial in ℓ, the minimum polynomial of a generator of the field of definition of a non-zero point in V_ℓ.

Using explicit estimates in Arakelov intersection theory it can be shown that such an algorithm exists. In fact, the actual algorithm is probabilistic with an expected running time polynomial in ℓ, but we shall ignore this aspect here. Let us describe the main idea, which is surprisingly simple. Consider a prime ℓ and let x be a non-zero point in V_ℓ. We want to compute the minimum polynomial of a generator of the field of definition of x. First of all, it can be shown that we can explicitly construct an effective divisor D of degree g on $X_1(\ell)$, supported on the cusps, such that x is equal to the class of $D' - D$ for a unique effective divisor $D' = P_1 + \cdots + P_g$ on $X_1(\ell)$. Here g is the genus of $X_1(\ell)$, which is a polynomial function of ℓ. Since the field of definition of D is small, we are reduced to finding the minimum polynomial of a generator of the field of definition K of D'. The essential idea is to do this by numerical methods. Using p-adic methods in the sense of Couveignes, or using numerical integration over the complex numbers, it is possible to write down an approximation $\tilde{D}'$ of D'. Having found this approximation, one obtains also an approximation of a generator α of the field of definition of D'. This is seen by the following lemma: there is an explicit finite sequence of morphisms $j_1, \ldots, j_N : X_1(\ell) \to \mathbb{P}^1$, defined over $\mathbb{Q}$, such that at least one j has the property that $j(P_1) + \cdots + j(P_g)$ generates K (in fact, for this we need to work on $X_1(5\ell)$, but we shall ignore this fact). It is virtually no extra effort to compute approximations to all Galois conjugates of α, and hence we find approximations of the rational numbers that form the coefficients of the minimum polynomial of α. If we could prove that the height of these coefficients is bounded by a polynomial in ℓ, we would have the required algorithm: indeed, the polynomial bound on the height implies that it is sufficient to carry out all the approximations in our earlier steps with an accuracy that is polynomial in ℓ, and hence they can be made to require a running time that is polynomial in ℓ. Now a bound on the height of the coefficients of the required shape follows from the following general proposition. The proof uses only arithmetic intersection theory as explained in Chapter 1.

Proposition 2.6.1. *Let X be a proper connected non-singular curve of genus $g \geq 1$ over $\overline{\mathbb{Q}}$, and let D be an effective divisor of degree g on X. For any torsion line bundle L on X that satisfies $h^0(L(D)) = 1$ we have then the following. Let K be a number field such that both L and D are defined over K, such that X has semi-stable reduction over K, and such that X has a rational point P over K. Let D' be the unique effective divisor on X such that L is isomorphic to $O_X(D' - D)$. Extend D, D' and P to horizontal divisors on the semi-stable model $p : \mathcal{X} \to B$ of X over K. Then for the Arakelov intersection $(D' - D, P)$ the upper bound*

$$(D' - D, P) \leq -\frac{1}{2}(D, D - \omega_{\mathcal{X}/B}) + 2g^2 \sum_b \nu_b \log \#k(b) + \frac{1}{2}\widehat{\deg} \det p_* \omega_{\mathcal{X}/B}$$
$$+ \sum_\sigma \log \|\vartheta\|_{\sigma,\sup} + \frac{g}{2}[K : \mathbb{Q}] \log(2\pi)$$

holds. Here ν_b is the number of irreducible components in the fiber at b if the fiber is reducible, and $\nu_b = 0$ otherwise.

It is not difficult to show that a polynomial bound for $(D' - D, P)$ in our set-up implies a polynomial bound for the height of the coefficients of the minimum polynomial of α. Hence we are done if we could see that the terms on the right hand side in the above lemma are bounded by a polynomial in ℓ. This is again not difficult, except for the first term $-\frac{1}{2}(D, D - \omega_{\mathcal{X}/B})$, which requires that we bound the Arakelov-Green function by a polynomial in ℓ. We have commented upon this particular problem in the previous section.

Chapter 3

Hyperelliptic Riemann surfaces I

The purpose of this and the next chapter is to make the analytic theory from Chapter 2 explicit in the case of a hyperelliptic Riemann surface X. We will prove two theorems (Theorem 3.1.2 and Theorem 3.1.3) expressing the Arakelov-Green function G of X, evaluated at pairs of Weierstrass points, in terms of the invariant $T(X)$ and a second natural invariant of X, which is introduced in Section 3.2 below. As corollaries, we find simple closed formulas for the invariant $T(X)$ and Faltings' delta-invariant $\delta(X)$ of X. The main part of the present chapter is devoted to a proof of Theorem 3.1.2. We finish with a section dealing with some more special results in the case $g = 2$. The proof of Theorem 3.1.3 will be given in the next chapter. Although our two theorems look very similar, the techniques used in the proofs are very different. The proof of Theorem 3.1.2 uses only complex function theory, but for the proof of Theorem 3.1.3 we need to take a broader perspective and consider hyperelliptic curves over arbitrary base schemes. A special role is then played by hyperelliptic curves which are defined over a discrete valuation ring with residue characteristic equal to 2.

3.1 Results

Let X be a hyperelliptic Riemann surface of genus $g \geq 2$. In Section 3.2 we introduce a non-zero invariant $\|\varphi_g\|(X)$, the Petersson norm of the modular discriminant associated to X. As we will see, this is a very natural invariant to consider for hyperelliptic Riemann surfaces. Unfortunately, it is not so clear how to extend its definition to the general Riemann surface of genus g.

Definition 3.1.1. We denote by G' the modified Arakelov-Green function

$$G'(P,Q) := S(X)^{-1/g^3} \cdot G(P,Q)$$

on $X \times X$.

In the present chapter we prove the following theorem dealing with G' and $T(X)$. Recall that the Weierstrass points of X are just the branch points of a hyperelliptic map $X \to \mathbb{P}^1$.

Theorem 3.1.2. *Let W be a Weierstrass point of X. Let $n = \binom{2g}{g+1}$. Consider the product $\prod_{W' \neq W} G'(W,W')$ running over all Weierstrass points W' different from W, ignoring their weights. Then $\prod_{W' \neq W} G'(W,W')$ is independent of the choice of W and the formula*

$$\prod_{W' \neq W} G'(W,W')^{(g-1)^2} = 2^{(g-1)^2} \pi^{2g+2} \cdot T(X)^{\frac{g+1}{g}} \cdot \|\varphi_g\|(X)^{\frac{1}{2n}}$$

holds.

The following theorem will be derived in the next chapter.

Theorem 3.1.3. *Let $m = \binom{2g+2}{g}$. Then we have*

$$\prod_{(W,W')} G'(W,W')^{n(g-1)} = \pi^{-2g(g+2)m} \cdot T(X)^{-(g+2)m} \cdot \|\varphi_g\|(X)^{-\frac{3}{2}(g+1)} ,$$

the product running over all ordered pairs of distinct Weierstrass points of X.

Combining the above two theorems yields a simple closed formula for the invariant $T(X)$ in terms of $\|\varphi_g\|(X)$. This formula should be compared with the formula in Theorem 2.3.3 above.

Theorem 3.1.4. *Let $\|\Delta_g\|(X)$ be the modified discriminant $\|\Delta_g\|(X) = 2^{-(4g+4)n} \cdot \|\varphi_g\|(X)$. Then the formula*

$$T(X) = (2\pi)^{-2g} \cdot \|\Delta_g\|(X)^{-\frac{3g-1}{8ng}}$$

holds.

From the viewpoint of arithmetic geometry, the modified invariant $\|\Delta_g\|$ is definitely the right one to consider. As we will see below, it has an integral structure which causes it to behave well in all characteristics. In this sense it is the right generalisation of the discriminant Δ for elliptic curves. The visual presence of the factors 2π and $\|\Delta_g\|$ in the above formula suggests the existence of a certain "motivic" interpretation of the invariant T. However, at present we do not know such an interpretation.

With Theorem 2.1.3 we obtain the following corollary.

Corollary 3.1.5. *For Faltings' delta-invariant $\delta(X)$ of X, the formula*

$$\exp(\delta(X)/4) = (2\pi)^{-2g} \cdot S(X)^{-(g-1)/g^2} \cdot \|\Delta_g\|(X)^{-\frac{3g-1}{8ng}}$$

holds.

We remark that in the case $g = 2$, an explicit formula for the delta-invariant has been given already by Bost [Bo]. We will turn to the relation between his and our formula in Section 3.8.

The idea of the proof of Theorem 3.1.2 is quite straightforward: we start with the formula for $T(X)$ in Proposition 2.2.7 and the formula for G in Theorem 2.1.2 and observe what happens if we let P approach the Weierstrass point W on X. Thus, we have to perform a local study around W of the function $\prod_{W'} \|\vartheta\|(gP - W')$ and of the functions $\|F_z\|(P)$ and $W_z(\omega)(P)$ for a suitable local coordinate z. In Section 3.3 we find a suitable local coordinate on an embedding of X into its jacobian. In Section 3.6 we collect the local information that we need in order to complete the proof in Section 3.7. Some preliminary work on this local information is carried out in the Sections 3.4 and 3.5. These two sections form the technical heart of the present chapter.

3.2 Modular discriminant

In this section we introduce the modular discriminant φ_g and its Petersson norm $\|\varphi_g\|$. The modular discriminant generalises the usual discriminant function Δ for elliptic curves.

Let $g \geq 2$ be an integer and let $\mathcal{H}_g$ be the Siegel upper half-space of symmetric complex $g \times g$-matrices with positive definite imaginary part. For $z \in \mathbb{C}^g$ (viewed as a column vector), a matrix $\tau \in \mathcal{H}_g$ and $\eta, \eta' \in \frac{1}{2}\mathbb{Z}^g$ we have the theta function with characteristic $\eta = [\begin{smallmatrix} \eta' \\ \eta'' \end{smallmatrix}]$ given by

$$\vartheta[\eta](z;\tau) := \sum_{n \in \mathbb{Z}^g} \exp(\pi i {}^t(n+\eta')\tau(n+\eta') + 2\pi i {}^t(n+\eta')(z+\eta'')) .$$

We agree that we always choose the entries of η' and η'' to be in the set $\{0, 1/2\}$. For an analytic theta characteristic η, the corresponding theta function $\vartheta[\eta](z;\tau)$ is either odd or even as a function of z. We call the analytic theta characteristic η odd if the corresponding theta function $\vartheta[\eta](z;\tau)$ is odd, and even if the corresponding theta function $\vartheta[\eta](z;\tau)$ is even.

For any subset S of $\{1, 2, \ldots, 2g+1\}$ we define a theta characteristic η_S as in [Mu2], Chapter IIIa: let

$$\begin{aligned} \eta_{2k-1} &= \begin{bmatrix} {}^t(0\,,\ldots,0\,,\frac{1}{2}\,,0\,,\ldots,0) \\ {}^t(\frac{1}{2}\,,\ldots,\frac{1}{2}\,,0\,,0\,,\ldots,0) \end{bmatrix}, \quad 1 \leq k \leq g+1\,, \\ \eta_{2k} &= \begin{bmatrix} {}^t(0\,,\ldots,0\,,\frac{1}{2}\,,0\,,\ldots,0) \\ {}^t(\frac{1}{2}\,,\ldots,\frac{1}{2}\,,\frac{1}{2}\,,0\,,\ldots,0) \end{bmatrix}, \quad 1 \leq k \leq g\,, \end{aligned}$$

where the non-zero entry in the top row occurs in the k-th position. Then we put $\eta_S := \sum_{k \in S} \eta_k$ where the sum is taken modulo 1.

Definition 3.2.1. (*Cf.* [Lo], Section 3.) Let $\mathcal{T}$ be the collection of subsets of $\{1, 2, \ldots, 2g+1\}$ of cardinality $g+1$. Write $U = \{1, 3, \ldots, 2g+1\}$ and let $\circ$ denote the symmetric difference. The modular discriminant φ_g is defined to be the function

$$\varphi_g(\tau) := \prod_{T \in \mathcal{T}} \vartheta[\eta_{T \circ U}](0;\tau)^8$$

on $\mathcal{H}_g$. The function φ_g is a modular form on $\Gamma_g(2) = \{\gamma \in \mathrm{Sp}(2g, \mathbb{Z}) | \gamma \equiv I_{2g} \bmod 2\}$ of weight $4r$ where $r = \binom{2g+1}{g+1}$.

Consider an equation $y^2 = f(x)$ where $f \in \mathbb{C}[X]$ is a monic and separable polynomial of degree $2g+1$. Write $f(x) = \prod_{k=1}^{2g+1}(x - a_k)$ and denote by $D = \prod_{k<l}(a_k - a_l)^2$ the discriminant of f. Let X be the hyperelliptic Riemann surface of genus g defined by $y^2 = f(x)$. Then X carries a basis of holomorphic differentials $\mu_k = x^{k-1}dx/2y$ where $k = 1, \ldots, g$. Further, in [Mu2], Chapter IIIa, §5 it is shown how, given an ordering of the roots of f, one can construct a canonical symplectic basis of the homology of X. Throughout this chapter, we will always work with such a canonical basis of homology, *i.e.*, a certain ordering of the roots of a hyperelliptic equation will always be taken for granted.

Let $(\mu|\mu')$ be the period matrix of the differentials μ_k with respect to a chosen canonical basis of homology, and let $\tau = \mu^{-1}\mu'$.

Proposition 3.2.2. *We have the formula*

$$D^n = \pi^{4gr} (\det \mu)^{-4r} \varphi_g(\tau)$$

relating the discriminant D of the polynomial f to the value $\varphi_g(\tau)$ of the modular discriminant.

Proof. We follow the proof of [Lo], Proposition 3.2. Let S be a subset of $\{1, 2, \ldots, 2g+1\}$ with $\#(S \circ U) = g+1$. Then Thomae's formula (*cf.* [Mu2], Chapter IIIa, §8) holds:

Theorem 3.2.3. *(Thomae's formula) We have*

$$\vartheta[\eta_S](0;\tau)^8 = (\det \mu)^4 \pi^{-4g} \prod_{\substack{k<l \\ k,l \in S \circ U}} (a_k - a_l)^2 \prod_{\substack{k<l \\ k,l \notin S \circ U}} (a_k - a_l)^2\,.$$

If $T \in \mathcal{T}$ then obviously $T \circ U$ is a set S with $\#(S \circ U) = g+1$, and conversely, every such set S can be obtained in this way by taking a $T \in \mathcal{T}$. Taking the product over all $T \in \mathcal{T}$ we obtain by Thomae's formula

$$\varphi_g(\tau) = (\det \mu)^{4r} \pi^{-4gr} \prod_{T \in \mathcal{T}} \left(\prod_{\substack{k<l \\ k,l \in T}} (a_k - a_l)^2 \prod_{\substack{k<l \\ k,l \notin T}} (a_k - a_l)^2 \right).$$

The number of times a term $(a_k - a_l)^2$ appears on the right hand side is easily seen to be n, hence $\varphi_g(\tau) = (\det \mu)^{4r} \pi^{-4gr} \prod_{k<l} (a_k - a_l)^{2n}$ which is what we wanted. $\square$

Definition 3.2.4. Let X be a hyperelliptic Riemann surface of genus $g \geq 2$ and let τ be a period matrix for X formed on a canonical symplectic basis, given by an ordering of the roots of an equation $y^2 = f(x)$ for X. Then we write $\|\varphi_g\|(\tau)$ for the Petersson norm $(\det \mathrm{Im}\tau)^{2r} \cdot |\varphi_g(\tau)|$ of $\varphi_g(\tau)$. This does not depend on the choice of τ and hence it defines an invariant $\|\varphi_g\|(X)$ of X.

It follows from Proposition 3.2.2 that the invariant $\|\varphi_g\|(X)$ is non-zero.

3.3 Local coordinate

For our local computations on our hyperelliptic Riemann surface we need a convenient local coordinate. We find one by embedding the Riemann surface into its jacobian and by taking one of the euclidean coordinates.

Let X be a hyperelliptic Riemann surface of genus $g \geq 2$, let $y^2 = f(x)$ with f monic of degree $2g+1$ be an equation for X, let μ_k be the differential given by $\mu_k = x^{k-1}dx/2y$ for $k = 1, \ldots, g$, and let $(\mu|\mu')$ be their period matrix formed on a canonical basis of homology. Let L be the lattice in $\mathbb{C}^g$ generated by the columns of $(\mu|\mu')$. We have an embedding $\iota : X \hookrightarrow \mathbb{C}^g/L$ given by integration $P \mapsto \int_\infty^P (\mu_1, \ldots, \mu_g)$. We want to express the coordinates $z_1, \ldots, z_g$, restricted to $\iota(X)$, in terms of a local coordinate about $0 = \iota(\infty)$. This is established by the following lemma. In general, we denote by $O(w_1, \ldots, w_s; d)$ a Laurent series in the variables $w_1, \ldots, w_s$ all of whose terms have total degree at least d.

Lemma 3.3.1. *The coordinate z_g is a local coordinate about 0 on $\iota(X)$, and we have*

$$z_k = \frac{1}{2(g-k)+1} z_g^{2(g-k)+1} + O(z_g; 2(g-k)+2)$$

on $\iota(X)$ for $k = 1, \ldots, g$.

Proof. We can choose a local coordinate t about ∞ on X such that $x = t^{-2}$ and $y = -t^{-(2g+1)} + O(t; -2g)$. For $P \in X$ in a small enough neighbourhood of ∞ on X and for a suitable integration path on X we then have

$$\begin{aligned} z_k(P) = \int_\infty^P \frac{x^{k-1}dx}{2y} &= \int_0^{t(P)} \frac{t^{-2(k-1)} \cdot (-2t^{-3}dt)}{-2t^{-(2g+1)} + O(t; -2g)} \\ &= \int_0^{t(P)} \left(t^{2(g-k)} + O(t; 2(g-k)+1) \right) dt \\ &= \frac{1}{2(g-k)+1} t(P)^{2(g-k)+1} + O(t(P); 2(g-k)+2) \,. \end{aligned}$$

By taking $k = g$ we find $z_g = t + O(t; 2)$ and for $k = 1, \ldots, g-1$ then

$$z_k = \frac{1}{2(g-k)+1} z_g^{2(g-k)+1} + O(z_g; 2(g-k)+2) \,,$$

which is what we wanted. $\square$

3.4 Schur polynomials

In this section we assemble some facts on Schur polynomials. We will need these facts at various places in the next sections. Fix a positive integer g. Consider the ring of symmetric polynomials with integer coefficients in the variables $x_1, \ldots, x_g$. Let e_r be the elementary symmetric functions given by the generating function $E(t) = \sum_{r\geq 0} e_r t^r = \prod_{k=1}^g (1 + x_k t)$.

Definition 3.4.1. Let d be a positive integer and let $\pi = \{\pi_1, \ldots, \pi_h\}$ with $\pi_1 \geq \ldots \geq \pi_h$ be a partition of d. The Schur polynomial associated to π is the polynomial

$$S_\pi := \det(e_{\pi'_k - k + l})_{1\leq k,l\leq h} ,$$

where h is the length of the partition π, and where π' is the conjugate partition of π given by $\pi'_k = \#\{l : \pi_l \geq k\}$, *i.e.*, the partition obtained by switching the associated Young diagram around its diagonal. The polynomial S_π is symmetric and has total degree d. We denote by S_g the Schur polynomial in g variables associated to the partition $\pi = \{g, g-1, \ldots, 2, 1\}$. Thus, the formula

$$S_g = \det\left(e_{g-2k+l+1}\right)_{1\leq k,l\leq g}$$

holds, and the polynomial S_g has total degree $g(g+1)/2$.

Let p_r be the elementary Newton functions (power sums) given by the generating function $P(t) = \sum_{r\geq 1} p_r t^{r-1} = \sum_{k\geq 1} x_k/(1 - x_k t)$. The following proposition is then a special case of Theorem 4.1 of [BEL2].

Proposition 3.4.2. *The Schur polynomial S_g can be expressed as a polynomial in the g functions $p_1, p_3, \ldots, p_{2g-1}$ only. This polynomial is unique.*

Definition 3.4.3. We define s_g to be the unique polynomial in g variables given by the above proposition.

The next proposition is a special case of Theorem 6.2 of [BEL2].

Proposition 3.4.4. *Let $s(x_1, \ldots, x_g) \in \mathbb{C}[x_1, \ldots, x_g]$ be a polynomial in g variables such that for any set of g complex numbers $w_1, \ldots, w_g$, the polynomial $s(z_1 - w, z_2 - w^3, \ldots, z_g - w^{2g-1})$ in w either has exactly g roots $w_1, \ldots, w_g$, or vanishes identically, if we give z the value $z = (p_1(w_1, \ldots, w_g), p_3(w_1, \ldots, w_g), \ldots, p_{2g-1}(w_1, \ldots, w_g))$. Then s is equal to the polynomial s_g up to a constant factor.*

Definition 3.4.5. We define σ_g to be the polynomial in g variables given by the equation

$$\sigma_g(z_1, \ldots, z_g) = s_g(z_g, 3z_{g-1}, \ldots, (2g-1)z_1) .$$

The following proposition is then the result of a simple calculation.

Proposition 3.4.6. *Up to a sign, the homogeneous part of least total degree of σ_g is equal to the Hankel determinant*

$$H(z) = \det \begin{pmatrix} z_1 & z_2 & \cdots & z_{(g+1)/2} \\ z_2 & z_3 & \cdots & z_{(g+3)/2} \\ \vdots & \vdots & \ddots & \vdots \\ z_{(g+1)/2} & z_{(g+3)/2} & \cdots & z_g \end{pmatrix}$$

if g is odd, or

$$H(z) = \det \begin{pmatrix} z_1 & z_2 & \cdots & z_{g/2} \\ z_2 & z_3 & \cdots & z_{(g+2)/2} \\ \vdots & \vdots & \ddots & \vdots \\ z_{g/2} & z_{(g+2)/2} & \cdots & z_{g-1} \end{pmatrix}$$

if g is even.

We conclude with some more general facts. These can all be found for example in Appendix A to [Fu].

Proposition 3.4.7. *Let $\pi = \{\pi_1, \ldots, \pi_h\}$ with $\pi_1 \geq \ldots \geq \pi_h$ be a partition. Then the formula*

$$S_\pi(1, \ldots, 1) = \prod_{k<l} \frac{\pi_k - \pi_l + l - k}{l - k}$$

holds. In particular, $S_g(1, \ldots, 1) = 2^{g(g-1)/2}$.

Definition 3.4.8. Let $\mathbf{i} = (i_1, \ldots, i_d)$ be a d-tuple of non-negative integers. The $\mathbf{i}$-th generalised Newton function $p^{(\mathbf{i})}$ is defined to be the polynomial

$$p^{(\mathbf{i})} := p_1^{i_1} \cdot p_2^{i_2} \cdot \ldots \cdot p_d^{i_d} ,$$

where the p_r are the elementary Newton functions.

Proposition 3.4.9. *The set of generalised Newton functions $p^{(\mathbf{i})}$, where $\mathbf{i}$ runs through the d-tuples $\mathbf{i} = (i_1, \ldots, i_d)$ of non-negative integers with $\sum \alpha i_\alpha = d$, forms a basis of the $\mathbb{Q}$-vector space of symmetric polynomials of total degree d with rational coefficients.*

Proposition 3.4.10. *For a partition π of d and a d-tuple $\mathbf{i} = (i_1, \ldots, i_d)$, denote by $\omega_\pi(\mathbf{i})$ the coefficient of the monomial $x_1^{\pi_1} \cdot \ldots \cdot x_d^{\pi_d}$ in $p^{(\mathbf{i})}$. Then the polynomial S_π can be expanded on the basis $\{p^{(\mathbf{i})}\}$ of generalised Newton functions of total degree d as $S_\pi = \sum_{\mathbf{i}} \frac{1}{z(\mathbf{i})} \cdot \omega_\pi(\mathbf{i}) \cdot p^{(\mathbf{i})}$. Here $z(\mathbf{i}) = i_1! 1^{i_1} \cdot i_2! 2^{i_2} \cdot \ldots \cdot i_d! d^{i_d}$.*

3.5 Sigma function

We consider again hyperelliptic Riemann surfaces of genus $g \geq 2$, defined by equations $y^2 = f(x)$ with f monic and separable of degree $2g+1$. We write $f(x) = x^{2g+1} + \lambda_1 x^{2g} + \cdots + \lambda_{2g} x + \lambda_{2g+1}$ and denote by λ the vector of coefficients $(\lambda_1, \ldots, \lambda_{2g+1})$. In this section we study the sigma function $\sigma(z; \lambda)$ with argument $z \in \mathbb{C}^g$ and parameter λ. This is a modified theta function, studied extensively in the nineteenth century. Klein observed that the sigma function serves very well to study the function theory of hyperelliptic Riemann surfaces. For us it will be a convenient technical tool for obtaining the local expansions that we need. We will give the definition of the sigma function, as well as its power series expansion in z, λ. For more details we refer to the *Enzyklopädie der mathematischen Wissenschaften*, Band II, Teil 2, Kapitel 7.XII. A modern reference is [BEL1], where one also finds applications of the sigma function in the theory of the Korteweg-de Vries differential equation.

As before, let μ_k be the holomorphic differential given by $\mu_k = x^{k-1} dx/2y$ for $k = 1, \ldots, g$, and let $(\mu|\mu')$ be their period matrix formed on a canonical basis of homology. Let L be the lattice in $\mathbb{C}^g$ generated by the columns of $(\mu|\mu')$. By the theorem of Abel-Jacobi we have a bijective map $\mathrm{Pic}_{g-1}(X) \xrightarrow{\sim} \mathbb{C}^g/L$ given by $\sum_k m_k P_k \longmapsto \sum_k m_k \int_\infty^{P_k} (\mu_1, \ldots, \mu_g)$. Denote by Θ the image of the theta divisor of classes of effective divisors of degree $g-1$, and let $q : \mathbb{C}^g \to \mathbb{C}^g/L$ be the projection map. Let $\tau = \mu^{-1}\mu'$. By Theorem 1.4.2, there exists a unique theta-characteristic δ such that $\vartheta[\delta](z; \tau)$ vanishes to order one precisely along $q^{-1}(\Theta)$. The characteristic δ is odd if $g \equiv 1$ or 2 mod 4, and even if $g \equiv 0$ or 3 mod 4.

Definition 3.5.1. Let ν be the matrix of A-periods of the differentials of the second kind $\nu_k = \frac{1}{4y} \sum_{l=k}^{2g-k} (l+1-k) \lambda_{l+k+1} x^k dx$ for $k = 1, \ldots, g$. These differentials have a second order pole at ∞ and no other poles. The sigma function is then the function

$$\sigma(z; \lambda) := \exp(-\frac{1}{2} z \nu \mu^{-1\,t} z) \cdot \vartheta[\delta](\mu^{-1} z; \tau) .$$

Using some of the facts on Schur polynomials from the previous section, we can give the power series expansion of $\sigma(z;\lambda)$. The result is probably well-known to specialists, but we couldn't find an explicit reference in the literature. For the special case $g=2$, a somewhat stronger version of the formula from the proposition below has been obtained by Grant, see [Gr], Theorem 2.11.

Proposition 3.5.2. *The power series expansion of $\sigma(z;\lambda)$ about $z=0$ is of the form*

$$\sigma(z;\lambda) = \gamma \cdot \sigma_g(z) + O(\lambda)\,,$$

where σ_g is the polynomial given by Definition 3.4.5 and where the symbol $O(\lambda)$ denotes a power series in z,λ in which each term contains a λ_k raised to a positive integral power. The constant γ satisfies the formula

$$\gamma^{8n} = \pi^{4g(r-n)}(\det\mu)^{-4(r-n)}\varphi_g(\tau)\,.$$

If we assign the variable z_k a weight $2(g-k)+1$, and the variable λ_k a weight $-2k$, then the power series expansion in z,λ of $\sigma(z;\lambda)$ is homogeneous of weight $g(g+1)/2$.

Proof. First of all, the homogeneity of the power series expansion in z,λ with respect to the assigned weights follows from an explicit formula for $\sigma(z;\lambda)$ given in [BEL3]. This homogeneity is also mentioned there, *cf.* the concluding remarks after Corollary 1. Write $\sigma(z;\lambda)=\sigma_0(z)+O(\lambda)$ where $O(\lambda)$ denotes a power series in z,λ in which each term contains a λ_k raised to a positive integral power. Because of the homogeneity, the series $\sigma_0(z)$ is necessarily a polynomial in the variables $z_1,\ldots,z_g$. By the Riemann vanishing theorem, there is a dense open subset $U\subset\mathbb{C}^{2g+1}$ such that for any $\lambda\in U$, the function $\sigma(z;\lambda)$ satisfies the following property: for any set of g points $P_1,\ldots,P_g$ on the hyperelliptic Riemann surface $X=X_\lambda$ corresponding to λ, the function $\sigma(z-\int_\infty^P(\mu_1,\ldots,\mu_g);\lambda)$ in P on X either has exactly g roots $P_1,\ldots,P_g$, or vanishes identically, when we give the argument z the value $z=\sum_k\int_\infty^{P_k}(\mu_1,\ldots,\mu_g)$. In the limit $\lambda\to 0$ we find then, as in the proof of Lemma 3.3.1, that for any set of g complex numbers $w_1,\ldots,w_g$ the polynomial

$$\sigma_0\left(\frac{1}{2g-1}\left(z_g-w^{2g-1}\right),\frac{1}{2g-3}\left(z_{g-1}-w^{2g-3}\right),\ldots,\frac{1}{3}\left(z_2-w^3\right),z_1-w\right)$$

in w either has exactly g roots $w_1,\ldots,w_g$, or vanishes identically, for the value

$$z=(p_1(w_1,\ldots,w_g),p_3(w_1,\ldots,w_g),\ldots,p_{2g-1}(w_1,\ldots,w_g))\,.$$

By Proposition 3.4.4, the polynomial σ_0 must be equal to the polynomial σ_g up to a constant factor γ. As to this constant γ, we find in [Ba], Section IX a calculation of a constant γ' such that $\sigma(z;\lambda)=\gamma'\cdot H(z)+O(z;\lfloor(g+3)/2\rfloor)$, where $H(z)$ is the Hankel determinant from Proposition 3.4.6 and where now we consider the power series expansion only with respect to the variables $z_1,\ldots,z_g$ and with respect to their usual weight $\deg(z_k)=1$. By Proposition 3.4.6, this γ' is equal to our constant γ, up to a sign. We just quote the result of Baker's computation:

$$\gamma^4=\vartheta(0;\tau)^4\cdot\prod_{\substack{k<l\\k,l\in U}}(a_k-a_l)^2/(\ell_1\ell_3\cdots\ell_{2g+1})\,,\text{ where }\ell_r:=-i\cdot\prod_{\substack{k\in U\\k\neq r}}(a_k-a_r)/\prod_{k\notin U}(a_k-a_r)\,.$$

By Thomae's formula Theorem 3.2.3 we have

$$\vartheta(0;\tau)^8=(\det\mu)^4\pi^{-4g}\prod_{\substack{k<l\\k,l\in U}}(a_k-a_l)^2\prod_{\substack{k<l\\k,l\notin U}}(a_k-a_l)^2\,.$$

Combining, we obtain $\gamma^8=D\cdot\pi^{-4g}\cdot(\det\mu)^4$. The formula for γ that we gave then follows from Proposition 3.2.2. $\square$

Example 3.5.3. By way of illustration, we have computed σ_g for small g:

g	σ_g
1	z_1
2	$-z_1 + \frac{1}{3} z_2^3$
3	$z_1 z_3 - z_2^2 - \frac{1}{3} z_2 z_3^3 + \frac{1}{45} z_3^6$
4	$z_1 z_3 - z_2^2 - z_3^2 z_4 + z_2 z_3 z_4^2 - \frac{1}{3} z_1 z_4^3 + \frac{1}{15} z_2 z_4^5 - \frac{1}{105} z_3 z_4^7 + \frac{1}{4725} z_4^{10}$

Remark 3.5.4. As can be seen from Proposition 3.4.6, the homogeneous part of least total degree (with respect to the usual weight $\deg(z_k) = 1$) of $\sigma_g(z)$ has degree $\lfloor (g+1)/2 \rfloor$. Hence, by a fundamental theorem of Riemann, the theta-characteristic δ gives rise to a linear system of dimension $\lfloor (g-1)/2 \rfloor$ on X.

3.6 Leading coefficients

In this section we calculate the leading coefficients of the power series expansions in z_g of the holomorphic functions $\vartheta[\delta](g\mu^{-1}z;\tau)|_{\iota(X)}$ and $W_{z_g}(\mu)$, the Wronskian in z_g of the basis $\{\mu_1, \ldots, \mu_g\}$.

Proposition 3.6.1. *The leading coefficient of the power series expansion of $\sigma(gz;\lambda)|_{\iota(X)}$, and hence of $\vartheta[\delta](g\mu^{-1}z;\tau)|_{\iota(X)}$, is equal to $\gamma \cdot 2^{g(g-1)/2}$, where γ is the constant from Proposition 3.5.2.*

Proof. By Lemma 3.3.1 and Proposition 3.5.2, the power series expansion of $\sigma(gz;\lambda)|_{\iota(X)}$ has the form

$$\sigma(gz;\lambda)|_{\iota(X)} = \gamma \cdot \sigma_g\left(\frac{g}{2g-1} z_g^{2g-1}, \frac{g}{2g-3} z_g^{2g-3}, \ldots, \frac{g}{3} z_g^3, g z_g\right) + O(z_g; g(g+1)/2+1)\,.$$

Hence we need to calculate $\sigma_g\left(\frac{g}{2g-1}, \frac{g}{2g-3}, \ldots, \frac{g}{3}, g\right)$. By Definition 3.4.5 this is $s_g(g,g,\ldots,g)$. But by Proposition 3.4.2 and Definition 3.4.3 we have $s_g(g,g,\ldots,g) = S_g(1,1,\ldots,1)$, and by Proposition 3.4.7 we have $S_g(1,\ldots,1) = 2^{g(g-1)/2}$. The proposition follows. $\square$

Proposition 3.6.2. *The leading coefficient of the power series expansion of the Wronskian $W_{z_g}(\mu)$ is equal to $\pm 2^{g(g-1)/2}$.*

Proof. Expanding the Wronskian yields

$$W_{z_g}(\mu) = \det\left(\frac{1}{(k-1)!}\frac{d^k z_l}{dz_g^l}\right)_{1\le k,l\le g} =$$
$$= \begin{pmatrix} z_g^{2g-2} & z_g^{2g-4} & \cdots & z_g^2 & 1 \\ (2g-2)z_g^{2g-3} & (2g-4)z_g^{2g-5} & \cdots & 2z_g & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{2g-2}{g-1} z_g^g & \binom{2g-4}{g-1} z_g^{g-2} & \cdots & 0 & 0 \end{pmatrix} + O(z_g; g(g-1)/2+1)\,.$$

Let A be the matrix of binomial coefficients $A := \left(\binom{2g-2k}{g-l}\right)_{1\le k,l\le g-1}$. From the expansion of the Wronskian it follows that, up to a sign, the required leading coefficient is equal to $\det A$. We will compute this number. First of all note that

$$\det A = \frac{(2g-2)!(2g-4)!\cdots 2!}{(g-1)!(g-2)!\cdots 1!} \det\left(\frac{1}{(g-2k+l)!}\right)_{1\le k,l\le g-1},$$

where we define $1/n! := 0$ for $n < 0$. Now let $d = g(g-1)/2$ and consider the ring of symmetric polynomials with integer coefficients in $g-1$ variables. It is well known that for the elementary symmetric functions e_r we have an expansion

$$e_r = \frac{1}{r!} \det \begin{pmatrix} p_1 & 1 & 0 & \cdots & 0 \\ p_2 & p_1 & 2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{r-1} & p_{r-2} & p_{r-3} & \cdots & r-1 \\ p_r & p_{r-1} & p_{r-2} & \cdots & p_1 \end{pmatrix},$$

with p_r the elementary Newton functions. From Definition 3.4.1 and this expansion it follows that $\det(1/(g-2k+l)!)$ is the coefficient of p_1^d in the expansion of S_{g-1} with respect to the basis of generalised Newton functions. By Proposition 3.4.10, this coefficient is equal to $\omega_{g-1}(d)/d!$, where $\omega_{g-1}(d)$ is the coefficient of $x_1^{g-1}x_2^{g-2}\cdots x_{g-1}^2 x_g$ in p_1^d. Writing this out, it immediately follows that $\det(1/(g-2k+l)!) = 1/(g-1)!(g-2)!\cdots 1!$. Combining one finds $\det A = 2^{g(g-1)/2}$. $\square$

3.7 Proof of Theorem 3.1.2

Now we are ready to prove Theorem 3.1.2. Let X be a hyperelliptic Riemann surface of genus $g \geq 2$, and let W be one of its Weierstrass points.

Proof of Theorem 3.1.2. Fix a hyperelliptic equation $y^2 = f(x)$ for X with f monic and separable of degree $2g+1$ that puts W at infinity. Choose a canonical basis of the homology of X, and form the period matrix $(\mu|\mu')$ of the differentials $x^{k-1}dx/2y$ for $k = 1, \ldots, g$ on this basis. Let L be the lattice in $\mathbb{C}^g$ generated by the columns of $(\mu|\mu')$, and embed X into $\mathbb{C}^g/L$ with base point W as in Section 3.3. We have the standard euclidean coordinates $z_1, \ldots, z_g$ on $\mathbb{C}^g/L$ and according to Lemma 3.3.1 we have that z_g is a local coordinate about W on X. The weight w of W is given by $w = g(g-1)/2$, *cf.* Remark 2.2.9. Consider then the following quantities:

$$\begin{aligned} A(W') &= \lim_{Q \to W} \frac{\|\vartheta\|(gQ - W')}{|z_g|^g} \quad \text{for Weierstrass points} \quad W' \neq W\,; \\ A(W) &= \lim_{Q \to W} \frac{\|\vartheta\|(gQ - W)}{|z_g|^{w+g}} = \lim_{Q \to W} \frac{\|F_{z_g}\|(Q)}{|z_g|^w}\,; \\ B(W) &= \lim_{Q \to W} \frac{|W_{z_g}(\omega)(Q)|}{|z_g|^w}\,, \end{aligned}$$

where $W_{z_g}(\omega)$ is the Wronskian in z_g of an orthonormal basis $\{\omega_1, \ldots, \omega_g\}$ of $H^0(X, \Omega_X^1)$. We have by Theorem 2.1.2

$$G'(W, W')^g = \frac{A(W')}{\prod_{W''} A(W'')^{w/g^3}} \quad \text{for Weierstrass points} \quad W' \neq W\,,$$

hence

$$\prod_{W' \neq W} G'(W, W')^g = \frac{1}{A(W)} \cdot \left(\prod_{W'} A(W') \right)^{\frac{g+1}{2g^2}}.$$

Further we have by Proposition 2.2.7, letting P approach W,

$$T(X) = A(W)^{-(g+1)} \cdot \left(\prod_{W'} A(W') \right)^{\frac{w(g-1)}{g^3}} \cdot B(W)^2\,.$$

Eliminating the factor $\prod_{W'} A(W')$ yields

$$\prod_{W' \neq W} G'(W,W')^{(g-1)^2} = A(W)^4 \cdot B(W)^{-\frac{2g+2}{g}} \cdot T(X)^{\frac{g+1}{g}} \, .$$

Now we use the results obtained in Section 3.6. Let $\tau = \mu^{-1}\mu'$. A simple calculation gives that $A(W)$ is $(\det \mathrm{Im}\tau)^{1/4}$ times the absolute value of the leading coefficient of the power series expansion of $\vartheta[\delta](g\mu^{-1}z;\tau)|_{\iota(X)}$ in z_g. Hence by Propositions 3.5.2 and 3.6.1 we have

$$A(W) = 2^{g(g-1)/2} \pi^{g\frac{r-n}{2n}} \cdot (\det \mathrm{Im}\tau)^{1/4} \cdot |\det \mu|^{-\frac{r-n}{2n}} \cdot |\varphi_g(\tau)|^{\frac{1}{8n}} \, .$$

Further we have by Proposition 1.4.1 that $\|\mu_1 \wedge \ldots \wedge \mu_g\|^2 = (\det \mathrm{Im}\tau) \cdot |\det \mu|^2$. This gives that $|W_{z_g}(\omega)| = |W_{z_g}(\mu)| \cdot (\det \mathrm{Im}\tau)^{-1/2} \cdot |\det \mu|^{-1}$. From Proposition 3.6.2 we derive then

$$B(W) = 2^{g(g-1)/2} \cdot (\det \mathrm{Im}\tau)^{-1/2} \cdot |\det \mu|^{-1} \, .$$

Plugging in our results for $A(W)$ and $B(W)$ finally gives the theorem. □

Remark 3.7.1. The fact that the product from Theorem 3.1.2 is independent of the choice of the Weierstrass point W follows *a fortiori* from the computations in the above proof. It would be interesting to have an *a priori* reason for this independence.

3.8 The case $g = 2$

We can say a little bit more if we specialise to the case of a Riemann surface X of genus $g = 2$. Note that such a Riemann surface is always hyperelliptic, and that it has 6 Weierstrass points, each of weight 1. The Arakelov theory of Riemann surfaces of genus 2 has been studied in quite some detail before, see especially the papers [Bo] and [BMM]. It will be convenient to work with the function

$$\varphi_2'(\tau) := \prod_{\eta \text{ even}} \vartheta[\eta](0;\tau)^2$$

on $\mathcal{H}_2$. This is a modular form on the full symplectic group $\mathrm{Sp}(4,\mathbb{Z})$ of weight 10. It relates to our φ_2 by the formula $\varphi_2 = (\varphi_2')^4$. If $\tau \in \mathcal{H}_2$ is associated to a Riemann surface X of genus 2 then we write $\|\varphi_2'\|(X) = (\det \mathrm{Im}\tau)^5 |\varphi_2'(\tau)|$. This definition is independent of the choice of τ. Also we will work with the modified $\|\Delta_2'\|(X) = 2^{-12}\|\varphi_2'\|(X)$. We remark that this $\|\Delta_2'\|$ is the $\|\Delta_2\|$ from the papers [Bo] and [BMM]. Our aim in this section is to prove the following two theorems. Recall the function $\|J\|$ from Definition 1.4.11 above.

Theorem 3.8.1. *Let W, W' be two Weierstrass points of X. Then the formula*

$$G'(W,W')^2 = 2^{1/4}\pi^{-2} \cdot \|\varphi_2'\|(X)^{-3/16} \cdot \|J\|(W,W')$$

holds.

Define the invariant $\|H\|(X)$ by

$$\log \|H\|(X) = \frac{1}{2} \int_{\mathrm{Pic}_1(X)} \log \|\vartheta\| \cdot \nu^2 \, .$$

This invariant has been introduced by Bost in [Bo].

Theorem 3.8.2. *The formula*

$$S(X) = \|\Delta_2'\|(X)^{-1/4} \cdot \|H\|(X)^4$$

holds.

From Theorems 3.8.1 and 3.8.2 we obtain the following corollary.

Corollary 3.8.3. *(i) Let W, W' be two Weierstrass points of X. Then the formula*

$$G(W, W')^2 = (2\pi)^{-2} \cdot \|\Delta_2'\|(X)^{-1/4} \cdot \|H\|(X) \cdot \|J\|(W, W')$$

holds.
(ii) For Faltings' delta-invariant $\delta(X)$ of X, the formula

$$\delta(X) = -16 \log(2\pi) - \log \|\Delta_2'\|(X) - 4 \log \|H\|(X)$$

holds.

Proof. The first statement is a consequence of Theorems 3.8.1 and 3.8.2 and the definition of G'. The second statement follows from Theorem 3.1.4, Theorem 3.8.2 and Theorem 2.1.3 relating the delta-invariant to the invariants $S(X)$ and $T(X)$. $\square$

The above corollary is also obtained by Bost in [Bo], Proposition 4, and is more or less proved in the appendix to [BMM]. Our approach is slightly different; in particular we think that in our approach the appearance of the function $\|J\|$ is more natural (see especially the proof of Lemma 3.8.6 below).

It would be interesting to have explicit formulas for $S(X)$ in higher genera. For this we probably need a generalisation of Lemma 3.8.8 below, but this seems difficult. A possible approach is suggested in [JK1].

We will frequently make use of Rosenhain's identity, *cf.* also Theorem 4.5.1 below.

Theorem 3.8.4. *(Rosenhain [Ro]) Let W, W' be two Weierstrass points of X. Then the formula*

$$\|J\|(W, W') = \pi^2 \cdot \prod_{W'' \neq W, W'} \|\vartheta\|(W - W' + W'')$$

holds.

Corollary 3.8.5. *(i) Let W, W' be two Weierstrass points of X. Then the formula*

$$G(W, W')^2 = 2^{-2} \cdot \|\Delta_2'\|(X)^{-1/4} \cdot \|H\|(X) \cdot \prod_{W'' \neq W, W'} \|\vartheta\|(W - W' + W'')$$

holds.
(ii) Suppose $y^2 = f(x)$ is a hyperelliptic equation for X with f monic of degree 5. Choose an ordering of its roots, and consider the canonical symplectic basis of homology that corresponds to this ordering. Let $(\mu|\mu')$ be the period matrix of the differentials $dx/2y, xdx/2y$ on this basis, and let $\tau = \mu^{-1}\mu'$. Let $W = (\alpha_1, 0)$ and let $W' = (\alpha_2, 0)$. Then the formula

$$G(W, W')^2 = \frac{(2\pi) \cdot |\alpha_1 - \alpha_2| \cdot \|H\|(X)}{|f'(\alpha_1) f'(\alpha_2)|^{1/4} (\det \mathrm{Im}\tau)^{1/4} |\det \mu|^{1/2}}$$

holds.

Proof. We obtain (i) from Corollary 3.8.3 by Rosenhain's formula. The formula in (ii) follows then from the first by an application of Thomae's formula, Theorem 3.2.3. $\square$

For the proof of Theorem 3.8.1 we need the following lemma. It is a specialisation to the case $g = 2$ of some of the results from Section 3.6. Choose a hyperelliptic equation $y^2 = f(x)$ for X with f monic and separable of degree 5. Choose an ordering of its roots, and consider the canonical symplectic basis of homology that corresponds to this ordering. Let $(\mu|\mu')$ be the period matrix of the differentials $dx/2y, xdx/2y$ on this basis, and let $\tau = \mu^{-1}\mu'$. Let L be the lattice in $\mathbb{C}^2$ generated by the columns of $(\mu|\mu')$, and make an embedding $\iota : X \hookrightarrow \mathbb{C}^2/L$ as in Section 3.5, taking the point at infinity as a base point. Let $z = (z_1, z_2)$ be the standard euclidean coordinates on $\mathbb{C}^2/L$. Let δ be the odd analytic theta characteristic such that $\vartheta[\delta](\mu^{-1}z;\tau)$ vanishes identically on $\iota(X)$. Let γ be the constant from Proposition 3.5.2.

Lemma 3.8.6. *We have*

$$\vartheta[\delta](2\mu^{-1}z;\tau)|_{\iota(X)} = 2\gamma z_2^3 + O(z_2;5)\,.$$

Further, for odd δ' different from δ let $J(\delta,\delta')(\tau)$ be the Jacobian

$$J(\delta,\delta')(\tau) = (\partial(\vartheta[\delta],\vartheta[\delta'])/\partial(z_1,z_2))(0;\tau)\,.$$

Then the expansion

$$\vartheta[\delta'](2\mu^{-1}z;\tau)|_{\iota(X)} = -2\gamma^{-1}J(\delta,\delta')(\tau)\cdot(\det\mu)^{-1}z_2 + O(z_2;3)$$

holds.

Proof. The first expansion follows directly from Propositions 3.5.2 and 3.6.1. As to the second, observe that

$$\vartheta[\delta'](2\mu^{-1}z;\tau) = 2\frac{\partial\vartheta[\delta'](\mu^{-1}z;\tau)}{\partial z_1}|_{z=0}\cdot z_1 + 2\frac{\partial\vartheta[\delta'](\mu^{-1}z;\tau)}{\partial z_2}|_{z=0}\cdot z_2 + O(z_1,z_2;3)$$

locally about 0. When restricted to $\iota(X)$, we know by Lemma 3.3.1 that z_2 becomes a local coordinate about 0 and that $z_1 = \frac{1}{3}z_2^3 + O(z_2;4)$ locally about 0. Thus when expanded with respect to the coordinate z_2 we get

$$\vartheta[\delta'](2\mu^{-1}z;\tau)|_{\iota(X)} = 2\frac{\partial\vartheta[\delta'](\mu^{-1}z;\tau)}{\partial z_2}|_{z=0}\cdot z_2 + O(z_2;3)$$

about 0. It remains to compute the constant $\frac{\partial\vartheta[\delta'](\mu^{-1}z;\tau)}{\partial z_2}|_{z=0}$. From Proposition 3.5.2 and the table accompanying this proposition we get that $\frac{\partial\vartheta[\delta](\mu^{-1}z;\tau)}{\partial z_2}|_{z=0} = 0$, and that $\frac{\partial\vartheta[\delta](\mu^{-1}z;\tau)}{\partial z_1}|_{z=0} = -\gamma$. This gives

$$-\gamma\cdot\frac{\partial\vartheta[\delta'](\mu^{-1}z;\tau)}{\partial z_2}|_{z=0} = \det\begin{pmatrix} \frac{\partial\vartheta[\delta](\mu^{-1}z;\tau)}{\partial z_1}|_{z=0} & \frac{\partial\vartheta[\delta'](\mu^{-1}z;\tau)}{\partial z_1}|_{z=0} \\ \frac{\partial\vartheta[\delta](\mu^{-1}z;\tau)}{\partial z_2}|_{z=0} & \frac{\partial\vartheta[\delta'](\mu^{-1}z;\tau)}{\partial z_2}|_{z=0} \end{pmatrix}.$$

But on the other hand we have

$$\det\begin{pmatrix} \frac{\partial\vartheta[\delta](\mu^{-1}z;\tau)}{\partial z_1}|_{z=0} & \frac{\partial\vartheta[\delta'](\mu^{-1}z;\tau)}{\partial z_1}|_{z=0} \\ \frac{\partial\vartheta[\delta](\mu^{-1}z;\tau)}{\partial z_2}|_{z=0} & \frac{\partial\vartheta[\delta'](\mu^{-1}z;\tau)}{\partial z_2}|_{z=0} \end{pmatrix} = (\det\mu)^{-1}\cdot J(\delta,\delta')(\tau)\,.$$

Together this gives the required constant. $\square$

Proof of Theorem 3.8.1. As in the proof of Theorem 3.1.2, we fix a hyperelliptic equation $y^2 = f(x)$

for X with f monic of degree 5 that puts W at infinity. We choose a canonical basis of the homology of X, and form the period matrix $(\mu|\mu')$ of the differentials $dx/2y, xdx/2y$ on this basis. Let $\tau = \mu^{-1}\mu'$ and let κ be the Riemann vector from Theorem 1.4.2 corresponding to infinity. The Abel-Jacobi map $t_\kappa \cdot u : \mathrm{Pic}_1(X) \xrightarrow{\sim} \mathbb{C}^2/\mathbb{Z}^2 + \tau\mathbb{Z}^2$ from Theorem 1.4.2 induces an identification of the set of Weierstrass points of X with the set of odd analytic theta characteristics in dimension 2, a Weierstrass point P corresponding to the characteristic $\eta = [{\eta' \atop \eta''}]$ such that $(t_\kappa \cdot u)(P) = [\eta' + \tau \cdot \eta'']$. In particular, the Weierstrass point W corresponds to the characteristic δ. Let δ' be the analytic theta characteristic corresponding to W', and for a general Weierstrass point W'', denote by δ'' the corresponding analytic theta characteristic. From the definition of G' and Theorem 2.1.2 it follows that

$$G'(W,W')^2 = \lim_{P \to W} \frac{\|\vartheta\|(2P - W')}{\left(\prod_{W''} \|\vartheta\|(2P - W'')\right)^{1/8}}\,.$$

We compute the right hand side with Lemma 3.8.6; we find that it is equal to

$$\frac{2|\gamma|^{-1}(\det \mathrm{Im}\tau)^{1/4} \cdot |J(\delta,\delta')(\tau)| \cdot |\det \mu|^{-1}}{\left(2|\gamma| \prod_{\delta'' \neq \delta} \left(2|\gamma|^{-1}(\det \mathrm{Im}\tau)^{1/4} \cdot |J(\delta,\delta'')(\tau)| \cdot |\det \mu|^{-1}\right)\right)^{1/8}}\,,$$

where γ is the constant from Proposition 3.5.2. Using the formula for γ from Proposition 3.5.2 we can rewrite this as

$$2^{1/4}\pi^{-3/4} \cdot \|\varphi_2'\|(X)^{-1/16} \left(\prod_{W'' \neq W} \|J\|(W,W'') \right)^{-1/8} \cdot \|J\|(W,W')\,.$$

Rosenhain's formula Theorem 3.8.4 gives that $\prod_{W'' \neq W} \|J\|(W,W'') = \pi^{10}\|\varphi_2'\|(X)$. Plugging this in finally gives the theorem. $\square$

We next proceed to the proof of Theorem 3.8.2. We will make use of the fact, special to the case $g = 2$, that the theta divisor in the jacobian of X can be identified with X itself. We need two lemmas.

Lemma 3.8.7. *Let W, W', W'' be distinct Weierstrass points on X. Then*

$$\lim_{P \to W} \frac{\|\vartheta\|(P - W + W')}{\|\vartheta\|(2\sigma(P) - W'')} = \frac{\|J\|(W,W')}{2\|J\|(W,W'')}\,,$$

where σ is the hyperelliptic involution of X.

Proof. This follows from the second expansion in Lemma 3.8.6. $\square$

The next lemma is Proposition 14 in [BMM]. The proof is by no means trivial, and seems difficult to generalise to higher genera.

Lemma 3.8.8. *Let W, W' be two distinct Weierstrass points of X. Then the equality*

$$\sum_{W'' \neq W, W'} \log \|\vartheta\|(W - W' + W'') - \int_{\Theta + W - W'} \log \|\vartheta\| \cdot \nu = 2\log 2 + 2\log \|H\|(X)$$

holds, the sum running over the Weierstrass points different from W and W'.

Proof of Theorem 3.8.2. Let R, R' be two points on X and let W'' be a Weierstrass point of X. We apply Green's formula Lemma 1.1.6 to the functions $f_1(P) = \|\vartheta\|(R - R' + P) = (\|\vartheta\| \cdot \phi_{R-R'})(P)$ and $f_2(P) = \|\vartheta\|(2P - W'') = (\|\vartheta\| \cdot \phi_{-W''})(P)$. Here for a divisor D on X we use the notation

φ_D introduced in Proposition 1.4.5. The divisor of f_1 on X is $R' + \sigma(R)$, and the divisor of f_2 on X is $\mathcal{W} + 2W''$, where $\mathcal{W}$ is the divisor of Weierstrass points on X. By Proposition 1.4.5 we have $\frac{1}{2\pi i}\partial\overline{\partial}\log f_1^2 = 2\mu$ and $\frac{1}{2\pi i}\partial\overline{\partial}\log f_2^2 = 8\mu$ outside the zeroes of f_1 and f_2, respectively. Green's formula gives

$$
\begin{aligned}
-8\int_X \log\|\vartheta\|(R - R' + P)\cdot\mu(P) + 2\int_X \log\|\vartheta\|(2P - W'')\cdot\mu(P) \\
= \log\|\vartheta\|(2R' - W'') + \log\|\vartheta\|(2\sigma(R) - W'') - \sum_{W\in\mathcal{W}} \log\|\vartheta\|(R - R' + W) \\
- 2\log\|\vartheta\|(R - R' + W''),
\end{aligned}
$$

in other words,

$$
\begin{aligned}
4\int_{\Theta+R-R'} \log\|\vartheta\|\cdot\nu + 2\log S(X) \\
= -\log\|\vartheta\|(2R' - W'') - \log\|\vartheta\|(2\sigma(R) - W'') + \sum_{W\in\mathcal{W}} \log\|\vartheta\|(R - R' + W) \\
+ 2\log\|\vartheta\|(R - R' + W''),
\end{aligned}
$$

where ν is the canonical translation invariant (1,1)-form on $\mathrm{Pic}_1(X)$ introduced in Section 1.4. We have used that Θ can be identified with X and that ν restricts to 2μ on Θ. Now fix two distinct Weierstrass points W, W'. Summing the above equation over the 4 Weierstrass points $W'' \neq W, W'$ we obtain

$$
\begin{aligned}
16\int_{\Theta+R-R'} \log\|\vartheta\|\cdot\nu + 8\log S(X) \\
= -\sum_{W''\neq W,W'} \log\|\vartheta\|(2R' - W'') - \sum_{W''\neq W,W'} \log\|\vartheta\|(2\sigma(R) - W'') \\
+ 4\sum_{W\in\mathcal{W}} \log\|\vartheta\|(R - R' + W) + 2\sum_{W''\neq W,W'} \log\|\vartheta\|(R - R' + W'') .
\end{aligned}
$$

Now let $R \to W$ and $R' \to W'$. We obtain from Lemma 3.8.7 and Theorem 3.8.4

$$
\begin{aligned}
16\int_{\Theta+W-W'} \log\|\vartheta\|\cdot\nu + 8\log S(X) \\
= \sum_{W''\neq W,W'} \log\left(\frac{\|J\|(W,W')^2}{4\|J\|(W,W'')\|J\|(W',W'')}\right) + 6\sum_{W''\neq W,W'} \log\|\vartheta\|(W - W' + W'') \\
= 16\sum_{W''\neq W,W'} \log\|\vartheta\|(W - W' + W'') - 32\log 2 - 2\log\|\Delta_2'\|(X) .
\end{aligned}
$$

The theorem then follows by plugging in the result mentioned in Lemma 3.8.8. $\square$

Chapter 4

Hyperelliptic Riemann surfaces II

In the present chapter we give a proof of Theorem 3.1.3. The idea will be to construct a certain isomorphism of line bundles on the moduli stack $\mathcal{I}_g$ of hyperelliptic curves of genus g. Over the complex numbers, these line bundles carry certain hermitian metrics, and we obtain Theorem 3.1.3 by computing the norm of corresponding sections in both line bundles. The proof is given in Section 4.4. In the sections 4.1 till 4.3 some preliminary work is done. As an application of Theorem 3.1.3 we prove in Section 4.5 a formula expressing a certain product of Jacobian Nullwerte, associated to a hyperelliptic Riemann surface, as a product of certain Thetanullwerte. In this way we prove a part of a conjecture formulated by Guàrdia [Gu2].

4.1 Hyperelliptic curves

In this section we assemble some general facts on hyperelliptic curves over an arbitrary base scheme. We will assume all our base schemes to be locally noetherian. The basic reference for this section is [LK].

Definition 4.1.1. Let B be a locally Noetherian scheme, and let $p : \mathcal{X} \to B$ be a smooth projective curve of genus $g \geq 2$. We call such a curve hyperelliptic if there exists an involution $\sigma \in \mathrm{Aut}_B(\mathcal{X})$ such that for every geometric point $\overline{b}$ of B, the quotient $\mathcal{X}_{\overline{b}}/\langle \sigma \rangle$ is isomorphic to $\mathbb{P}^1_{\kappa(\overline{b})}$. For a hyperelliptic curve $p : \mathcal{X} \to B$, the involution σ is uniquely determined. This is well-known for $B = \mathrm{Spec}(k)$ with k an algebraically closed field, and the general case follows from this by the fact that $\mathrm{Aut}_B(\mathcal{X})$ is unramified over B (*cf.* [DM], Theorem 1.11). We call σ the hyperelliptic involution of $\mathcal{X}$.

Definition 4.1.2. By a twisted $\mathbb{P}^1_B$ we mean a smooth, projective curve of genus 0 over B. By [LK], Corollary 3.4, if $q : \mathcal{Y} \to B$ is a twisted $\mathbb{P}^1_B$, there exists an étale surjective morphism $B' \to B$ such that $\mathcal{Y}_{B'} \cong \mathbb{P}^1_{B'}$.

A hyperelliptic curve $p : \mathcal{X} \to B$ carries a relative dualising sheaf (or, what amounts to the same in this case, a sheaf of relative differentials) $\omega_{\mathcal{X}/B}$. We will sometimes leave out the subscript $\mathcal{X}/B$ if the context is clear.

Proposition 4.1.3. *Let $p : \mathcal{X} \to B$ be a hyperelliptic curve of genus $g \geq 2$ and let σ be the hyperelliptic involution of $\mathcal{X}$. The following properties hold: (i) the quotient map $\mathcal{X} \to \mathcal{X}/\langle \sigma \rangle$ is a finite, faithfully flat B-morphism $h : \mathcal{X} \to \mathcal{Y}$ of degree 2, where $\mathcal{Y} \to B$ is a twisted $\mathbb{P}^1_B$; (ii) after an étale surjective base change $B' \to B$ we obtain from this a finite, faithfully flat B'-morphism $h : \mathcal{X}_{B'} \to \mathbb{P}^1_{B'}$ of degree 2; (iii) the image $\mathcal{Y}'$ of the canonical morphism $\pi : \mathcal{X} \to \mathbb{P}(p_* \omega_{\mathcal{X}/B})$ is a twisted $\mathbb{P}^1_B$, and its formation commutes with arbitrary base change; (iv) there exists a closed* *embedding $j : \mathcal{Y} \to \mathbb{P}(p_*\omega_{\mathcal{X}/B})$ such that $\pi = j \cdot h$. After a faithfully flat base change $B' \to B$, the embedding j is isomorphic to the Veronese morphism $\mathbb{P}^1_{B'} \hookrightarrow \mathbb{P}^{g-1}_{B'}$.*

Proof. See [LK], Theorem 5.5, Lemmas 5.6 and 5.7, and Remark 5.11. □

4.2 Canonical isomorphisms

In this section we construct a canonical isomorphism on the base which involves the relative dualising sheaf and the Weierstrass subscheme of a hyperelliptic curve $p : \mathcal{X} \to B$. We make use of the Deligne bracket and its canonical metrisation, defined in Section 1.3. Lemma 4.2.1 and Corollary 4.2.2 are modelled on [BMM], Proposition 1 and Proposition 2, respectively.

Lemma 4.2.1. *Let $p : \mathcal{X} \to B$ be a hyperelliptic curve of genus $g \geq 2$ and let σ be the hyperelliptic involution of $\mathcal{X}$. For any σ-invariant section $P : B \to \mathcal{X}$ of p we have a unique isomorphism*

$$\omega_{\mathcal{X}/B} \xrightarrow{\sim} O_{\mathcal{X}}((2g-2)P) \otimes p^*\langle P, P\rangle^{\otimes -(2g-1)}$$

which induces by pulling back along P the adjunction isomorphism $\langle P, \omega_{\mathcal{X}/B}\rangle \xrightarrow{\sim} \langle P, P\rangle^{\otimes -1}$. Its formation commutes with arbitrary base change. In the case $B = \mathrm{Spec}(\mathbb{C})$, this isomorphism is an isometry if one endows both members with their canonical Faltings-Arakelov metrics.

Proof. First of all, let P be *any* section of p. Let $h : \mathcal{X} \to \mathcal{Y}$ be the morphism from Proposition 4.1.3(i) with $\mathcal{Y}$ a twisted $\mathbb{P}^1_B$ with structure morphism $q : \mathcal{Y} \to B$. By composing P with h we obtain a section Q of q, and hence we can write $\mathcal{Y} \cong \mathbb{P}(V)$ for some locally free sheaf V of rank 2 on B (*cf.* [LK], Proposition 3.3). On the other hand, consider the canonical morphism $\pi : \mathcal{X} \to \mathbb{P}(p_*\omega)$. We have a natural isomorphism $\omega \cong \pi^*(O_{\mathbb{P}(p_*\omega)}(1))$. Let $j : \mathcal{Y} \to \mathbb{P}(p_*\omega)$ be the closed embedding given by Proposition 4.1.3(iv). By that same proposition, and by using a faithfully flat descent argument, we have a natural isomorphism $j^*(O_{\mathbb{P}(p_*\omega)}(1)) \cong O_{\mathbb{P}(V)}((g-1))$. By [EGA], II.4.2.7 there is a unique line bundle L on B such that $O_{\mathbb{P}(V)}((g-1)) \cong O_{\mathbb{P}(V)}((g-1)\cdot Q) \otimes q^*L$. By pulling back along h, we find a natural isomorphism $\omega \xrightarrow{\sim} O_{\mathcal{X}}((g-1)\cdot(P+\sigma(P))) \otimes p^*L$. In the special case where P is σ-invariant, this can be written as a natural isomorphism $\omega \xrightarrow{\sim} O_{\mathcal{X}}((2g-2)P) \otimes p^*L$. Pulling back along P we find that $L \cong \langle \omega, P\rangle \otimes \langle P, P\rangle^{\otimes -(2g-2)}$ and with the adjunction formula $\langle P, P\rangle \cong \langle -P, \omega\rangle$ then finally $L \cong \langle P, P\rangle^{\otimes -(2g-1)}$. It is now clear that we have an isomorphism $\omega \xrightarrow{\sim} O_{\mathcal{X}}((2g-2)P) \otimes p^*\langle P, P\rangle^{\otimes -(2g-1)}$ which induces by pulling back along P an isomorphism $\langle P, \omega_{\mathcal{X}/B}\rangle \xrightarrow{\sim} \langle P, P\rangle^{\otimes -1}$. Possibly after multiplying with a unique global section of O_B^*, we can establish that the latter isomorphism be the canonical adjunction isomorphism. The commutativity with base change is clear from the general base change properties of the relative dualising sheaf and the Deligne bracket. Turning now to the case $B = \mathrm{Spec}(\mathbb{C})$, note that since both members of the isomorphism have admissible metrics with the same curvature form, the isomorphism must multiply the Arakelov metrics by a constant on $\mathcal{X}$. Since the adjunction isomorphism is an isometry, the isomorphism is an isometry at P, hence everywhere. □

Corollary 4.2.2. *Let $p : \mathcal{X} \to B$ be a hyperelliptic curve of genus $g \geq 2$. For any two σ-invariant sections P, Q of p we have a canonical isomorphism of line bundles on B*

$$\langle \omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}\rangle \xrightarrow{\sim} \langle P, Q\rangle^{\otimes -4g(g-1)},$$

and its formation commutes with arbitrary base change. In the case $B = \mathrm{Spec}(\mathbb{C})$, this isomorphism is an isometry if one endows both members with their canonical Faltings metrics.

Proof. By Lemma 4.2.1, we have canonical isomorphisms $\omega \xrightarrow{\sim} O_{\mathcal{X}}((2g-2)P) \otimes p^*\langle P, P\rangle^{\otimes -(2g-1)}$ and $\omega \xrightarrow{\sim} O_{\mathcal{X}}((2g-2)Q) \otimes p^*\langle Q, Q\rangle^{\otimes -(2g-1)}$. It follows that $O_{\mathcal{X}}((2g-2)(P-Q))$ comes from the base, and hence $\langle (2g-2)(P-Q), P-Q \rangle$ is canonically trivial on B. Expanding, this gives a canonical isomorphism $\langle P, P\rangle^{\otimes 2g-2} \otimes \langle Q, Q\rangle^{\otimes 2g-2} \xrightarrow{\sim} \langle P, Q\rangle^{\otimes 2(2g-2)}$ of line bundles on B. Expanding next the right hand member of the canonical isomorphism

$$\langle \omega, \omega \rangle \xrightarrow{\sim} \langle O_{\mathcal{X}}((2g-2)P) \otimes p^* \langle P, P\rangle^{\otimes -(2g-1)}, O_{\mathcal{X}}((2g-2)Q) \otimes p^* \langle Q, Q\rangle^{\otimes -(2g-1)} \rangle$$

gives then the result. The commutativity with base change is clear. Finally it is readily verified that all isomorphisms here become isometries when restricted to $B = \mathrm{Spec}(\mathbb{C})$. Indeed, the Arakelov metric on $O_{\mathcal{X}}((2g-2)(P-Q))$ becomes a constant metric when one trivialises it; as a consequence the metric on $\langle (2g-2)(P-Q), P-Q \rangle$ becomes the trivial metric. That the other isomorphisms are isometries follows from Lemma 4.2.1. □

Definition 4.2.3. Let $p : \mathcal{X} \to B$ be a hyperelliptic curve with hyperelliptic involution σ. Then we call Weierstrass subscheme of $\mathcal{X}$ the fixed point subscheme of $\mathcal{X}$ under the action of $\langle \sigma \rangle$. It is denoted by $W_{\mathcal{X}/B}$. We recall at this point that in general locally, on an affine scheme with ring R, the fixed point scheme for the action of a finite group G is defined by the ideal I_G of R generated by the set $\{r - g(r) | r \in R, g \in G\}$.

Proposition 4.2.4. *The following properties hold: the Weierstrass subscheme $W_{\mathcal{X}/B}$ of $\mathcal{X}$ is the subscheme associated to an effective Cartier divisor on $\mathcal{X}$ relative to B. It is finite and flat over B of degree $2g+2$, and its formation commutes with arbitrary base change. The Weierstrass subscheme $W_{\mathcal{X}/B}$ is étale over a point $b \in B$ if and only if the residue characteristic of b is not 2. After a faithfully flat base change, the Weil divisor given by the Weierstrass subscheme $W_{\mathcal{X}/B}$ can be written as a sum $W_1 + \cdots + W_{2g+2}$ of (not necessarily distinct) sections of p.*

Proof. See [LK], Proposition 6.3, Proposition 6.5, Corollary 6.8 and Theorem 7.3. □

We call Weierstrass divisor the Weil divisor on $\mathcal{X}$ given by the Weierstrass subscheme; we will also denote it by $W_{\mathcal{X}/B}$, and no confusion is to be expected here. We will sometimes leave out the subscript $\mathcal{X}/B$ if the context is clear.

Example 4.2.5. Consider the genus 2 curve $p : \mathcal{X} \to B = \mathrm{Spec}(\mathbb{Z}[1/5])$ given by the affine equation $y^2 + x^3 y = x$. One checks that it has good reduction everywhere, hence $p : \mathcal{X} \to B$ is a hyperelliptic curve according to our definition. Over the ring $R' = R[\zeta_5, \sqrt[5]{2}]$ it acquires six σ-invariant sections $W_0, \ldots, W_5$ where W_0 is given by $x = 0$ and W_k is given by $x = -\zeta_5^k \sqrt[5]{4}$ for $k = 1, \ldots, 5$. One can check by hand that these σ-invariant sections do not meet over points of residue characteristic $\neq 2$, so that indeed the Weierstrass subscheme W is étale over such points. Over a prime of characteristic 2, all σ-invariant sections meet in one point W_0 given in coordinates by $x = y = 0$. The degree 2 quotient map $h : \mathcal{X}_{\mathbb{F}_2} \to \mathcal{Y} = \mathcal{X}_{\mathbb{F}_2}/\langle \sigma \rangle \cong \mathbb{P}^1_{\mathbb{F}_2}$ is ramified only in this point W_0.

Remark 4.2.6. In general, if B is the spectrum of a field of characteristic 2, then $h : \mathcal{X} \to \mathcal{X}/\langle \sigma \rangle$ ramifies in at most $g+1$ distinct points (*cf.* [LK], Remark on p. 104).

In the following proposition we relate the relative dualising sheaf and the Weierstrass subscheme by a canonical isomorphism of line bundles on the base.

Proposition 4.2.7. *Let $p : \mathcal{X} \to B$ be a hyperelliptic curve of genus $g \geq 2$. Then we have a canonical isomorphism of line bundles*

$$\nu : \langle \omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B} \rangle^{\otimes (2g+2)(2g+1)} \xrightarrow{\sim} \langle W_{\mathcal{X}/B}, W_{\mathcal{X}/B} \otimes \omega_{\mathcal{X}/B} \rangle^{\otimes -4g(g-1)},$$

whose formation commutes with arbitrary base change, and which is an isometry in the case $B = \mathrm{Spec}(\mathbb{C})$.

Proof. By Proposition 4.2.4, after a faithfully flat base change we can write W as a sum of sections $W = W_1 + \cdots + W_{2g+2}$. By the adjunction formula for the Deligne bracket we then have a canonical isomorphism $\langle W, W \otimes \omega\rangle \xrightarrow{\sim} \bigotimes_{i \neq j} \langle W_i, W_j\rangle$, which is an isometry in the case $B = \mathrm{Spec}(\mathbb{C})$. On the other hand, by Corollary 4.2.2 we have $\langle W_i, W_j\rangle^{\otimes -4g(g-1)} \xrightarrow{\sim} \langle \omega, \omega\rangle$ for each $i \neq j$, which is again an isometry in the case $B = \mathrm{Spec}(\mathbb{C})$. The general case follows by faithfully flat descent. $\square$

4.3 Canonical sections

The purpose of this section is to prove the following two propositions.

Proposition 4.3.1. *Let B be a regular scheme and let $p : \mathcal{X} \to B$ be a hyperelliptic curve of genus $g \geq 2$. Then the line bundle $(\det p_* \omega_{\mathcal{X}/B})^{\otimes 8g+4}$ has a canonical trivialising global section $\Lambda_{\mathcal{X}/B}$. In the case $B = \mathrm{Spec}(\mathbb{C})$, let $n = \binom{2g}{g+1}$, let $r = \binom{2g+1}{g+1}$ and let $\|\Delta_g\|(X) = 2^{-(4g+4)n} \|\varphi_g\|(X)$ where X is the Riemann surface $\mathcal{X}(\mathbb{C})$. Then for the norm $\|\Lambda\|$ of Λ the formula $\|\Lambda\|^n = (2\pi)^{4g^2 r} \|\Delta_g\|(X)^g$ holds.*

Proposition 4.3.2. *Let B be an irreducible regular scheme with generic characteristic $\neq 2$, and let $p : \mathcal{X} \to B$ be a hyperelliptic curve of genus $g \geq 2$. Then the line bundle $\langle W_{\mathcal{X}/B}, W_{\mathcal{X}/B} \otimes \omega_{\mathcal{X}/B}\rangle$ has a canonical trivialising global section $\Xi_{\mathcal{X}/B}$. In the case $B = \mathrm{Spec}(\mathbb{C})$, the norm $\|\Xi\|$ of Ξ satisfies $\|\Xi\| = 2^{-(2g+2)} \prod_{(W,W')} G(W, W')$, the product running over all ordered pairs of Weierstrass points of the Riemann surface $\mathcal{X}(\mathbb{C})$.*

For the proofs we need three lemmas. At this point we follow [Ka], Section 6 rather closely. Let $B = \mathrm{Spec}(R)$ with R a discrete valuation ring with quotient field K and residue field k. Assume that $\mathrm{char}(K) \neq 2$. The quotient map $R \to k$ is denoted, as usual, by a bar $\bar{\ }$. Let $p : \mathcal{X} \to B$ be a hyperelliptic curve of genus $g \geq 2$.

Lemma 4.3.3. *After a finite étale surjective base change with a discrete valuation ring R' dominating R, there exists an open affine subscheme $U \cong \mathrm{Spec}(E)$ of $\mathcal{X}'$ with $E = A[y]/(y^2 + ay + b)$, where $A = R'[x]$ and $a, b \in A$, such that $f := a^2 - 4b \in K'[x]$ is separable of degree $2g + 2$ and such that $\deg a \leq g + 1$ and $\deg b \leq 2g + 2$. For the reduced polynomials $\overline{a}, \overline{b} \in k'[x]$ we have $\deg \overline{a} = g + 1$ or $\deg \overline{b} \geq 2g + 1$.*

Proof. After a finite étale surjective base change with a discrete valuation ring R' dominating R, we have by Proposition 4.1.3 a finite faithfully flat R'-morphism $h' : \mathcal{X}' \to \mathbb{P}^1_{R'}$ of degree two. Choose a point $\infty \in \mathbb{P}^1_{K'}$ such that $\mathcal{X}_{K'} \to \mathbb{P}^1_{K'}$ is unramified above ∞, and let x be a coordinate on $V = \mathbb{P}^1_{K'} - \infty$. We can then describe $U := h'^{-1}(V)$ as $U \cong \mathrm{Spec}(E)$ with $E = A[y]/(y^2 + ay + b)$ where $A = R'[x]$ and $a, b \in A$. Moreover, if we assume the degree of a to be minimal, we have $\deg a \leq g + 1$ and $\deg b \leq 2g + 2$. Next let us consider the degree of f. By Proposition 4.2.4, the Weierstrass subscheme $W_{\mathcal{X}'/B'}$ is finite and flat over B' of degree $2g + 2$. By definition, the ideal of $W_{\mathcal{X}'/B'}$ is generated by $y - \sigma(y) = 2y + a$ on U. Note that $(2y + a)^2 = a^2 - 4b = f$, which defines the norm under h' of $W_{\mathcal{X}'/B'}$ in $\mathbb{P}^1_{R'}$. Since this norm is also finite and flat of degree $2g + 2$ over B', and since $W_{\mathcal{X}'/B'}$ is entirely supported in U by our choice of ∞, we obtain that $\deg(f) = 2g + 2$. Since the norm of $W_{\mathcal{X}'/B'}$ in $\mathbb{P}^1_{R'}$ is étale over K' by Proposition 4.2.4, the polynomial $f \in K'[x]$ is separable. Consider finally the reduced polynomials $\overline{a}, \overline{b} \in k'[x]$. Regarding y as an element of $k'(X_{k'})$, we have $\mathrm{div}(y) \geq -\min(\deg \overline{a}, \frac{1}{2} \deg \overline{b}) \cdot h'^*(\overline{\infty})$ by the equation for y. On the other hand it follows from Riemann-Roch that y has a pole at both points of $h'^*(\overline{\infty})$ of order strictly larger than g. This gives then the last statement of the lemma. $\square$

Lemma 4.3.4. *Suppose we have an open affine subscheme $U \cong \mathrm{Spec}(E)$ on $\mathcal{X}$ as in Lemma 4.3.3. Then the differentials $x^i dx/(2y + a)$ for $i = 0, \ldots, g - 1$ are nowhere vanishing on U and extend to regular global sections of $\omega_{\mathcal{X}/B}$.*

Proof. Let F be the polynomial $y^2+ay+b \in A[y]$, and let F_x and F_y be its derivatives with respect to x and y, respectively. It is readily verified that the morphism $\Omega_{E/R} = (Edx+Edy)/(F_x dx+F_y dy) \to E$ given by $dx \mapsto F_y, dy \mapsto -F_x$, is an isomorphism of E-modules. This gives that the differentials $x^i dx/(2y+a)$ for $i = 0, \ldots, g-1$ are nowhere vanishing on U. For the second part of the lemma, it suffices to show that the differentials $x^i dx/(2y+a)$ for $i = 0, \ldots, g-1$ on the generic fiber U_K extend to global sections of $\Omega_{\mathcal{X}_K/K}$—but this is well-known to be true. $\square$

Suppose that a polynomial $f \in K[x]$ of degree d factors over an extension of K as $f = H\prod_{i=1}^{d}(x-\alpha_i)$. Then its discriminant $D(f)$ is given as $D(f) = H^{2d-2}\prod_{i\neq j}(\alpha_i - \alpha_j)$. Recall that this element lies in R if the coefficients of f lie in R.

Lemma 4.3.5. *Suppose we have an open affine subscheme $U \cong \mathrm{Spec}(E)$ on $\mathcal{X}$ as in Lemma 4.3.3. Then the modified discriminant $\Delta(f) = 2^{-(4g+4)} \cdot D(f)$ is a unit of R.*

Proof. In the case that the characteristic of k is $\neq 2$, this is not hard to see: we know that $W_{\mathcal{X}_k/k}$ is étale of degree $2g+2$ by Proposition 4.2.4, and hence f remains separable of degree $2g+2$ in $k[x]$ under the reduction map. So let us assume from now on that the characteristic of k equals 2. If C is any ring, and if $P(T) = \sum_{i=0}^{n} u_i T^i$ and $Q(T) = \sum_{i=0}^{m} v_i T^i$ are two polynomials in $C[T]$, we denote by $R_T^{n,m}(P,Q) \in C$ the resultant of P and Q. Recall the following property of the resultant: suppose that at least one of u_n, v_m is non-zero, and that C is a field. Then $R_T^{n,m}(P,Q) = 0$ if and only if P and Q have a root in common in an extension field of C. Let F be the polynomial $y^2 + a(x)y + b(x)$ in $A[y]$ with $A = R[x]$, and let F_x and F_y be its derivatives with respect to x and y, respectively. We set $Q = R_y^{2,1}(F, F_x)$ and $P = R_y^{2,1}(F, F_y) = 4b - a^2 = -f$. Let $H \in R$ be the leading coefficient of P, and abbreviate the modified discriminant $\Delta(f)$ of f by Δ. A calculation (*cf.* [Lo], Section 1) shows that $R_x^{2g+2,4g+2}(P,Q) = (H \cdot \Delta)^2$. We should read this equation as a formal identity between certain universal polynomials in the coefficients of $a(x)$ and $b(x)$. Doing so, we may conclude that $\Delta \in R$ and that H^2 divides $R_x^{2g+2,4g+2}(P,Q)$ in R. To finish the argument, we distinguish two cases. First assume that $\overline{H} \neq 0$. Then $\deg \overline{P} = 2g+2$ and again a calculation shows that $R_x^{2g+2,4g+2}(\overline{P}, \overline{Q}) = (\overline{H} \cdot \overline{\Delta})^2$. The fact that X_k is smooth implies that $R_x^{2g+2,4g+2}(\overline{P}, \overline{Q})$ is non-zero, and altogether we obtain that $\overline{\Delta}$ is non-zero. Next assume that $\overline{H} = 0$. Then since $\overline{P} = \overline{a}^2$ we obtain that $\deg \overline{a} \leq g$ and hence $\deg \overline{P} \leq 2g$. By Lemma 4.3.3 we have then $2g+1 \leq \deg \overline{b} \leq 2g+2$. But then from $2\deg(y) = \deg(\overline{a}y + \overline{b})$ and $\deg(y) > g$ (*cf.* the proof of Lemma 4.3.3) it follows that in fact $\deg \overline{b} = 2g+2$ and hence $\deg \frac{d\overline{b}}{dx} = 2g$ since we are in characteristic 2. This implies that $\deg \overline{Q} = 4g$. A calculation shows that $R_x^{2g,4g}(\overline{P}, \overline{Q}) = \overline{\Delta}^2$. Again by smoothness of X_k we may conclude that $R_x^{2g,4g}(\overline{P}, \overline{Q})$ is non-zero. This finishes the proof. $\square$

Example 4.3.6. Consider once more the curve over $R = \mathbb{Z}[1/5]$ given in Example 4.2.5 above. In the notation from Lemma 4.3.3, we have $a = x^3, b = -x$. We compute $D(a^2 - 4b) = D(x^6 + 4x) = 2^{12}5^5$ so that $\Delta(f) = 5^5$ which is indeed a unit in R.

We can now prove Propositions 4.3.1 and 4.3.2.

Proof of Proposition 4.3.1. Possibly after a faithfully flat base change we may assume, by Proposition 4.1.3, that p is a morphism $p : \mathcal{X} \to \mathbb{P}_B^1$. The scheme $\mathcal{X}$ is covered by affine schemes $U \cong \mathrm{Spec}(E)$ with $E = A[y]/(y^2 + ay + b)$ and A a polynomial ring $R[x]$. For such an affine scheme U, consider $V := \mathrm{Spec}(A)$. In the line bundle $(\det p_* \omega_{U/V})^{\otimes 8g+4}$ we have a rational section

$$\Lambda_{U/V} := \Delta(f)^g \cdot \left(\frac{dx}{2y+a} \wedge \ldots \wedge \frac{x^{g-1}dx}{2y+a} \right)^{\otimes 8g+4} ,$$

where $\Delta(f)$ is as in Lemma 4.3.5. One can check that this element does not depend on any choice of affine equation $y^2 + ay + b$ for U, and moreover, these sections coincide on overlaps. Hence they build a canonical rational section $\Lambda_{\mathcal{X}/B}$ of $(\det p_*\omega_{\mathcal{X}/B})^{\otimes 8g+4}$. By Lemma 4.3.4 and Lemma 4.3.5, this $\Lambda_{\mathcal{X}/B}$ is a global trivialising section. The general case follows by faithfully flat descent. Now consider the case $B = \mathrm{Spec}(\mathbb{C})$. In that case, we can make a change of coordinates $y' := 2y + a$, $x' := x$ so that we may write $y'^2 = f(x')$ as an equation for $\mathcal{X}$, with $f := a^2 - 4b$. We have

$$\Lambda = \left(2^{-(4g+4)} \cdot D\right)^g \left(\frac{dx'}{y'} \wedge \ldots \wedge \frac{x'^{g-1}dx'}{y'}\right)^{\otimes 8g+4} ,$$

where D is the discriminant of f. Consider the Riemann surface $X = \mathcal{X}(\mathbb{C})$. Using the differentials $\mu_1 = dx'/2y', \ldots, \mu_g = x'^{g-1}dx'/2y'$ corresponding to the equation $y' = f(x')$ and a canonical basis of the homology of X, we form a period matrix $(\mu|\mu')$ and the associated matrix $\tau = \mu^{-1}\mu'$ in the Siegel upper half space. According to Proposition 3.2.2 we have the formula $D^n = \pi^{4gr}(\det \mu)^{-4r}\varphi_g(\tau)$. We put $\Delta_g = 2^{-(4g+4)n} \cdot \varphi_g$. Let $z_1, \ldots, z_g$ be the standard euclidean coordinates on $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$. Recall that we have a canonical isomorphism j : $\det H^0(X, \Omega^1_X) \xrightarrow{\sim} \det H^0(\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g, \Omega^g)$. Then we have

$$\begin{aligned}
j^{\otimes(8g+4)n}(\Lambda^{\otimes n}) &= j^{\otimes(8g+4)n}\left(\left(2^{-(4g+4)} \cdot D\right)^{gn} \left(\frac{dx'}{y'} \wedge \ldots \wedge \frac{x'^{g-1}dx'}{y'}\right)^{\otimes(8g+4)n}\right) \\
&= j^{\otimes(8g+4)n}\left(2^{-(4g+4)gn}\pi^{4g^2r}(\det \mu)^{-4gr}\varphi_g(\tau)^g \left(\frac{dx'}{y'} \wedge \ldots \wedge \frac{x'^{g-1}dx'}{y'}\right)^{\otimes(8g+4)n}\right) \\
&= j^{\otimes(8g+4)n}\left((2\pi)^{4g^2r}(\det \mu)^{-4gr}\Delta_g(\tau)^g \left(\frac{dx'}{2y'} \wedge \ldots \wedge \frac{x'^{g-1}dx'}{2y'}\right)^{\otimes(8g+4)n}\right) \\
&= (2\pi)^{4g^2r}\Delta_g(\tau)^g(dz_1 \wedge \ldots \wedge dz_g)^{\otimes(8g+4)n} .
\end{aligned}$$

The claim on the norm of Λ follows since $\|dz_1 \wedge \ldots \wedge dz_g\| = \sqrt{\det \mathrm{Im}\tau}$. □

Proof of Proposition 4.3.2. By Propositions 4.1.3 and 4.2.4 we can assume, possibly after a faithfully flat base change, that the image of the canonical map $h : \mathcal{X} \to \mathcal{X}/\langle\sigma\rangle$ is isomorphic to $\mathbb{P}^1_B$, and that W is a sum of sections $W = W_1 + \cdots + W_{2g+2}$. The latter gives by the adjunction formula for the Deligne bracket a canonical isomorphism $\langle W, W \otimes \omega\rangle \xrightarrow{\sim} \bigotimes_{i\neq j}\langle W_i, W_j\rangle$. The latter line bundle contains a canonical rational section $\Xi' := \bigotimes_{i\neq j}\langle s_i, s_j\rangle$ with s_i and s_j canonical sections of $O_{\mathcal{X}}(W_i)$ and $O_{\mathcal{X}}(W_j)$, respectively. We claim that $\Xi := 2^{-(2g+2)} \cdot \Xi'$ is a trivialising global section. The first statement of the proposition follows then by faithfully flat descent. In order to prove the claim, we assume that $B = \mathrm{Spec}(R)$ with R a discrete valuation ring with $\mathrm{char}(K) \neq 2$. Also we assume that its discrete valuation v is normalised in the sense that $v(K^*) = \mathbb{Z}$. Then the valuation $v(\Xi')$ of Ξ' at the closed point b of B is given by the sum $\sum_{i\neq j}(W_i, W_j)$ of the local intersection multiplicities (W_i, W_j) above b of pairs of W_i, W_j. Suppose that W_i is given by a polynomial $x - \alpha_i$, and write α_i as a shorthand for the corresponding section of $\mathbb{P}^1_R$. By the projection formula (*cf.* [Li], Theorem 9.2.12) we have for the local intersection multiplicities that $4(W_i, W_j) = (2W_i, 2W_j) = (h^*\alpha_i, h^*\alpha_j) = 2(\alpha_i, \alpha_j)$ for each $i \neq j$ hence $(W_i, W_j) = \frac{1}{2}(\alpha_i, \alpha_j)$ for each $i \neq j$. Now the local intersection multiplicity (α_i, α_j) on $\mathbb{P}^1_R$ above b is calculated to be $v(\alpha_i - \alpha_j)$. This gives that $v(\Xi') = \sum_{i\neq j}(W_i, W_j) = \frac{1}{2}\sum_{i\neq j} v(\alpha_i - \alpha_j) = \frac{1}{2}v(D(f))$. By Lemma 4.3.5 we have $v(D(f)) = (4g + 4)v(2)$ hence if we define $\Xi = 2^{-(2g+2)} \cdot \Xi'$ we obtain a section with $v(\Xi) = 0$, which is what we wanted. Turning finally to the case $B = \mathrm{Spec}(\mathbb{C})$, we see, since the adjunction isomorphism is an isometry, that $\|\Xi\| = 2^{-(2g+2)}\prod_{i\neq j}\|\langle s_i, s_j\rangle\|$. The required formula follows. □

4.4 Proof of Theorem 3.1.3

In this section we give a proof of Theorem 3.1.3. We will work on the universal hyperelliptic curve of genus g. To be precise, let $\mathcal{I}_g$ be the category with objects the hyperelliptic curves $p : \mathcal{X} \to B$ of genus g, and morphisms given by cartesian diagrams. Then $\mathcal{I}_g$ is an algebraic stack in the sense of Deligne-Mumford [DM], and according to [LL], Theorem 3, the stack $\mathcal{I}_g$ is a smooth, closed substack of dimension $2g-1$ of the moduli stack $\mathcal{M}_g$ of smooth curves of genus g. We denote the universal family on $\mathcal{I}_g$ by $\mathcal{U}_g$.

The idea of the proof will be to apply our results from the previous section to the universal map $p : \mathcal{U}_g \to \mathcal{I}_g$, in order to obtain a canonical isomorphism of line bundles on $\mathcal{I}_g$. We obtain the theorem by comparing the norms of corresponding sections in these line bundles. We will need one more lemma.

Lemma 4.4.1. *We have that* $H^0(\mathcal{I}_g, \mathbb{G}_{\mathrm{m}}) = \{-1, +1\}$.

Proof. It suffices to see that $H^0(\mathcal{I}_g \otimes \mathbb{C}, \mathbb{G}_{\mathrm{m}}) = \mathbb{C}^*$ for then the lemma follows since $\mathcal{I}_g \to \mathrm{Spec}(\mathbb{Z})$ is smooth and surjective. We can describe $\mathcal{I}_g \otimes \mathbb{C}$ as the space of $(2g+2)$-tuples of distinct points on $\mathbb{P}^1$ modulo projective equivalence, that is we can write $\mathcal{I}_g \otimes \mathbb{C} = ((\mathbb{C} \setminus \{0,1\})^{2g-1} - \Delta)/S_{2g+2}$ (in the orbifold sense) where Δ denotes the fat diagonal and where S_{2g+2} is the symmetric group acting by permutation on $2g+2$ points on $\mathbb{P}^1$. According to [HM], the first homology of $(\mathbb{C} \setminus \{0,1\})^{2g-1} - \Delta$ is isomorphic to the irreducible representation of S_{2g+2} corresponding to the partition $\{2g, 2\}$ of $2g+2$; in particular it does not contain a trivial representation of S_{2g+2}. This proves that $H_1(\mathcal{I}_g \otimes \mathbb{C}, \mathbb{Q})$ is trivial, and hence $H^0(\mathcal{I}_g \otimes \mathbb{C}, \mathbb{G}_{\mathrm{m}}) = \mathbb{C}^*$. $\square$

Proposition 4.4.2. *Let $p : \mathcal{U}_g \to \mathcal{I}_g$ be the universal hyperelliptic curve of genus g. Then there is a canonical isomorphism*

$$\psi : (\det p_* \omega)^{\otimes 12(8g+4)(2g+2)(2g+1)} \xrightarrow{\sim} \langle W, W \otimes \omega \rangle^{-\otimes 4(8g+4)g(g-1)}$$

of line bundles on $\mathcal{I}_g$. This isomorphism has the property that

$$\psi(\Lambda^{\otimes 12(2g+2)(2g+1)}) = \pm \Xi^{-\otimes 4(8g+4)g(g-1)} .$$

Over the complex numbers, the norm of ψ is equal to $\left((2\pi)^{-4g} \exp(\delta)\right)^{(8g+4)(2g+2)(2g+1)}$.

Proof. By Theorem 1.6.1, we have on $\mathcal{I}_g$ a canonical isomorphism $\mu : (\det p_* \omega)^{\otimes 12} \xrightarrow{\sim} \langle \omega, \omega \rangle$. Further, by Proposition 4.2.7 we have a canonical isomorphism

$$\nu : \langle \omega, \omega \rangle^{\otimes (2g+2)(2g+1)} \xrightarrow{\sim} \langle W, W \otimes \omega \rangle^{\otimes -4g(g-1)} .$$

Combining, we obtain a canonical isomorphism ψ as required. According to Theorem 1.6.1, the isomorphism μ has norm $(2\pi)^{-4g} \exp(\delta)$ over the complex numbers, and by Proposition 4.2.7 the isomorphism ν is an isometry. This easily implies the statement on the norm of ψ. Now let us consider the canonical sections Λ and Ξ of Propositions 4.3.1 and 4.3.2. Since $\mathcal{I}_g$ is smooth, these are trivialising global sections. We conclude that $\psi(\Lambda^{\otimes 12(2g+2)(2g+1)}) = \Xi^{-\otimes 4(8g+4)g(g-1)}$ only up to an element of $H^0(\mathcal{I}_g, \mathbb{G}_{\mathrm{m}})$. However, we know by Lemma 4.4.1 that the latter group is just $\{-1, +1\}$. $\square$

We can now give the proof of Theorem 3.1.3.

Proof of Theorem 3.1.3. Let $p : \mathcal{X} \to B = \mathrm{Spec}(\mathbb{C})$ be the complex hyperelliptic curve such that $X = \mathcal{X}(\mathbb{C})$. By taking norms on both sides of the isomorphism in Proposition 4.4.2 we arrive at the following fundamental formula:

$$\left((2\pi)^{-4g}\exp(\delta(X))\right)^{(8g+4)(2g+2)(2g+1)}\|\Lambda\|(B)^{12(2g+2)(2g+1)} = \|\Xi\|(B)^{-4g(g-1)(8g+4)}\,.$$

Now let us see what we have for the individual terms from this formula. First, by Proposition 4.3.1 we have $\|\Lambda\|(B)^n = (2\pi)^{4g^2r}\|\Delta_g\|(X)^g$, where $\|\Delta_g\|(X) = 2^{-(4g+4)n}\cdot\|\varphi_g\|(X)$. Second, by Proposition 4.3.2 and the definition of G' we have

$$\|\Xi\|(B) = 2^{-(2g+2)}\prod_{(W,W')} G(W,W') = 2^{-(2g+2)}S(X)^{(2g+2)(2g+1)/g^3}\prod_{(W,W')} G'(W,W')\,,$$

the product running over all ordered pairs of distinct Weierstrass points. Finally we have by Theorem 2.1.3 that $\exp(\delta(X)/4) = S(X)^{-(g-1)/g^2}\cdot T(X)$. We find the theorem by plugging in these results. $\square$

Remark 4.4.3. We have not been able to find in general a formula for $G'(W,W')$ with W,W' just two Weierstrass points. It follows from Corollary 3.8.5 above that in the case $g=2$ we have

$$G'(W,W')^2 = 2^{1/4}\cdot\|\varphi_2\|(X)^{-3/64}\cdot\prod_{W''\neq W,W'}\|\vartheta\|(W-W'+W'')\,.$$

We guess that in general we have

$$(??)\quad G'(W,W')^g = A(X)\cdot\prod_{\substack{S=\{W_1,\ldots,W_{g-1}\}\\ W,W'\notin S}}\|\vartheta\|(W-W'+W_1+\cdots+W_{g-1})\,,$$

with $A(X)$ some invariant of X. Such a result is consistent with Theorems 3.1.2 and 3.1.3 above.

4.5 Jacobian Nullwerte

In this section we derive from Theorem 3.1.3 a relation between certain products of Jacobian Nullwerte and certain products of Thetanullwerte, associated to hyperelliptic Riemann surfaces. The theme of this section finds its origin in Jacobi's derivative formula, discovered by Jacobi around 1830: let η_1 be the odd analytic theta characteristic in genus one, and let η_2,η_3,η_4 be the even ones. We then have an equality

$$\vartheta[\eta_1]'(0;\tau) = -\pi\vartheta[\eta_2](0;\tau)\vartheta[\eta_3](0;\tau)\vartheta[\eta_4](0;\tau)$$

of functions of τ in the complex upper half plane. It is natural to ask for generalisations of this identity to higher dimensions. For this one considers the following so-called Jacobian Nullwerte: let $\eta_1,\ldots,\eta_g$ be g odd theta characteristics in dimension g. Then we put

$$J(\eta_1,\ldots,\eta_g)(\tau) := (\partial(\vartheta[\eta_1],\ldots,\vartheta[\eta_g])/\partial(z_1,\ldots,z_g))(0;\tau)\,.$$

These Jacobian Nullwerte are modular forms on the Siegel upper half space $\mathcal{H}_g$. A first generalisation of Jacobi's derivative formula was stated by Rosenhain around 1850.

Theorem 4.5.1. *(Rosenhain [Ro]) Let $\eta_1,\ldots,\eta_6$ be the six odd theta characteristics in $g=2$. Let $\tau\in\mathcal{H}_2$. Then for every pair of distinct η_k,η_l, the identity*

$$J(\eta_k,\eta_l)(\tau) = \pm\pi^2\prod_{m\neq k,l}\vartheta[\eta_k+\eta_l-\eta_m](0;\tau)$$

holds. Here the sum $\eta_k + \eta_l - \eta_m$ is taken modulo 1, and the characteristics $\eta_k + \eta_l - \eta_m$ occurring in the product are even.

After that, some scattered generalisations of Jacobi's derivative formula were obtained by, among others, Riemann, Thomae [Th] and Frobenius [Fr]. A general result was proved by Igusa. In order to state this result, we need the notion of a fundamental system of theta characteristics. This notion was already employed by the nineteenth century authors.

Definition 4.5.2. For an analytic theta characteristic η, we put $e(\eta) = 1$ if η is even, and $e(\eta) = -1$ if η is odd. Given three analytic theta characteristics η_1, η_2, η_3 we define
$e(\eta_1, \eta_2) = e(\eta_1)e(\eta_2)e(\eta_1 + \eta_2)$ and $e(\eta_1, \eta_2, \eta_3) = e(\eta_1)e(\eta_2)e(\eta_3)e(\eta_1 + \eta_2 + \eta_3)$. We say that the triplet $\{\eta_1, \eta_2, \eta_3\}$ is azygetic (resp. zygetic) if $e(\eta_1, \eta_2, \eta_3) = -1$ (resp. $e(\eta_1, \eta_2, \eta_3) = 1$). A fundamental system of analytic theta characteristics is a set $\mathcal{S} = \{\eta_1, \ldots, \eta_{2g+2}\}$ of $2g + 2$ theta characteristics such that the $\eta_1, \ldots, \eta_g$ are odd, the $\eta_{g+1}, \ldots, \eta_{2g+2}$ are even, and every triplet $\{\eta_k, \eta_l, \eta_m\} \subset \mathcal{S}$ is azygetic.

Theorem 4.5.3. *(Igusa [Ig2]) Let $g \geq 2$ be an integer. Let $\eta_1, \ldots, \eta_g$ be odd analytic theta characteristics such that the function $J(\eta_1, \ldots, \eta_g)(\tau)$ on $\mathcal{H}_g$ is not identically zero and is contained in the $\mathbb{C}$-algebra generated by the Thetanullwerte $\vartheta[\eta](0;\tau)$, with η running through the even characteristics. Then $\eta_1, \ldots, \eta_g$ can be completed to form a fundamental system, and:*

$$J(\eta_1, \ldots, \eta_g)(\tau) = \pi^g \sum_{\{\eta_{g+1}, \ldots, \eta_{2g+2}\} \in \mathcal{S}} \pm \prod_{k=g+1}^{2g+2} \vartheta[\eta_k](0;\tau) \,,$$

where $\mathcal{S}$ is the set of all $(g+2)$-tuples $\{\eta_{g+1}, \ldots, \eta_{2g+2}\}$ of even theta characteristics such that $\{\eta_1, \ldots, \eta_g, \eta_{g+1}, \ldots, \eta_{2g+2}\}$ form a fundamental system. If τ is a period matrix of a hyperelliptic Riemann surface, then there is exactly one non-zero term in the sum at the right hand side of the equality.

Now, consider a hyperelliptic Riemann surface X of genus $g \geq 2$. Fix an ordering $W_1, \ldots, W_{2g+2}$ of the Weierstrass points of X. Consider an equation $y^2 = f(x)$ with f monic of degree $2g + 1$ that puts W_{2g+2} at infinity. We have then as usual the period matrix $(\mu|\mu')$ of the differentials $dx/2y, \ldots, x^{g-1}dx/2y$ on a canonical symplectic basis of the homology of X. Let $\tau = \mu^{-1}\mu'$, let $\kappa \in \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ be the Riemann vector associated to infinity by Theorem 1.4.2, and consider the Abel-Jacobi map $t_\kappa \cdot u : \mathrm{Pic}_{g-1}(X) \xrightarrow{\sim} \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ from Theorem 1.4.2. As was explained there, the Abel-Jacobi map induces an identification of the set of classes of semi-canonical divisors on X (*i.e.*, divisors D with $2D$ linearly equivalent to a canonical divisor) with the set of theta characteristics, the class $[D]$ of D corresponding to $\eta = [{\eta' \atop \eta''}]$ where $(t_\kappa \cdot u)([D]) = [\eta' + \tau \cdot \eta'']$. We can be even more precise. Recall the definition of the analytic theta characteristics η_k for $k = 1, \ldots, 2g+1$ from Section 3.2. We there defined for a subset S of $\{1, 2, \ldots, 2g+1\}$ a theta characteristic η_S by putting $\eta_S := \sum_{k \in S} \eta_k$. We extend this definition here to subsets S of $\{1, 2, \ldots, 2g+2\}$ by putting $\eta_S := \sum_{\substack{k \in S \\ k \neq 2g+2}} \eta_k$. Further, as before we put $U = \{1, 3, \ldots, 2g+1\}$ and we let $\circ$ denote the symmetric difference.

Lemma 4.5.4. *Let X be a hyperelliptic Riemann surface of genus $g \geq 2$ and fix an ordering $W_1, \ldots, W_{2g+2}$ of its Weierstrass points. Consider the identification of the set of classes of semi-canonical divisors with the set of analytic theta characteristics as above. (i) The semi-canonical divisor $W_{i_1} + \cdots + W_{i_{g-1}}$ corresponds to the characteristic $\eta_{T \circ U}$ where $T = \{i_1, \ldots, i_{g-1}\}$. (ii) The semi-canonical divisor $W_{i_1} + \cdots + W_{i_g} - W_{i_{g+1}}$ corresponds to the characteristic $\eta_{T \circ U}$ where $T = \{i_1, \ldots, i_{g+1}\}$.*

Proof. See [Mu2], Chapter IIIa, Proposition 6.2. $\square$

In [Gu2], Guàrdia proves the following result.

Lemma 4.5.5. *Let X be a hyperelliptic Riemann surface of genus $g \geq 2$ and fix an ordering $W_1, \ldots, W_{2g+2}$ of its Weierstrass points. Consider the identification of the set of classes of semi-canonical divisors with the set of analytic theta characteristics as above. Let $\{i_1, \ldots, i_{2g+2}\}$ be a permutation of the set $\{1, \ldots, 2g+2\}$. Denote by η_k for $k = 1, \ldots, g$ the analytic theta characteristic corresponding to the semi-canonical divisor $\sum_{\substack{l=1 \\ l \neq k}}^{g} W_{i_l}$, and denote by η_k for $k = g+1, \ldots, 2g+2$ the analytic theta characteristic corresponding to the semi-canonical divisor $(\sum_{l=1}^{g} W_{i_l}) - W_{i_k}$. Then the set $\{\eta_1, \ldots, \eta_{2g+2}\}$ is a fundamental system of theta characteristics.*

It follows from Lemma 4.5.4 that the set $\mathcal{F}_g$ that one gets by letting $\{i_1, \ldots, i_{2g+2}\}$ range over the permutations of $\{1, \ldots, 2g+2\}$, is independent of the chosen ordering of the Weierstrass points, and even independent of X. In fact, the cardinality of $\mathcal{F}_g$ is $\binom{2g+2}{g}$ and an element $\{\eta_1, \ldots, \eta_g, \eta_{g+1}, \ldots, \eta_{2g+2}\} \in \mathcal{F}_g$ is determined by the set $\{\eta_1, \ldots, \eta_g\}$.

Considering Igusa's result Theorem 4.5.3, Guàrdia states in [Gu2] the following conjecture:

Conjecture 4.5.6. *(Guàrdia [Gu2]) Let $g \geq 2$ be an integer. Let $\{\eta_1, \ldots, \eta_g, \eta_{g+1}, \ldots, \eta_{2g+2}\}$ be a fundamental system contained in $\mathcal{F}_g$, and let τ be a period matrix associated to a hyperelliptic Riemann surface. Then the formula*

$$J(\eta_1, \ldots, \eta_g)(\tau) = \pm \pi^g \prod_{k=g+1}^{2g+2} \vartheta[\eta_k](0;\tau)$$

holds.

Using the results from this chapter we are able to prove an easy consequence of Conjecture 4.5.6, mentioned by Guàrdia himself already in [Gu2].

Theorem 4.5.7. *Let $\mathcal{T}$ be the set of subsets of $\{1, 2, \ldots, 2g+1\}$ of cardinality $g+1$. Let $\tau \in \mathcal{H}_g$ be a period matrix associated to a hyperelliptic Riemann surface. Let $m = \binom{2g+2}{g}$. Then the formula*

$$\prod_{\{\eta_1, \ldots, \eta_g, \eta_{g+1}, \ldots, \eta_{2g+2}\} \in \mathcal{F}_g} J(\eta_1, \ldots, \eta_g)(\tau) = \pm \pi^{gm} \prod_{T \in \mathcal{T}} \vartheta[\eta_{T \circ U}](0;\tau)^{2g+2}$$

holds.

If we take the product over all $\{\eta_1, \ldots, \eta_g, \eta_{g+1}, \ldots, \eta_{2g+2}\} \in \mathcal{F}_g$ in the formula from Conjecture 4.5.6, we obtain the formula from Theorem 4.5.7.

In order to prove Theorem 4.5.7, we focus on a fixed hyperelliptic Riemann surface X of genus $g \geq 2$, marked with an ordering $W_1, \ldots, W_{2g+2}$ of its Weierstrass points. Associated to these data we have an equation $y^2 = f(x)$ putting W_{2g+2} at infinity; we have then the period matrix $(\mu|\mu')$, the matrix τ and the identification $\mathrm{Pic}_{g-1}(X) \xrightarrow{\sim} \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ as explained above.

Recall the definition of the function $\|J\|$ from Definition 1.4.11. From Theorem 1.4.12 we derive the following proposition, which, in combination with Theorem 3.1.3, easily gives Theorem 4.5.7.

Proposition 4.5.8. *We have*

$$\prod_{(W,W')} G'(W,W')^{n(g-1)} = T(X)^{-(g+2)m} \cdot \|\varphi_g\|(X)^{(g^2-1)/2} \cdot \prod_{\{i_1,\ldots,i_g\}} \|J\|(W_{i_1}, \ldots, W_{i_g})^{-(2g+4)},$$

the first product running over all ordered pairs of distinct Weierstrass points, the second product running over the subsets of $\{1, \ldots, 2g+2\}$ of cardinality g.

Proof. From Theorem 1.4.12 and Theorem 2.1.3 we derive the relation

$$\prod_{k=1}^{g} G'(P_k,Q)^{2g-2} = \frac{1}{T(X)} \cdot \frac{\|\vartheta\|(P_1+\cdots+P_g-Q)^{2g-2}}{\|J\|(P_1,\ldots,P_g)^2} \cdot \prod_{k\neq l} G'(P_k,P_l)$$

for generic $P_1,\ldots,P_g,Q \in X$. Let $W_{i_1},\ldots,W_{i_g}$ be g distinct Weierstrass points. We obtain

$$\begin{aligned}&\prod_{W\notin\{W_{i_1},\ldots,W_{i_g}\}} \prod_{k=1}^{g} G'(W_{i_k},W)^{2g-2} \\ &= \frac{1}{T(X)^{g+2}} \cdot \frac{\prod_{W\notin\{W_{i_1},\ldots,W_{i_g}\}} \|\vartheta\|(W_{i_1}+\cdots+W_{i_g}-W)^{2g-2}}{\|J\|(W_{i_1},\ldots,W_{i_g})^{2g+4}} \cdot \prod_{k\neq l} G'(W_{i_k},W_{i_l})^{g+2}\,.\end{aligned}$$

Taking the product over all sets of indices $\{i_1,\ldots,i_g\}$ of cardinality g we find

$$\begin{aligned}&\prod_{(W,W')} G'(W,W')^{n(g-1)} \\ &\qquad = \frac{1}{T(X)^{(g+2)m}} \cdot \prod_{\{i_1,\ldots,i_g\}} \frac{\prod_{W\notin\{W_{i_1},\ldots,W_{i_g}\}} \|\vartheta\|(W_{i_1}+\cdots+W_{i_g}-W)^{2g-2}}{\|J\|(W_{i_1},\ldots,W_{i_g})^{2g+4}}\,.\end{aligned}$$

Since $W_{i_1}+\cdots+W_{i_g}-W_{i_{g+1}} \sim W_{i'_1}+\cdots+W_{i'_g}-W_{i'_{g+1}}$ if and only if $\{i_1,\ldots,i_{g+1}\}=\{i'_1,\ldots,i'_{g+1}\}$ or $\{i_1,\ldots,i_{g+1}\}\cup\{i'_1,\ldots,i'_{g+1}\}=\{1,\ldots,2g+2\}$ (*cf.* [Mu2], Chapter IIIa, Proposition 6.1), we have, by Lemma 4.5.4(ii) and the definition of the modular discriminant from Section 3.2, that

$$\prod_{\{i_1,\ldots,i_{g+1}\}} \|\vartheta\|(W_{i_1}+\cdots+W_{i_g}-W_{i_{g+1}})^4 = \|\varphi_g\|(X)\,,$$

where the product runs over the subsets of $\{1,\ldots,2g+2\}$ of cardinality $g+1$. Hence we have

$$\prod_{\{i_1,\ldots,i_g\}} \prod_{W\notin\{W_{i_1},\ldots,W_{i_g}\}} \|\vartheta\|(W_{i_1}+\cdots+W_{i_g}-W)^{2g-2} = \|\varphi_g\|(X)^{(g^2-1)/2}\,.$$

Plugging this in finishes the proof. □

Proof of Theorem 4.5.7. Let X be a hyperelliptic Riemann surface of genus $g\geq 2$ and fix an ordering $W_1,\ldots,W_{2g+2}$ of its Weierstrass points. A comparison of Proposition 4.5.8 with Theorem 3.1.3 immediately yields that

$$\prod_{\{i_1,\ldots,i_g\}} \|J\|(W_{i_1},\ldots,W_{i_g})^4 = \pi^{4gm}\|\varphi_g\|(X)^{g+1}\,,$$

where the product runs over the subsets of $\{1,\ldots,2g+2\}$ of cardinality g. Dividing left and right by the appropriate power of $\det \mathrm{Im}\tau$ and using the definitions of $\|J\|$ and $\mathcal{F}_g$ we find that

$$\prod_{\{\eta_1,\ldots,\eta_g,\eta_{g+1},\ldots,\eta_{2g+2}\}\in\mathcal{F}_g} |J(\eta_1,\ldots,\eta_g)(\tau)|^4 = \pi^{4gm}|\varphi_g(\tau)|^{g+1}\,,$$

where τ is a period matrix associated to X. Taking 4th roots and applying the maximum principle we find

$$\prod_{\{\eta_1,\ldots,\eta_g,\eta_{g+1},\ldots,\eta_{2g+2}\}\in\mathcal{F}_g} J(\eta_1,\ldots,\eta_g)(\tau) = \varepsilon\pi^{gm}\prod_{T\in\mathcal{T}} \vartheta[\eta_{T\circ U}](0;\tau)^{2g+2}$$

with ε a complex number of modulus 1. We find the right value of ε by considering Fourier expansions on the left and the right as in [Ig1], pp. 86–88. $\square$

4.6 Jacobi's derivative formula

The arguments leading to the proof of Theorem 4.5.7 specialise to the case $g = 1$ with only little modifications. In the present section we spell out the details, leading to a proof of Jacobi's derivative formula mentioned at the beginning of Section 4.5. In contrast to traditional proofs, which rely on special analytic or combinatorial properties of the theta function, such as the heat equation, the present proof gives insight into the algebraico-geometric structure behind the formula.

Proof of Jacobi's derivative formula. We consider the universal elliptic curve $p : \mathcal{U}_1 \to \mathcal{M}_1$. Let ω be the relative dualising sheaf of p. By Theorem 1.6.1, we have on $\mathcal{M}_1$ an isomorphism

$$\mu : (p_*\omega)^{\otimes 12} \xrightarrow{\sim} \langle \omega, \omega \rangle \, ,$$

which is unique up to sign. Now it is easy to see that the canonical homomorphism $p^* p_* \omega \to \omega$ is in fact an isomorphism; in particular ω is pulled back from the base. This implies that $\langle \omega, \omega \rangle$ is canonically isomorphic to the trivial line bundle $O_{\mathcal{M}_1}$, also as a metrised line bundle. This means that we have a canonical global trivialising section Ξ of $\langle \omega, \omega \rangle$, with unit length over the complex numbers. Consider next the line bundle $(p_*\omega)^{\otimes 12}$. It follows from Proposition 4.3.1 that this line bundle carries a canonical global trivialising section Λ, which we can write as

$$\Lambda = 2^4 \pi^{12} (\vartheta[\eta_2](0;\tau)\vartheta[\eta_3](0;\tau)\vartheta[\eta_4](0;\tau))^8 (dz)^{\otimes 12}$$

on the elliptic curve $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$. As in Lemma 4.4.1, we have $H^0(\mathcal{M}_1, \mathbb{G}_{\mathrm{m}}) = \{-1, +1\}$, hence, as in the proof of Proposition 4.4.2, we have $\mu(\Lambda) = \pm \Xi$. Now by Theorem 1.6.1 the norm of μ is equal to $(2\pi)^{-4} \exp(\delta)$. Further, the norm of Λ is $\|\Lambda\| = 2^4 \pi^{12} (\mathrm{Im}\tau)^6 |\vartheta[\eta_2](0;\tau)\vartheta[\eta_3](0;\tau)\vartheta[\eta_4](0;\tau)|^8$ and the norm of Ξ is $\|\Xi\| = 1$. Combining gives that

$$\pi^8 (\mathrm{Im}\tau)^6 |\vartheta[\eta_2](0;\tau)\vartheta[\eta_3](0;\tau)\vartheta[\eta_4](0;\tau)|^8 \exp(\delta(X)) = 1 \, .$$

By Theorem 2.1.3 and Proposition 2.3.3 we have for the delta-invariant

$$\exp(-\delta(X)) = (\mathrm{Im}\tau)^6 |\vartheta[\eta_1]'(0;\tau)|^8 \, ,$$

and plugging this in gives

$$|\vartheta[\eta_1]'(0;\tau)| = \pi |\vartheta[\eta_2](0;\tau)\vartheta[\eta_3](0;\tau)\vartheta[\eta_4](0;\tau)| \, .$$

By the maximum principle we find an equality of holomorphic functions

$$\vartheta[\eta_1]'(0;\tau) = \varepsilon \pi \vartheta[\eta_2](0;\tau)\vartheta[\eta_3](0;\tau)\vartheta[\eta_4](0;\tau)$$

of τ in the complex upper half plane, where ε is a complex constant of modulus 1. We find the right value $\varepsilon = -1$ by considering q-expansions as in [Mu2], Chapter I, §13. $\square$

Chapter 5

Elliptic curves

In this chapter we give a rather self-contained and fairly elementary discussion of the Arakelov theory of elliptic curves. Many results on the Arakelov theory of elliptic curves are already known by [Fa2] and Szpiro's paper [Sz], but our approach is different. In particular, we base our discussion on a projection formula for the Arakelov-Green function on Riemann surfaces of genus 1 related by an isogeny. From this formula we derive a projection formula for Arakelov intersections, as well as a formula for the so-called "energy of an isogeny". Both of these formulas seem new. In fact, the latter formula provides an answer to a question posed by Szpiro in [Sz].

Using the new results, we give alternative proofs of several of the earlier results. For example, we arrive at explicit formulas for the Arakelov-Green function on an elliptic curve, for the canonical norm in the holomorphic cotangent bundle, and for the self-intersection of a point. We also give an elementary proof of a recent result due to Autissier [Au] on the average height of the quotients of an elliptic curve by its cyclic subgroups of a fixed order.

5.1 Analytic projection formula

We start by studying the fundamental (1,1)-form μ with respect to isogenies. Let X and X' be Riemann surfaces of genus 1, and suppose that $f : X \to X'$ is an isogeny, say of degree N. Let μ_X and $\mu_{X'}$ be the fundamental (1,1)-forms of X and X', respectively.

Proposition 5.1.1. *(i) We have $f^*\mu_{X'} = N \cdot \mu_X$;*
(ii) the canonical isomorphism $f^* : H^0(X', \Omega^1_{X'}) \xrightarrow{\sim} H^0(X, \Omega^1_X)$ given by inclusion has norm $\sqrt{N}$.

Proof. We identify X with a complex torus $\mathbb{C}/\Lambda$, and obtain X' as the quotient of $\mathbb{C}/\Lambda$ by a finite subgroup Λ'/Λ. Hence we may identify X' with $\mathbb{C}/\Lambda'$. A small computation shows that the differentials $dz/\sqrt{\mathrm{vol}(\Lambda)}$ and $dz/\sqrt{\mathrm{vol}(\Lambda')}$ are orthonormal bases of $H^0(X, \Omega^1_X)$ and $H^0(X', \Omega^1_{X'})$, respectively. We obtain the proposition by observing that $N = \mathrm{vol}(\Lambda)/\mathrm{vol}(\Lambda')$. $\square$

Proposition 5.1.1 gives rise to a projection formula for the Arakelov-Green function.

Theorem 5.1.2. *(Analytic projection formula) Let X and X' be Riemann surfaces of genus 1 and let G_X and $G_{X'}$ be the Arakelov-Green functions of X and X', respectively. Suppose we have an isogeny $f : X \to X'$. Let D be a divisor on X'. Then the canonical isomorphism of line bundles*

$$f^*O_{X'}(D) \xrightarrow{\sim} O_X(f^*D)$$

is an isometry. In particular we have a projection formula: for any $P \in X$ the formula

$$G_X(f^*D, P) = G_{X'}(D, f(P))$$

holds.

Proof. Let N be the degree of f. By Proposition 5.1.1 we have

$$\mathrm{curv} f^* O_{X'}(D) = f^*(\mathrm{curv} O_{X'}(D)) = f^*((\deg D)\cdot \mu_{X'}) = N\cdot(\deg D)\cdot \mu_X = \deg(O_X(f^*D))\cdot \mu_X\,,$$

which means that $f^*O_{X'}(D)$ is an admissible line bundle on X. Hence by Proposition 1.1.3 we have $\|f^*(s_D)\|_{f^*O_{X'}(D)} = c\cdot \|s_{f^*D}\|_{O_X(f^*D)}$ for some constant c where s_D and s_{f^*D} are the canonical sections of $O_{X'}(D)$ and $O_X(f^*D)$, respectively. But since

$$\int_X \log\|f^*(s_D)\|_{f^*O_{X'}(D)}\cdot \mu_X = \frac{1}{N}\int_X \log\|f^*(s_D)\|_{f^*O_{X'}(D)}\cdot f^*\mu_{X'} = \int_{X'} \log\|s_D\|_{O_{X'}(D)}\cdot \mu_{X'} = 0\,,$$

this constant is equal to 1. □

5.2 Energy of an isogeny

At this point, we recall some notation from Section 2.3. Let τ be an element of the complex upper half plane, and write $q = \exp(2\pi i\tau)$. Then we have the eta-function $\eta(\tau) = q^{1/24}\prod_{k=1}^{\infty}(1-q^k)$ and the modular discriminant $\Delta(\tau) = \eta(\tau)^{24} = q\prod_{k=1}^{\infty}(1-q^k)^{24}$. The latter is the unique normalised cusp form of weight 12 on $\mathrm{SL}(2,\mathbb{Z})$. Suppose that we have a Riemann surface X of genus 1 identified with a complex torus $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$. Then we put $\|\eta\|(X) = (\mathrm{Im}\tau)^{1/4}\cdot|\eta(\tau)|$ and $\|\Delta\|(X) = \|\eta\|(X)^{24} = (\mathrm{Im}\tau)^6\cdot|\Delta(\tau)|$. These definitions do not depend on the choice of τ, and hence they define invariants of X.

In [Sz], Szpiro proves the following statement (*cf.* Théorème 1): let E and E' be semi-stable elliptic curves defined over a number field K, and suppose we have an isogeny $f : E \to E'$. Then the formula

$$\sum_{\sigma : K \hookrightarrow \mathbb{C}} \sum_{\substack{P_\sigma \in \mathrm{Ker} f_\sigma,\\ P_\sigma \neq 0}} \log G(0, P_\sigma) = \frac{[K:\mathbb{Q}]}{2}\log N + \sum_{\sigma : K\hookrightarrow \mathbb{C}} \log\frac{\|\eta\|(E'_\sigma)^2}{\|\eta\|(E_\sigma)^2}$$

holds, where N is the degree of f and where the sum is over the complex embeddings of K. Szpiro then asks whether a similar statement holds without the sum over the complex embeddings. The following theorem gives a positive answer to that question. The terminology "energy of an isogeny" is adopted from [Sz].

Theorem 5.2.1. *(Energy of an isogeny) Let X and X' be Riemann surfaces of genus 1. Suppose we have an isogeny $f : X \to X'$. Then we have*

$$\prod_{P\in \mathrm{Ker} f, P\neq 0} G(0,P) = \frac{\sqrt{N}\cdot\|\eta\|(X')^2}{\|\eta\|(X)^2}\,,$$

where N is the degree of f.

It is the purpose of the present section to prove Theorem 5.2.1. *En passant* we make the Arakelov-Green function and the canonical norm on the holomorphic cotangent bundle explicit, see Propositions 5.2.5 and 5.2.6. These formulas are also given in [Fa2], but the proofs there rely on a consideration of the eigenvalues and eigenfunctions of the Laplace operator. Our approach is more elementary.

Definition 5.2.2. Let X be a Riemann surface of genus 1. Let ω be a holomorphic differential of norm 1 in $H^0(X,\Omega_X^1)$. Then we put $A(X) := \|\omega\|_{\mathrm{Ar}}$ for the norm of ω in Ω_X^1.

Proposition 5.2.3. *Let $f : X \to X'$ be an isogeny of degree N. Then the formula*

$$\prod_{P \in \mathrm{Ker} f, P \neq 0} G(0,P) = \frac{\sqrt{N} \cdot A(X)}{A(X')}$$

holds.

Proof. Let ν be the norm of the isomorphism of line bundles $f^*\Omega^1_{X'} \tilde{\to} \Omega^1_X$ given by the usual inclusion. We will compute ν in two ways. First of all, consider an $\omega' \in H^0(X', \Omega^1_{X'})$ of norm 1, so that ω' has norm $A(X')$ in $\Omega^1_{X'}$. Then by Proposition 5.1.1 we have that $f^*(\omega')$ has norm $\sqrt{N}$ in $H^0(X, \Omega^1_X)$, hence it has norm $\sqrt{N} \cdot A(X)$ in Ω^1_X. This gives

$$\nu = \frac{\sqrt{N} \cdot A(X)}{A(X')} \, .$$

On the other hand, by Theorem 5.1.2, the canonical isomorphism $f^*(O_{X'}(0)) \tilde{\to} O_X(\mathrm{Ker} f)$ is an isometry. Tensoring with the isomorphism $f^*\Omega^1_{X'} \tilde{\to} \Omega^1_X$ gives an isomorphism

$$f^*(\Omega^1_{X'}(0)) \xrightarrow{\sim} \Omega^1_X(0) \otimes \bigotimes_{P \in \mathrm{Ker} f, P \neq 0} O_X(P)$$

of norm ν given in local coordinates by

$$f^*(\frac{dz}{z}) \mapsto \frac{dz}{z} \otimes s$$

where s is the canonical section of $\bigotimes_{P \in \mathrm{Ker} f, P \neq 0} O_X(P)$. By the definition of the canonical norm on the holomorphic cotangent bundle, the dz/z have norm 1, so we find

$$\nu = \prod_{P \in \mathrm{Ker} f, P \neq 0} G(0,P) \, .$$

Together with the earlier formula for ν this implies the proposition. $\square$

The following corollary seems to be well-known, see for instance [SU], Lemme 6.2.

Corollary 5.2.4. *Denote by $X[N]$ the kernel of the multiplication-by-N map $X \to X$. Then the formula*

$$\prod_{P \in X[N], P \neq 0} G(0,P) = N$$

holds.

Proof. Immediate from Proposition 5.2.3. $\square$

Let τ be an element of the complex upper half plane. We recall the identities

$$(a) \qquad \left(\exp(\pi i \tau/4) \cdot \vartheta(0;\tau)\vartheta(1/2;\tau)\vartheta(\tau/2;\tau)\right)^8 = 2^8 \cdot \Delta(\tau)$$

and

$$(b) \qquad \left(\exp(\pi i \tau/4) \cdot \frac{\partial \vartheta}{\partial z}\left(\frac{1+\tau}{2};\tau\right)\right)^8 = (2\pi)^8 \cdot \Delta(\tau) \, .$$

The first can be proved by the fact that the left hand side is a cusp form on $\mathrm{SL}(2,\mathbb{Z})$ of weight 12. The second follows then from the first by an application of Jacobi's derivative formula which we proved in Section 4.6.

Proposition 5.2.5. *(Faltings [Fa2]) Let X be a Riemann surface of genus 1, and write $X = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ with τ in the complex upper half plane. For the Arakelov-Green function G on X the formula*

$$G(0, z) = \frac{\|\vartheta\|(z + (1 + \tau)/2; \tau)}{\|\eta\|(X)}$$

holds.

Proof. It is not difficult to check that $\|\vartheta\|(z + (1 + \tau)/2)$ vanishes only at $z = 0$, with order 1. Also it is not difficult to check that $\partial_z\overline{\partial}_z \log \|\vartheta\|(z + (1 + \tau)/2)^2 = 2\pi i\mu_X$ for $z \neq 0$. By the defining properties of the Arakelov-Green function we have from this that $G(0, z) = c \cdot \|\vartheta\|(z + (1 + \tau)/2; \tau)$ where c is some constant. It remains to compute this constant. If we apply Corollary 5.2.4 with $N = 2$ we obtain

$$c^3 \cdot \|\vartheta\|(0; \tau)\|\vartheta\|(1/2; \tau)\|\vartheta\|(\tau/2; \tau) = G(0, 1/2)G(0, \tau/2)G(0, (1 + \tau)/2) = 2\,.$$

Combining this with identity (a) we obtain $c = \|\eta\|(X)^{-1}$. ☐

Proposition 5.2.6. *(Faltings [Fa2]) For the invariant $A(X)$, the formula*

$$A(X) = \frac{1}{(2\pi)\|\eta\|(X)^2}$$

holds.

Proof. We follow the argument from [Fa2]: writing $X = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ we can take $dz/\sqrt{\mathrm{Im}\tau}$ as an orthonormal basis of $H^0(X, \Omega^1_X)$. By the definition of the canonical metric on Ω^1_X we have $\|dz/\sqrt{\mathrm{Im}\tau}\|_{\mathrm{Ar}} = (\sqrt{\mathrm{Im}\tau})^{-1} \cdot \lim_{z\to 0} |z|/G(0, z)$. We obtain the required formula by using the explicit formula for $G(0, z)$ in Proposition 5.2.5 and the identity (b) mentioned above. ☐

Proof of Theorem 5.2.1. Immediate from Propositions 5.2.3 and 5.2.6. ☐

We conclude this section with a corollary, dealing with the value of the Arakelov-Green function on pairs of 2-torsion points.

Proposition 5.2.7. *Let X be a Riemann surface of genus 1 and suppose that $y^2 = 4x^3 - px - q =: f(x)$ is a Weierstrass equation for X. Write $f(x) = 4(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$. Let $P_1 = (\alpha_1, 0)$, $P_2 = (\alpha_2, 0)$ and $P_3 = (\alpha_3, 0)$. Then the formulas*

$$G(P_1, P_2)^{12} = \frac{16 \cdot |\alpha_1 - \alpha_2|^2}{|\alpha_1 - \alpha_3| \cdot |\alpha_2 - \alpha_3|}\,,$$

$$G(P_1, P_3)^{12} = \frac{16 \cdot |\alpha_1 - \alpha_3|^2}{|\alpha_1 - \alpha_2| \cdot |\alpha_3 - \alpha_2|}\,,$$

$$G(P_2, P_3)^{12} = \frac{16 \cdot |\alpha_2 - \alpha_3|^2}{|\alpha_2 - \alpha_1| \cdot |\alpha_3 - \alpha_1|}$$

hold.

Proof. This follows directly from an application of Thomae's formula Theorem 3.2.3 and the explicit formula for $G(0, z)$ in Proposition 5.2.5. ☐

We remark that this proposition has been obtained by Szpiro in [Sz] in the special case that X is the Riemann surface associated to a Frey curve $y^2 = x(x + a)(x - b)$, where a, b are non-zero integers with $2^4|a$ and $b \equiv -1 \mod 4$ (*cf.* [Sz], Section 1.3).

5.3 Arakelov projection formula

In this section we prove a projection formula for Arakelov intersections on arithmetic surfaces of genus 1. The essential idea is to use the analytic projection formula from Theorem 5.1.2; the rest of the proof is quite straightforward. We will use the Arakelov projection formula in Section 5.5.

Definition 5.3.1. Let $p : \mathcal{E} \to B$ and $p' : \mathcal{E}' \to B$ be arithmetic surfaces of genus 1, and suppose there exists a proper B-morphism $f : \mathcal{E} \to \mathcal{E}'$. Let D be an Arakelov divisor on $\mathcal{E}$, and write $D = D_{\mathrm{fin}} + \sum_\sigma \alpha_\sigma \cdot E_\sigma$. The pushforward f_*D of D is defined to be the Arakelov divisor $f_*D := f_*D_{\mathrm{fin}} + d \cdot \sum_\sigma \alpha_\sigma \cdot E'_\sigma$ on $\mathcal{E}'$, where f_*D_{fin} is the usual pushforward of the Weil divisor D_{fin}. Next let D' be an Arakelov divisor on $\mathcal{E}'$. The pullback f^*D' of D' is to be the Arakelov divisor $f^*D' := f^*D'_{\mathrm{fin}} + \sum_\sigma \alpha'_\sigma \cdot E_\sigma$ on $\mathcal{E}$, where $f^*D'_{\mathrm{fin}}$ is the pullback of the Weil divisor D'_{fin} on $\mathcal{E}'$, defined in the usual way using Cartier divisors.

Theorem 5.3.2. *(Arakelov projection formula) Let E and E' be elliptic curves defined over a number field K, and let $p : \mathcal{E} \to B$ and $p' : \mathcal{E}' \to B$ be arithmetic surfaces over the ring of integers of K with generic fibers isomorphic to E and E', respectively. Suppose we have an isogeny $f : E \to E'$, and suppose that f extends to a B-morphism $f : \mathcal{E} \to \mathcal{E}'$. Let D be an Arakelov divisor on $\mathcal{E}$ and let D' be an Arakelov divisor on $\mathcal{E}'$. Then the equality of intersection products $(f^*D', D) = (D', f_*D)$ holds.*

For the proof we need a lemma. Recall the definition of the principal Arakelov divisor (f) of a non-zero rational function f from Section 1.2.

Lemma 5.3.3. *Let g be a non-zero function in $K(E')$. (i) We have $(f^*g)_{\mathrm{inf}} = f^*(g)_{\mathrm{inf}}$. (ii) We have $f^*(g) = (f^*g)$, and hence f^* descends to a group homomorphism $\widehat{\mathrm{Cl}}(\mathcal{E}') \to \widehat{\mathrm{Cl}}(\mathcal{E})$.*

Proof. In order to prove (i), let σ be a complex embedding. Let N be the degree of f. Then we have $-\int_{E_\sigma} \log|f^*g|_\sigma \cdot \mu_{E_\sigma} = -\frac{1}{N}\int_{E_\sigma} \log|f^*g|_\sigma \cdot f^*\mu_{E'_\sigma} = -\int_{E'_\sigma} \log|g|_\sigma \cdot \mu_{E'_\sigma}$, and this means $(f^*g)_{\mathrm{inf}} = f^*(g)_{\mathrm{inf}}$. This gives (i). Next it is clear that $f^*(g)_{\mathrm{fin}} = (f^*g)_{\mathrm{fin}}$. Together with (i) this gives (ii). ∎

Remark 5.3.4. Under the canonical isomorphism $\widehat{\mathrm{Cl}} \xrightarrow{\sim} \widehat{\mathrm{Pic}}$ from Theorem 1.2.7, the group homomorphism $f^* : \widehat{\mathrm{Cl}}(\mathcal{E}') \to \widehat{\mathrm{Cl}}(\mathcal{E})$ is just the canonical group homomorphism $f^* : \widehat{\mathrm{Pic}}(\mathcal{E}') \to \widehat{\mathrm{Pic}}(\mathcal{E})$ defined by pullback of metrised line bundles.

Proof of Theorem 5.3.2. We may restrict to the case where both D and D' are Arakelov divisors with trivial contributions "at infinity". By the moving lemma on $\mathcal{E}'$ (*cf.* [Li], Corollary 9.1.10) we can find a function $g \in K(E')$ such that $D'' := D' + (g)_{\mathrm{fin}}$ and f_*D have no components in common. Obviously $D'' + (g)_{\mathrm{inf}}$ is Arakelov linearly equivalent to D', and hence by Lemma 5.3.3(ii) the pull-back $f^*D'' + f^*(g)_{\mathrm{inf}}$ is linearly equivalent to f^*D'. By Lemma 5.3.3(i) this means that $f^*D'' + (f^*g)_{\mathrm{inf}}$ is linearly equivalent to f^*D'. It is therefore sufficient to prove that $(f^*D''+(f^*g)_{\mathrm{inf}}, D) = (D''+(g)_{\mathrm{inf}}, f_*D)$. It is clear that $((f^*g)_{\mathrm{inf}}, D) = ((g)_{\mathrm{inf}}, f_*D)$, so it remains to prove that $(f^*D'', D) = (D'', f_*D)$. By the traditional projection formula (*cf.* [Li], Theorem 9.2.12 and Remark 9.2.13) we have $(f^*D'', D)_{\mathrm{fin}} = (D'', f_*D)_{\mathrm{fin}}$. For the contributions at infinity we can reduce to the case where D and D'' are sections of $\mathcal{E} \to B$ and $\mathcal{E}' \to B$, respectively. Let σ be a complex embedding of K. Let D_σ and D''_σ be the points corresponding to D and D'' on E_σ and E'_σ. Then for the local intersection at σ we have $(f^*D'', D)_\sigma = (D'', f_*D)_\sigma$ by the analytic projection formula from Proposition 5.1.2. The theorem follows. ∎

Remark 5.3.5. In general, an isogeny $f : E \to E'$ may not extend to a morphism $\mathcal{E} \to \mathcal{E}'$. However, if $\mathcal{E}'$ is a minimal arithmetic surface, then it contains the Néron model of E'/K, and hence by the universal property of the Néron model, any isogeny f extends. In any case we can achieve that f extends after blowing up finitely many closed points on $\mathcal{E}$.

The following corollary appears in Szpiro's paper [Sz].

Corollary 5.3.6. *(Szpiro [Sz]) Let D_1, D_2 be Arakelov divisors on $\mathcal{E}'$. Let N be the degree of f. Then the formula*

$$(f^*D_1, f^*D_2) = N \cdot (D_1, D_2)$$

holds.

Proof. It is not difficult to see (*cf.* [Li], Theorem 7.2.18 and Proposition 9.2.11) that $f_*f^*D_2 = N \cdot D_2$. Theorem 5.3.2 then gives $(f^*D_1, f^*D_2) = (D_1, f_*f^*D_2) = (D_1, N \cdot D_2) = N \cdot (D_1, D_2)$. □

5.4 Self-intersection of a point

Let $p : \mathcal{E} \to B$ be an arithmetic surface of genus 1. In the present section we compute the self-intersection (P, P) of a section of p.

Theorem 5.4.1. *(Szpiro [Sz]) Let E be a semi-stable elliptic curve over a number field K, and let $p : \mathcal{E} \to B$ be its regular minimal model over the ring of integers of K. Let $P : B \to \mathcal{E}$ be a section of p, and denote by $\Delta(E/K)$ the minimal discriminant ideal of E/K. Then the formula*

$$(P, P) = -\frac{1}{12} \log |N_{K/\mathbb{Q}}(\Delta(E/K))|$$

holds.

Before we give the proof, we recall two geometric results.

Proposition 5.4.2. *Let $p : \mathcal{E} \to B$ be a minimal arithmetic surface of genus 1. Then the canonical homomorphism $p^*p_*\omega_{\mathcal{E}/B} \to \omega_{\mathcal{E}/B}$ is an isomorphism.*

Proof. See [Li], Corollary 9.3.27. □

Proposition 5.4.3. *Let $p : \overline{\mathcal{U}}_1 \to \overline{\mathcal{M}}_1$ be the universal stable elliptic curve. Then there is a canonical isomorphism $(p_*\omega)^{\otimes 12} \xrightarrow{\sim} O(\Delta)$ of line bundles on $\overline{\mathcal{M}}_1$. Let Λ be the canonical global section of $(p_*\omega)^{\otimes 12}$ given by this isomorphism. Then for a Riemann surface $X = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ of genus 1 we can write $\Lambda = (2\pi)^{12}\Delta(\tau)(dz)^{\otimes 12}$.*

Proof. The canonical isomorphism follows from the theory of the Tate elliptic curve. Over $\mathcal{M}_1$, the section Λ is to be identified with the Λ from Proposition 4.3.1 above, which is also applicable in our case. The formula follows from the proof of Proposition 4.3.1. □

Proof of Proposition 5.4.1. By the adjunction formula we have to prove that $12 \widehat{\deg} P^*\omega_{\mathcal{E}/B} = \log |N_{K/\mathbb{Q}}(\Delta(E/K))|$. By Proposition 5.4.2 we have a canonical isomorphism $p_*\omega_{\mathcal{E}/B} \xrightarrow{\sim} P^*\omega_{\mathcal{E}/B}$, and what we will do is consider the image of the section $\Lambda_{\mathcal{E}/B}$, given by Proposition 5.4.3, in $(P^*\omega_{\mathcal{E}/B})^{\otimes 12}$, and compute its Arakelov degree. As is clear from the canonical isomorphism in Proposition 5.4.3, the finite places yield a contribution $\log |N_{K/\mathbb{Q}}(\Delta(E/K))|$. As to the infinite places, recall that by Proposition 5.2.6 we have $\|dz\|_{\mathrm{Ar}} = \sqrt{\mathrm{Im}\tau}/((2\pi)\|\eta\|(X)^2)$ for a Riemann surface $X = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ of genus 1. Together with the formula in Proposition 5.4.3 we obtain that $\|\Lambda_\sigma\|_{\mathrm{Ar}} = 1$ for each complex embedding σ, and hence the infinite contributions vanish. This gives the proposition. □

The proof of Theorem 5.4.1 given in [Sz] is much more involved. The above proof in fact answers a question raised in [Sz] on the norm $\|\Lambda\|_{\mathrm{Ar}}$ of Λ in $\Omega^{\otimes 12}$. Note that Proposition 5.4.2 also proves that $(\omega_{\mathcal{E}/B}, \omega_{\mathcal{E}/B}) = 0$ on a minimal arithmetic surface $p : \mathcal{E} \to B$ of genus 1, a fact observed by Faltings [Fa2] for the semi-stable case.

5.5 Average height of quotients

In this final section we study the average height of quotients of an elliptic curve by its cyclic subgroups of fixed order. Using our results from the previous sections, we give an alternative proof of a formula due to Autissier [Au]. A slightly less general result appears in [SU], and in fact our method is very much in the spirit of this latter paper. The main difference is perhaps that in our approach we do not need to consider the distribution of torsion points on the bad fibers. In fact we do not need any non-trivial arithmetic information at all; the main ingredients are the Arakelov projection formula from Theorem 5.3.2, the formula for the "energy of an isogeny" from Theorem 5.2.1, and the formula for the self-intersection of a point from Theorem 5.4.1. Amusingly, we shall mention at the end of this section how a purely arithmetic result, namely the injectivity of torsion, follows from our Arakelov-theoretic results.

We start with an explicit formula for the Faltings height $h_F(E)$ of an elliptic curve E (*cf.* Definition 1.5.6). This formula is certainly well-known, *cf.* [Si], Proposition 1.1.

Proposition 5.5.1. *Let E be a semi-stable elliptic curve over a number field K. Let $\Delta(E/K)$ be the minimal discriminant ideal of E/K. Then the formula*

$$h_F(E) = \frac{1}{[K:\mathbb{Q}]}\left(\frac{1}{12}\log|N_{K/\mathbb{Q}}(\Delta(E/K))| - \frac{1}{12}\sum_\sigma \log((2\pi)^{12}\|\Delta\|(E_\sigma))\right)$$

holds. Here the sum runs over the complex embeddings of K.

Proof. This follows directly from Proposition 5.4.3. □

Example 5.5.2. Proposition 5.5.1 makes it possible to compute the Faltings height of elliptic curves explicitly. The answers that we get in the case of elliptic curves with complex multiplication are given by the celebrated Chowla-Selberg formula. This is described for instance in [De1], §1. Let E be an elliptic curve defined over a number field K. Suppose that $E/\overline{K}$ has complex multiplication by the full ring of integers of an imaginary quadratic field F. It is well-known that E then has potentially everywhere good reduction. As a consequence, the formula

$$12\,[K:\mathbb{Q}]\,h_F(E) = -\sum_\sigma \log((2\pi)^{12}\cdot\|\Delta\|(\tau_\sigma))$$

holds, where the sum is over the complex embeddings of K. The Chowla-Selberg formula evaluates the right-hand side of this expression. Let $-D$ be the discriminant of F, let h be the class number of F, and let w be the number of roots of unity in F. The result is then that

$$h_F(E) = -\frac{1}{2}\log\left(\frac{\pi}{\sqrt{D}}\cdot\left(\prod_{0<m<D}\Gamma(m/D)^{(\frac{D}{m})}\right)^{w/2h}\right),$$

where $(\frac{D}{\cdot})$ is the Dirichlet character of conductor D. For instance, for the elliptic curve $E_1/\mathbb{Q}$ given by $y^2 = x^3 - x$ (with $j = 1728$), which has complex multiplication by the ring of integers of $F = \mathbb{Q}(\sqrt{-1})$, we have $D = 4$, $h = 1$ and $w = 4$ hence

$$h_F(E_1) = -\log\left(\frac{\Gamma(1/4)\cdot\sqrt{\pi}}{\Gamma(3/4)\cdot\sqrt{2}}\right) = -1.3105329259115095183...$$

For the elliptic curve $E_2/\mathbb{Q}$ given by $y^2 = x^3 - 1$ (with $j = 0$), which has complex multiplication by the ring of integers of $F = \mathbb{Q}(\sqrt{-3})$, we have $D = 3$, $h = 1$ and $w = 6$ hence

$$h_F(E_2) = -\frac{1}{2}\log\left(\left(\frac{\Gamma(1/3)}{\Gamma(2/3)}\right)^3 \cdot \frac{\pi}{\sqrt{3}}\right) = -1.3211174284280379150...$$

Actually, this is the infimum of h_F on $\mathcal{M}_1(\overline{\mathbb{Q}})$.

Now let's turn to the result of Autissier. First we introduce some notations. Let N be a positive integer. Then we denote by e_N the number of cyclic subgroups of order N on an elliptic curve defined over $\mathbb{C}$, which is

$$e_N = N\prod_{p|N}\left(1+\frac{1}{p}\right),$$

where the product is over the primes dividing N. Further we put

$$\lambda_N = \sum_{\substack{p|N \\ p^r\|N}} \frac{p^r-1}{p^{r-1}(p^2-1)}\log p,$$

where the notation $p^r\|N$ means that $p^r|N$ and $p^{r+1} \nmid N$. For an elliptic curve E and a finite subgroup C of E we denote by E^C the quotient of E by C.

In [SU] we find the following theorem.

Theorem 5.5.3. *(Szpiro-Ullmo [SU]) Let E be a semi-stable elliptic curve defined over a number field K. Suppose that E has no complex multiplication over $\overline{K}$ and that the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ acts transitively on the points of order N on E. Let C be a cyclic subgroup of order N on E. Then the formula*

$$h_F(E^C) = h_F(E) + \frac{1}{2}\log N - \lambda_N$$

holds.

One may wonder what one can say without the assumption that $\mathrm{Gal}(\overline{K}/K)$ acts transitively. In [Au] we find a proof of the following statement. The price we pay for dropping the assumption on $\mathrm{Gal}(\overline{K}/K)$ is that we can only deal with the average over all C.

Theorem 5.5.4. *(Autissier [Au]) Let E be an elliptic curve defined over a number field K. Then the formula*

$$\frac{1}{e_N}\sum_C h_F(E^C) = h_F(E) + \frac{1}{2}\log N - \lambda_N$$

holds, where the sum runs over the cyclic subgroups of E of order N.

In fact, this formula was already stated in [SU] under the restriction that N is squarefree. Autissier's proof uses the Hecke correspondence T_N and a generalised intersection theory for higher-dimensional arithmetic varieties. The disadvantage of this approach is that the analytic machinery needed to deal with the contributions at infinity becomes quite complicated. We will give a proof of Theorem 5.5.4 which is much more elementary. Besides this merit, we also think that the structure of the somewhat strange constant λ_N becomes more clear through our approach. It would be interesting to have a generalisation of Theorem 5.5.4 to abelian varieties of higher dimension.

Theorem 5.5.4 follows directly from the following two propositions, by using the explicit formula for h_F in Proposition 5.5.1. The next proposition occurs as Lemme 5.4 in [SU].

Proposition 5.5.5. *Let E be a semi-stable elliptic curve over a number field K and suppose that all N-torsion points are K-rational. Then one has*

$$\sum_C \left(\log|N_{K/\mathbb{Q}}(\Delta(E/K))| - \log|N_{K/\mathbb{Q}}(\Delta(E^C/K))|\right) = 0\,.$$

Here the sum runs over the cyclic subgroups of E of order N.

Proposition 5.5.6. *Let X be a Riemann surface of genus 1. Then*

$$\frac{1}{e_N}\sum_C\left(\frac{1}{12}\log\|\Delta\|(X)-\frac{1}{12}\log\|\Delta\|(X^C)\right)=\frac{1}{2}\log N-\lambda_N\,,$$

where the sum runs over the cyclic subgroups of X of order N.

Our first step is to reduce these two propositions to the following two:

Proposition 5.5.7. *Let E be a semi-stable elliptic curve over a number field K and suppose that all N-torsion points are K-rational. Extend all N-torsion points of E over the regular minimal model of E/K. Then one has*

$$\sum_C\sum_{\substack{Q\in C\\ Q\neq O}}(Q,O)=0\,,$$

where the first sum runs over the cyclic subgroups of E of order N, and the second sum runs over the non-zero points in C.

Proposition 5.5.8. *Let X be a Riemann surface of genus 1. Then one has*

$$\frac{1}{e_N}\sum_C\sum_{\substack{Q\in C\\ Q\neq 0}}\log G(Q,0)=\lambda_N\,.$$

Here the first sum runs over the cyclic subgroups of X of order N, and the second sum runs over the non-zero points in C.

The latter proposition is an improvement of Proposition 6.5 in [SU], which gives an analogous statement, but only with the left hand side summed over the complex embeddings of K, and divided by $[K:\mathbb{Q}]$. Our result holds in full generality for an arbitrary Riemann surface of genus 1.

Proof of Proposition 5.5.5 from Proposition 5.5.7. Let C be any cyclic subgroup of E, and let O' be the zero-section of E^C. Extend it over the minimal regular model of E^C/K. We then have

$$\frac{1}{12}\log|N_{K/\mathbb{Q}}(\Delta(E/K))|-\frac{1}{12}\log|N_{K/\mathbb{Q}}(\Delta(E^C/K))|=(O',O')-(O,O)$$

by Theorem 5.4.1. The latter is equal to $\sum_{\substack{Q\in C\\ Q\neq O}}(Q,O)$ by Theorem 5.3.2. Summing over all cyclic subgroups of E of order N and using Proposition 5.5.7 we find the result. $\square$

Proof of Proposition 5.5.6 from Proposition 5.5.8. By Theorem 5.2.1 we have for any subgroup C of X of order N that

$$\frac{1}{12}\log\|\Delta\|(X)-\frac{1}{12}\log\|\Delta\|(X^C)=\frac{1}{2}\log N-\sum_{\substack{Q\in C\\ Q\neq 0}}\log G(Q,0)\,.$$

The statement of Proposition 5.5.6 is then immediate from Proposition 5.5.8. $\square$

In order to prove Proposition 5.5.7, we make use of the following combinatorial lemma.

Lemma 5.5.9. *Let M be a positive integer with $M|N$. Let E be an elliptic curve defined over an algebraically closed field of characteristic zero. Then each cyclic subgroup of E of order M is contained in exactly e_N/e_M cyclic subgroups of order N.*

Proof. This follows easily by fixing a basis for the N-torsion and then considering the induced natural transitive left action of $\mathrm{SL}(2,\mathbb{Z})$ on the set of cyclic subgroups of order M and order N. ☐

Proof of Proposition 5.5.7. Let $\overline{E}[M]$ be the set of points of exact order M on E. By Lemma 5.5.9 we have

$$\sum_{C} \sum_{\substack{Q \in C \\ Q \neq O}} (Q,O) = \sum_{\substack{M|N \\ M>1}} \frac{e_N}{e_M} \sum_{Q \in \overline{E}[M]} (Q,O)\,.$$

We claim that for any positive integer M, we have $\sum_{Q \in \overline{E}[M]} (Q,O) = 0$. Indeed, we have

$$\sum_{Q \in E[M], Q \neq O} (Q,O) = 0$$

for all M by Theorem 5.3.2 and then the claim follows by Möbius inversion. ☐

Also for the proof of Proposition 5.5.8 we will need a lemma. For a Riemann surface X of genus 1, and $M > 1$ an integer, we put

$$t(M) = \sum_{Q \in \overline{X}[M]} \log G(Q,0)\,,$$

the sum running over the set $\overline{X}[M]$ of points of exact order M on X.

Part of the following lemma is also given in [SU], *cf.* Lemme 6.2.

Lemma 5.5.10. *We have*

$$t(p^r) = \log p$$

for any prime integer p and any positive integer r. Moreover we have $t(M) = 0$ for any positive integer M which is not a prime power.

Proof. By Corollary 5.2.4 we have

$$\sum_{Q \in X[M], Q \neq 0} \log G(Q,0) = \log M\,.$$

The lemma follows from this by Möbius inversion. ☐

Proof of Proposition 5.5.8. For any divisor $M|N$, let $\overline{X}[M]$ be the set of points of exact order M on X and let $t(M) = \sum_{Q \in \overline{X}[M]} \log G(Q,0)$ as in Lemma 5.5.10 where it is understood that $t(1) = 0$. Then by Lemma 5.5.9 we can write

$$\frac{1}{e_N} \sum_{C} \sum_{\substack{Q \in C \\ Q \neq 0}} \log G(Q,0) = \frac{1}{e_N} \sum_{M|N} \frac{e_N}{e_M} \cdot t(M)\,.$$

Lemma 5.5.10 gives us that

$$\frac{1}{e_N} \sum_{M|N} \frac{e_N}{e_M} \cdot t(M) = \sum_{\substack{p|N \\ p^r \| N}} \left(\frac{1}{e_p} + \cdots + \frac{1}{e_{p^r}} \right) \log p\,.$$

Finally note that $e_{p^k} = p^k(1 + 1/p)$ which gives

$$\frac{1}{e_p} + \cdots + \frac{1}{e_{p^r}} = \frac{p^r - 1}{p^{r-1}(p^2 - 1)}\,.$$

From this the result follows. ☐

Remark 5.5.11. An alternative proof of Proposition 5.5.6 can be given by classical methods using modular forms identities, see for instance [CT], Proposition VII.3.5(b) for the case that N is a prime, and [Au], Lemme 2.2 and Lemme 2.3 for the general case. We preferred to give an argument using Arakelov theory, indicating that Arakelov theory can sometimes be used to derive analytic results on Riemann surfaces in a short manner. We have seen another instance of this in Section 4.5, where we gave an Arakelov theoretic proof of a certain higher-dimensional modular forms identity.

We finish with a corollary from the results above. This corollary gives another interpretation to the constant λ_N.

Corollary 5.5.12. *Let E be a semi-stable elliptic curve over a number field K and suppose that all N-torsion points are K-rational. Extend these torsion points over the minimal regular model of E/K. Then one has*

$$\frac{1}{[K:\mathbb{Q}]}\frac{1}{e_N}\sum_C \sum_{\substack{Q\in C\\ Q\neq O}} (Q,O)_{\mathrm{fin}} = \lambda_N \, ,$$

where the first sum runs over the cyclic subgroups of E of order N, and the second sum runs over the non-zero points in C.

Proof. Let C be a finite cyclic subgroup of E. Note that by definition of the Arakelov intersection product

$$\sum_{\substack{Q\in C\\ Q\neq O}} (Q,O) = \sum_{\substack{Q\in C\\ Q\neq O}} (Q,O)_{\mathrm{fin}} - \sum_{\substack{Q\in C\\ Q\neq O}} \sum_{\sigma} \log G(Q^\sigma, 0) \, .$$

The corollary follows therefore easily from Proposition 5.5.7 and Proposition 5.5.8. ∎

Note that Corollary 5.5.12 is purely arithmetical in nature. It should also be possible to give a direct proof, but probably this would require a more *ad hoc* approach, making for instance a case distinction between the supersingular and the ordinary primes for E/K. Also note that Corollary 5.5.12 immediately gives the classical arithmetic result that, for any prime number p, the p-torsion points are injective on a fiber at a prime of characteristic different from p. Indeed, take $N = p$ in the formula from Corollary 5.5.12, then the right hand side is a rational multiple of $\log p$, and so the same holds for the left hand side. This means that the local intersections $(Q,O)_{\mathrm{fin}}$, which are always non-negative, are in fact zero at primes of characteristic different from p. Hence, each p-torsion point Q stays away from O on fibers above such primes. Of course the argument can be repeated with O replaced by any other p-torsion point.

Chapter 6

Numerical examples

As was explained in the Introduction, it is important to know how to calculate Arakelov invariants explicitly. Our Theorems 2.1.2 and 2.1.3 provide a solution to this problem. We illustrate this in the present chapter by computing examples of Arakelov invariants of hyperelliptic curves of small genus. In Section 6.1 we say some words on implementation. In Section 6.2 we focus on curves of genus 2. The computational aspects of this case are well-documented in [BMM]. Our approach in Section 6.2 will be different, but we do not pretend to be able to attain significantly better results. In Section 6.3 we consider a hyperelliptic curve of genus 3. In particular we find an explicit result for its delta-invariant. As far as we know, no explicit values of Arakelov invariants in genus 3 have been obtained so far, and it seems that the method and results in Section 6.3 are new.

6.1 Implementation

The difficulties in computing Arakelov invariants are usually caused by the analytic contributions at infinity. In this section we explain what we need to compute exactly, and how one can do this, given the results in this thesis.

Let X be a compact and connected Riemann surface of genus $g > 0$. First of all we need a period matrix $(\Omega_1|\Omega_2)$ for X. It is well-known that if X has many automorphisms, it is possible to compute such a period matrix purely theoretically. For example, there is a beautiful theory dealing with periods of elliptic curves with complex multiplication, as we saw in the previous chapter. An exact period matrix for the genus 2 Riemann surface associated to the equation $y^2 + y = x^5$, which visibly admits at least 10 automorphisms, was given in [BMM].

Next, when exact computations turn out not to be possible, one can often resort to a long tradition going back at least to Gauss which is concerned with finding algorithms to give rapidly converging series of approximations to periods. These algorithms can be very efficient for special types of curves. In general, however, there is no other method than to approximate the occurring line integrals directly. If one does this, one has various numerical integration methods at one's disposal, and nowadays many of these have been implemented in computer algebra packages such as Maple or Mathematica. These allow one to approximate periods very efficiently.

Once one has a period matrix, one has the associated matrix $\tau = \Omega_1^{-1}\Omega_2$ in the Siegel upper half space of degree g and if the period matrix was on the basis $\{\omega_1, \ldots, \omega_g\}$ of $H^0(X, \Omega_X^1)$, we also find the length of $\omega_1 \wedge \ldots \wedge \omega_g$ with respect to the Faltings metric on $\wedge^g H^0(X, \Omega_X^1)$, by the formula

$$\|\omega_1 \wedge \ldots \wedge \omega_g\|^2 = (\det \mathrm{Im}\tau) \cdot |\det \Omega_1|^2$$

from Proposition 1.4.10. These results allow one to calculate the analytic contributions to the

Faltings height of a curve.

Next we want to calculate the delta-invariant and certain values of the Arakelov-Green function. These we need in order to be able to calculate Arakelov intersection numbers, such as the self-intersection of the relative dualising sheaf, or the height of a rational point. A suitable formula for the self-intersection of the relative dualising sheaf follows for instance from the proof of Proposition 2.5.4.

As is clear from Theorems 2.1.2, 2.1.3 and 2.2.8, we need to be able to calculate certain values of the function $\|\vartheta\|$ on $\mathrm{Pic}_{g-1}(X)$ and of the function $\|J\|$ on $\mathrm{Sym}^g(X)$, but also we need to calculate the integral

$$\log S(X) = -\int_X \log \|\vartheta\|(gP - Q) \cdot \mu(P)$$

over the Riemann surface X.

The first problem is not difficult by the explicit formulas for $\|\vartheta\|$ and $\|J\|$ given in Chapter 1. We work with the usual identification

$$\mathrm{Pic}_{g-1}(X) \xrightarrow{\sim} \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g \quad , \quad \sum m_k P_k \mapsto \sum m_k \int_{P_0}^{P_k} (\eta_1, \ldots, \eta_g) + \kappa(P_0) \, ,$$

with $\kappa(P_0)$ the Riemann vector for a base point P_0. Here the basis $\{\eta_1, \ldots, \eta_g\}$ is given as $\{\eta_1, \ldots, \eta_g\} = \{\omega_1, \ldots, \omega_g\} \cdot {}^t\Omega_1^{-1}$, and the Riemann vector $\kappa(P_0) = (\kappa(P_0)_1, \ldots, \kappa(P_0)_g)$ can be made explicit by the classical formula

$$\kappa(P_0)_k = \frac{1 + \tau_{kk}}{2} - \sum_{l \neq k} \int_{A_l} \eta_l(x) \int_{P_0}^{x} \eta_k \quad \text{for } k = 1, \ldots, g \, ,$$

see [Fay], p. 43. The $A_1, \ldots, A_g$ are the A-chains in homology leading to the part Ω_1 of the period matrix. Using the explicit formulas in Chapter 1 it is not difficult to carry out an *a priori* investigation which shows how many terms in the defining series for ϑ and $\frac{\partial \vartheta}{\partial z}$ we have to compute in order to approximate a value of $\|\vartheta\|$ or $\|J\|$ with a prescribed accuracy.

The second problem, to calculate the integral, is more difficult. First of all, one needs to make the form μ explicit. This can be done using our basis $\{\omega_1, \ldots, \omega_g\}$ of holomorphic differentials: it follows from the definition and Riemann's bilinear relations that if one puts $h = \left(\overline{\Omega_1}(\mathrm{Im}\tau)^t\Omega_1\right)^{-1}$ then the form μ can be written as $\mu = \frac{i}{2g}\sum_{k,l=1}^{g} h_{kl} \cdot \omega_k \wedge \overline{\omega}_l$. Using a local coordinate and writing out the differentials $\omega_1, \ldots, \omega_g$ in this local coordinate one next tries to convert the integral into an integral over a domain in $\mathbb{C}$, using the standard euclidean coordinates. The main problem is, however, that the integrand has singularities at the Weierstrass points of X. This means that any numerical approximation has to take special care of these points. If the weights of the Weierstrass points are not too large, one can perhaps safely resort to the defining equation of $\log S(X)$. Otherwise, one probably does better by using the formula in Proposition 2.2.6, which involves a similar integral, but this integral has only a singularity at the chosen point P, and the order of vanishing of $\|\vartheta\|(gP-Q)$ at $Q = P$ is equal to g. However, one has to note that the error produced in calculating the integral will be multiplied by g^2 if one wants to obtain $\log S(X)$ in this way. In the computer algebra package Mathematica, it is possible to specify the points in an integration domain at which the evaluation of an integral needs special care, for instance because of the presence of logarithmic singularities in the integrand. There are special packages available particulary suited for integrands with logarithmic singularities, also in 2 dimensions.

Let's make the above more explicit in the case of hyperelliptic Riemann surfaces, which seems the easiest case from the computational point of view. Our numerical examples in Sections 6.2 and 6.3 below deal with this case. Suppose that we deal with a hyperelliptic Riemann surface X of genus $g \geq 2$ given by an equation $y^2 = f(x)$ with $f(x) \in \mathbb{C}[x]$ separable of degree $2g+1$. Fix an ordering of the roots of f. Recall that in [Mu2], Chapter IIIa, §5 a traditional and canonical way is given to build a symplectic basis $\{A_1, \ldots, A_g, B_1, \ldots, B_g\}$ for the homology of X. We take this basis as a starting point, and with Mathematica we compute the periods of, say, the differentials $\omega_1 = dx/y, \ldots, \omega_g = x^{g-1}dx/y$. This involves making appropriate branch cuts in $\mathbb{C}$, and then taking line integrals over paths that become the loops $A_1, \ldots, B_g$ on the 2-sheeted cover X of $\mathbb{P}^1$, reversing the orientation each time one crosses a branch cut. The line integrals involved in the Abel-Jacobi map are carried out in a similar way. We only still need the Riemann vector, but this is done in [Mu2], Chapter IIIa, §5: if we take ∞ as a base-point on X, then κ is given by $\kappa = \kappa_1 + \tau \cdot \kappa_2 \mod \mathbb{Z}^g + \tau\mathbb{Z}^g$ with $\kappa_1 = \left(\frac{g}{2}, \frac{g-1}{2}, \ldots, 1, \frac{1}{2}\right)$ and $\kappa_2 = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$. We will turn to specific details concerning the computation of $\log S(X)$ in the sections below.

6.2 Example with $g = 2$

In broad lines, the computational aspects of Arakelov theory for genus 2 curves have been discussed already in [BMM]. For concrete calculations, however, the authors specialise to the case of semi-stable arithmetic surfaces whose singular fibers are irreducible curves with a single double point, *cf.* Section 3 of [BMM]. We want to give formulas for the Arakelov invariants of an arbitrary semi-stable arithmetic surface of genus 2. Although not worked out in detail in [BMM], it is certainly well-known among experts how to do this.

For a Riemann surface X of genus 2, we denote by $\|\Delta_2'\|(X)$ the invariant of X defined in Section 3.8. This is the $\|\Delta_2\|(X)$ occurring in [BMM].

Proposition 6.2.1. *Let $p : \mathcal{X} \to B = \mathrm{Spec}(R)$ be a semi-stable arithmetic surface of genus 2 with good reduction at all primes dividing 2. Then the formulas*

$$10 \,\widehat{\deg} \det p_* \omega_{\mathcal{X}/B} = \sum_b \varepsilon_b \delta_b \log \#k(b) - \sum_\sigma \log((2\pi)^{20} \|\Delta_2'\|(X_\sigma))$$

and

$$(\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) = \sum_b (\frac{6}{5}\varepsilon_b - 1)\delta_b \log \# \log k(b) + \frac{1}{20} \sum_\sigma \log \|\Delta_2'\|(X_\sigma) + 4 \sum_\sigma \log S(X_\sigma)$$

hold, where b runs through the closed points of B and where $\varepsilon_b = 2$ if the stable geometric fiber at b is the union of two curves of genus 1 meeting at a single point, and $\varepsilon_b = 1$ otherwise.

Proof. We can assume that the generic fiber of $\mathcal{X}$ is given by an equation $y^2 = f(x)$, with $f(x)$ a separable polynomial of degree 6 defined over the quotient field of R. Let D be the discriminant of f. In [Ue], Proposition 2.1 it is shown that the element $\Lambda_{\mathcal{X}/B} = D \cdot (dx/y \wedge xdx/y)^{\otimes 10}$ defines a rational section of $(\det p_* \omega_{\mathcal{X}/B})^{\otimes 10}$ independent of the choice of equation $y^2 = f(x)$. By an argument as in Lemma 4.3.1 to deal with the infinite contributions we obtain

$$10 \,\widehat{\deg} \det p_* \omega_{\mathcal{X}/B} = \sum_b d_b \log \#k(b) - \sum_\sigma \log((2\pi)^{20} \|\Delta_2'\|(X_\sigma)) \,,$$

where $d_b = \mathrm{ord}_b(\Lambda_{\mathcal{X}/B})$. According to the Table in §5 of [Ue] one has $d_b = \varepsilon_b \delta_b$ with ε_b as in the proposition. This gives the first formula. The second formula follows from Noether's formula, where we eliminate the factor $\sum_\sigma \delta(X_\sigma)$ by using Corollary 3.1.5. □

Let us turn to a concrete example. We take smooth projective curves X_t given by the equation

$$y^2 + (x^2 + 1)y = x^5 + t \,,$$

with $t \in \mathbb{Z}$. One can check that the X_t are curves of genus 2 defined over $\mathbb{Q}$ and having good reduction at 2. Moreover, if $t \not\equiv 3 \mod 7$ then X_t has semi-stable reduction over $\mathbb{Q}$. Contrary to the family of curves considered in Section 3 of [BMM], various types of reduction will occur.

Let us specialise for example to $t = 7$. We find (in the standard Kodaira notation employed in [Ue]) reduction I_{1-1-0} (an irreducible curve with 2 double points) at 3, reduction I_{2-0-0} (a union of a smooth curve of genus 1 and a $\mathbb{P}^1$ of self-intersection -2) at 5, and reduction I_{1-0-0} (an irreducible curve with a single double point) at 29 and 339617. Thus, $\delta_3 = \delta_5 = 2$, $\delta_{29} = \delta_{339617} = 1$ and all ε's are 1.

Let us proceed by computing the Arakelov invariants of $X = X_7$. We take an equation $y^2 = f(x)$ for X with f monic and separable of degree 5. We compute the period matrix $(\Omega_1|\Omega_2)$ on the differentials dx/y and xdx/y as described in Section 6.1. We obtain

$$\|\Delta_2'\|(X) = 2.067079790957566... \cdot 10^{-5}$$

With Proposition 6.2.1 we find

$$h_F(X) = -0.44517827222228057...$$

Using Theorem 3.1.4 we compute

$$\log T(X) = -3.9806368335392663...$$

In order to calculate $\log S(X)$ we make use of the formula

$$\log S(X) = -4 \int_X \log \|\vartheta\|(2P - Q) \cdot \mu(Q) + \frac{1}{2} \sum_{W \in \mathcal{W}} \log \|\vartheta\|(2P - W)$$

derived from Corollary 2.2.6. We do this since the integrand in the defining equation of $\log S(X)$ diverges at infinity. Write $x = u + iv$ with u, v real. We want to express μ in terms of the coordinates u, v. This is done by the following lemma.

Lemma 6.2.2. *Let h be the matrix given by $h = \left(\overline{\Omega_1}(\mathrm{Im}\tau)^t\Omega_1\right)^{-1}$. Then we can write*

$$\mu = (h_{11} + 2h_{12}u + h_{22}(u^2 + v^2)) \cdot \frac{dudv}{2|f|}$$

in the coordinates u, v.

Proof. Let $\omega_k = x^{k-1}dx/y$ for $k = 1, 2$. As we have noted above, the form μ is given by $\mu = \frac{i}{4}\sum_{k,l=1}^{2} h_{kl} \cdot \omega_k \wedge \overline{\omega_l}$. Expanding this expression gives the result, where we note that the matrix h is real symmetric, since our defining equation for X is defined over the real numbers. □

We can now carry out the integral, choosing an arbitrary point P and taking care of the singularity of the integrand at this point P. We find the approximation

$$\log S(X) = 0.77...$$

leading to

$$\delta(X) = -16.69...$$

by Theorem 2.1.3 and finally to

$$e(X) = 4.53...$$

by Proposition 6.2.1.

We have checked the computation by also calculating the invariant $\log \|H\|(X)$ and using the formulas in Section 3.8. It turns out that calculating the invariant $\log \|H\|$ is done much faster by Mathematica. Hence, it seems that for the computations on the analytic side in genus 2 it is better to stick to the approach in [BMM].

In [BMM] the curve $Y/\mathbb{Q}$ given by $y^2 + y = x^5$ is discussed. The results imply that

$$\|\Delta_2'\|(Y) = 2.07046497... \cdot 10^{-5}$$

and

$$\delta(Y) = -16.68...$$

The reader will notice that these values are rather close to the values for $\|\Delta_2'\|(X)$ and $\delta(X)$ found above. This is no coincidence: a calculation shows that the family X_t over $\mathbb{P}^1(\mathbb{C})$ has potentially good reduction at infinity, with smooth fiber isomorphic to Y.

Using Proposition 6.2.1 and the fact that Y has potentially everywhere good reduction, one finds (as in [BMM])

$$h_F(Y) = -2.597239125..., \quad e(Y) = 0.2152...$$

On the other hand, for $t \in \mathbb{Z}$ one finds that $h_F(X_t)$ and $e(X_t)$ tend to infinity as $|t|$ tends to infinity. This illustrates the complicated behaviour of the functions h_F and e on the moduli space of curves.

Finally, we remark that a PARI program for computing the reduction and the potential stable reduction of curves of genus 2 defined over $\mathbb{Q}$ is available at the homepage of Qing Liu.

6.3 Example with $g = 3$

In this section we turn again to the methods developed in Section 3 of [BMM]. We generalise some of the results there to hyperelliptic curves of higher genera, and conclude with a numerical example in genus 3.

First of all, we prove a result on the self-intersection of the relative dualising sheaf. Let $p : \mathcal{X} \to B$ be a semi-stable arithmetic surface whose generic fiber is a hyperelliptic curve of genus $g \geq 2$. According to [DM], Theorem 1.11, the hyperelliptic involution on the generic fiber extends uniquely to an involution $\sigma \in \mathrm{Aut}_B(\mathcal{X})$.

Proposition 6.3.1. *Assume that $p : \mathcal{X} \to B$ has two σ-invariant sections $P, Q : B \to \mathcal{X}$. Assume furthermore that the fibers of p are irreducible. Then the formula*

$$(\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) = -4g(g-1) \cdot (P, Q)$$

holds.

Proof. We follow the argument in [BMM], Section 1.3. Let U be the largest open subset of B over which p is smooth. According to Lemma 4.2.1, the line bundle $\omega_{\mathcal{X}/B} \otimes O_{\mathcal{X}}(-(2g-2)P) \otimes p^*\langle P, P\rangle^{\otimes(2g-1)}$ has a nowhere vanishing section s when restricted to $\mathcal{X}_U$. Thus, this s can be seen as a rational section of that same bundle on $\mathcal{X}$. Let V_P its divisor. Its support is disjoint with $\mathcal{X}_U$, and we have a canonical isomorphism

$$\omega_{\mathcal{X}/B} \xrightarrow{\sim} O_{\mathcal{X}}((2g-2)P) \otimes p^*\langle P, P\rangle^{\otimes -(2g-1)} \otimes O_{\mathcal{X}}(V_P) \, .$$

Pulling back along P we find a canonical isomorphism $\langle \omega_{\mathcal{X}/B}, P\rangle \xrightarrow{\sim} \langle P, P\rangle^{\otimes -1} \otimes \langle P, O_{\mathcal{X}}(V_P)\rangle$, extending the canonical adjunction isomorphism $\langle \omega_{\mathcal{X}/B}, P\rangle_U \xrightarrow{\sim} \langle P, P\rangle_U^{\otimes -1}$ over U. But we know the adjunction isomorphism extends over B, so we must have that $\langle P, O_{\mathcal{X}}(V_P)\rangle$ is trivial. Since $\mathcal{X}$

is normal and since by assumption all fibers of p are irreducible, we find that $V_P = 0$. The formula follows then by a calculation as in the proof of Corollary 4.2.2 above. $\square$

Let K be a number field, and let A be its ring of integers. Let $F \in A[x]$ be monic of degree $2g+1$ with $F(0)$ and $F(1)$ a unit in A. Put $R(x) = x(x-1) + 4F(x)$. Suppose that the following conditions hold for R: (i) the discriminant D of R is non-zero; (ii) for every prime $\wp$ of A of residue characteristic $\neq 2$ we have $v_\wp(D) = 0$ or 1; (iii) if $\mathrm{char}(\wp) \neq 2$ and $v_\wp(D) = 1$, then $R \pmod{\wp}$ has a unique multiple root, and its multiplicity is 2. As in [BMM], Section 3 one may then prove the following statement.

Proposition 6.3.2. *The equation*

$$C_F \,:\, y^2 = x(x-1)R(x)$$

defines a hyperelliptic curve of genus g over K. It extends to a semi-stable arithmetic surface $p : \mathcal{X} \to B = \mathrm{Spec}(A)$. We have that $\mathcal{X}$ has bad reduction at $\wp$ if and only if $\mathrm{char}(\wp) \neq 2$ and $v_\wp(D) = 1$. In this case, the bad fiber is an irreducible curve with a single double point. The differentials $dx/y, \ldots, x^{g-1}dx/y$ form a basis of the O_B-module $p_*\omega_{\mathcal{X}/B}$. The points W_0, W_1 on C_F given by $x = 0$ and $x = 1$ extend to disjoint σ-invariant sections of p.

As for the Arakelov invariants of C_F, we find from this the following result.

Proposition 6.3.3. *At a complex embedding $\sigma : K \hookrightarrow \mathbb{C}$, let $\Omega_\sigma = (\Omega_{1\sigma}|\Omega_{2\sigma})$ be a period matrix for the Riemann surface corresponding to $C_F \otimes_{\sigma,K} \mathbb{C}$, formed on the basis $dx/y, \ldots, x^{g-1}dx/y$. Further, let $\tau_\sigma = \Omega_{1\sigma}^{-1}\Omega_{2\sigma}$. Then*

$$\widehat{\deg} \det p_*\omega_{\mathcal{X}/B} = -\frac{1}{2}\sum_\sigma \log\left(|\det \Omega_{1\sigma}|^2 (\det \mathrm{Im}\tau_\sigma)\right) \,,$$

where the sum runs over the complex embeddings of K. Further, the formula

$$(\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) = 4g(g-1)\sum_\sigma \log G_\sigma(W_0, W_1)$$

holds.

Proof. The first statement follows by Lemma 1.4.1 and Proposition 6.3.2. The second follows from Proposition 6.3.1 and Proposition 6.3.2. $\square$

For our numerical example, we choose the polynomial $F(x) = x^5 + 6x^4 + 4x^3 - 6x^2 - 5x - 1$ defined over $\mathbb{Q}$. Then the corresponding $R(x) = x(x-1) + 4F(x)$ satisfies the conditions described above. The corresponding hyperelliptic curve (which we will call X from now on) of genus 3 has bad reduction at the primes $p = 37, p = 701$ and $p = 14717$. An equation is given by

$$X \,:\, y^2 = x(x-1)(4x^5 + 24x^4 + 16x^3 - 23x^2 - 21x - 4)\,.$$

We choose an ordering of the Weierstrass points of X. We construct from this a canonical symplectic basis of the homology of (the Riemann surface corresponding to) X. Using Mathematica, we compute the periods of the differentials $dx/y, xdx/y, x^2dx/y$. This leads, by Proposition 6.3.3, to the numerical approximation

$$h_F(X) = -1.280295247656532068...$$

With Theorem 3.1.4 we find the following numerical approximation to $\log T(X)$:

$$\log T(X) = -4.44361200473681284...$$

It remains then to calculate the invariant $\log S(X)$. Again we compute it by using Corollary 2.2.6. Write $x = u + iv$ with u, v real. The analogue of Lemma 6.2.2 is as follows, with basically the same proof.

Lemma 6.3.4. *Let h be the 3×3-matrix given by $h = \left(\overline{\Omega}_1(\mathrm{Im}\tau)^t\Omega_1\right)^{-1}$. Then we can write*

$$\mu = \left(h_{11} + 2h_{12}u + 2h_{13}(u^2 - v^2) + h_{22}(u^2 + v^2) + 2h_{23}u(u^2 + v^2) + h_{33}(u^2 + v^2)^2\right) \cdot \frac{dudv}{3|f|}$$

in the coordinates u, v.

Using this, and taking care of the singularities of the integrand, we find the approximation

$$\log S(X) = 17.57...$$

In order to check this result, we have taken several choices for P. Also, to exclude a possible systematic error, we have checked that μ integrates to 1 over X.

By Theorem 2.1.3 we have

$$\delta(X) = -33.40...$$

and using Theorem 2.1.2 we can approximate, by taking $Q = W_1$ and letting P approach W_0,

$$G(W_0, W_1) = 2.33...$$

By Proposition 6.3.3 we finally find

$$e(X) = 20.32...$$

The running times of the computations were negligible, except for the computation of the integral involved in $\log S(X)$, which took about 7 hours on the author's laptop.

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Curriculum vitae

After finishing the Christelijk Lyceum of Delft in 1994, the author of the present thesis went to Leiden and started studying mathematics and astronomy. In 1995 he completed the Propedeuse in both subjects. He graduated in mathematics in 1999, and attended the MRI organised Master Class "Arithmetic Algebraic Geometry" in the next academic year. Meanwhile he continued his philosophy studies in Leiden, and graduated in 2001. In that same year the author started working on his Ph.D. project under the supervision of Gerard van der Geer. The present thesis is the result of his work done in that project. In October and November 2004 the author worked at the Institut des Hautes Études Scientifiques in Bures-sur-Yvette. After the defense of his thesis he will take up a two-year post-doc position at the University of Leiden, under the direction of Bas Edixhoven.