The generalized Sato-Tate conjecture for the twists of the Fermat quartic curve.

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The original Sato-Tate conjecture.

Let

$$E:\ y^2 = x^3 + ax + b$$

be an elliptic curve defined over a number field k (finite extension of $\mathbb{Q}$) and without complex multiplication ($End_{\bar{k}}(E) = \mathbb{Z}$). Let $\mathfrak{p}$ be a prime of k of good reduction of E. And let $Frob_{\mathfrak{p}}$ be any Frobenius element in $G_k := Gal\left(\bar{k}/k\right)$ acting on the Tate module

$$V_l(E) = T_l(E) \otimes \mathbb{Q}_l \simeq \mathbb{Q}_l \times \mathbb{Q}_l.$$

The characteristic polynomial of that endomorphism is

$$L_{\mathfrak{p}}(E/k,\ T) = 1 - a_1(\mathfrak{p})T + qT^2.$$

Where $q = N\mathfrak{p}$ and $a_1(\mathfrak{p}) \in \mathbb{Z}$.

The original Sato-Tate conjecture.

Moreover,

$$tr\left(Frob_{\mathfrak{p}}\right) = a_1\left(\mathfrak{p}\right) = 1 + q - \left|E\left(\mathbb{F}_q\right)\right|$$

and the two zeros α, $\bar{\alpha}$ of $L_{\mathfrak{p}}\left(E/k,\ T\right) = 1 - a_1(\mathfrak{p})T + qT^2$ have norm $\left(N\mathfrak{p}\right)^{-1/2}$ and $a_1(\mathfrak{p})/q^{1/2} \in [-2, 2]$. One can think of

$$\theta_{\mathfrak{p}} := arccos\frac{a_1(\mathfrak{p})}{2q^{1/2}}$$

as a random variable on the set of primes of good reduction of E taking values on $[0, \pi]$. We will call the distribution of this random variable the Sato-Tate distribution of the elliptic curve E.

The original Sato-Tate conjecture.

Conjecture *(Sato-Tate)*: The random variable $\theta_{\mathfrak{p}}$ is equidistributed with respect to the measure $\frac{2}{\pi} sin^2\theta d\theta$ on $[0, \pi]$.

An equivalent formulation is:

Conjecture *(Sato-Tate)*: the distribution of the polynomials $L_{\mathfrak{p}}\left(E/k, q^{-1/2}T\right)$ corresponds to the distribution, with the Haar measure, of the characteristic polynomial of a random matrix in $USp\,(2)$.

The conjecture is proven only when k is totally real.

The original Sato-Tate conjecture.

A natural question is asking what happens when E has complex multipication.

- If E has CM defined over k, one takes the uniform measure on the interval $[0, \pi]$, or equivalently the Haar measure on the group $U(1)$.
- If E has CM not defined over k, one takes half of the uniform measure plus half of a discrete measure concentrated at $\pi/2$ on the interval $[0, \pi]$ (this is because half of the times the elliptic curve happens to be supersingular). Or equivalently, one takes the Haar measure on the normalizer group of $U(1)$ in $SU(2)$.

The generalized Sato-Tate conjecture.

The next natural question is asking what happens when one considers abelian varieties of arbitrary dimension g.
Let A/k be an abelian variety of dimension g. And let

$$L_{\mathfrak{p}}(A/k,\ T) = \sum_{i=0}^{2g} (-1)^i a_i(\mathfrak{p})\, T^i$$

be its local factor at a prime $\mathfrak{p}$ of k of good reduction. One can think now of a_i as a random variable on the set of primes of good reduction of A taking values on $\left[-q^{i/2}\binom{2g}{i},\ q^{i/2}\binom{2g}{i}\right]$ and of

$$\theta_i(\mathfrak{p}) = arccos\left(\frac{\bar{a}_i}{\binom{2g}{i} q^{i/2}}\right)$$

as a random variable taking values on $[0,\ \pi]$.

The generalized Sato-Tate conjecture.

Let us now define the so called Sato-Tate group associated to a given abelian variety.
The action of G_k on $V_l(A)$ defines a continuous homomorphism:

$$\rho_l : G_k \longrightarrow GSp_{2g}(\mathbb{Q}_l),$$

we define $G_l := \rho_l(G_k)$ and we denote by G_l^{Zar} its Zariski closure and we write $G_{l,i}^{Zar} := G_l^{Zar} \otimes_i \mathbb{C}$, where we have fixed an embedding $i : \mathbb{Q}_l \hookrightarrow \mathbb{C}$. Define a continuous map $N : G_{l,i}^{Zar} \longrightarrow \mathbb{C}^*$ by $N\left(\rho_l\left(Frob_{\mathfrak{p}}^{-1}\right)\right) = N\mathfrak{p}$.

Definition

The Sato-Tate group $ST_k(A)$ is a maximal Zariski compact algebraic subgroup of the kernel of N.

The generalized Sato-Tate conjecture.

Conjecture *(Generalized Sato-Tate)*: For every abelian variety A the polynomials $L_{\mathfrak{p}}\left(A/k,\, q^{-1/2}T\right)$ are equidistributed with respect to the Haar measure on the group $ST_k(A)$.

Recently, in [FKRS11] the authors treat the dimension 2 case. They have computed all the Sato-Tate groups for the dimension 2 abelian varieties, the distribution in these groups and have checked numerically that these distributions match with the Sato-Tate distributions of abelian varieties with the prescribed Sato-Tate group.

They find 52 different Sato-Tate groups (up to conjugacy) of which 34 can appear when $k = \mathbb{Q}$.

The generalized Sato-Tate conjecture.

Some interesting facts are that:

- The S-T group is completely determined by

$$\left(G = Gal\left(K/k\right),\ End_{\bar{k}}\left(A\right) \otimes \mathbb{R},\ \rho\right).$$

  Where K is the minimal field of definition of $End_{\bar{k}}\left(A\right)$ and $\rho : Gal\left(K/k\right) \longrightarrow End_{\bar{k}}\left(A\right) \otimes \mathbb{R}$ is the representation afforded by the natural action of $Gal\left(K/k\right)$ on $End_{\bar{k}}\left(A\right) \otimes \mathbb{R}$.
- for every Sato-Tate group there exists a genus 2 curve such that its Jacobian realizes such S-T group.
- given a genus 2 curve with many automorphims if we vary the field of definition k and we consider different twists, then we can obtain all, except the generic, possibilities for the S-T groups in dimension 2.

The Sato-Tate conjecture for the twists of the Fermat curve.

The Fermat quartic curve is the non-hyperelliptic genus 3 curve:

$$C: \ x^4 + y^4 + z^4 = 0.$$

Let us consider it defined over $\mathbb{Q}$. Its automorphism group has order 96 and is defined over $\mathbb{Q}(i)$.
And its Jacobian, $J(C)$, is isomorphic over $\mathbb{Q}$ to E^3, where $E: \ y^2 = x^3 + x$, that has CM by $\mathbb{Q}(i)$. Thus,

$$End^0_{\overline{\mathbb{Q}}}(J(C)) \simeq \mathcal{M}_3(\mathbb{Q}(i)).$$

Let us consider as an example the twist:

$$C': \ a^2x^4 + y^4 + z^4 = 0.$$

Where $a \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$. Thus, $J(C') \sim_{\mathbb{Q}} E \times (E')^2$.

The Sato-Tate conjecture for the twists of the Fermat curve.

Theorem *([Fit11])*: *Let A/k be an abelian variety s.t. $A \sim_K E^g$, where E/k is an elliptic curve with CM not defined over k. Let $\eta(A)$ denote the representation afforded by the $M(G)-$module $End_K^0(A)$, where M denotes the center of $End_K^0(A)$. Then*

$$M_{2n}(a_1(A/k)) = \frac{1}{|G|}\sum_{\sigma\in G}(Tr\,\eta(A)(\sigma))^n \cdot \frac{1}{2}\tilde{c}_n.$$

Where $\tilde{c}_n = \begin{pmatrix} 2n \\ n \end{pmatrix}$.
In our example $k = \mathbb{Q}$, $M = \mathbb{Q}(i)$, $K = \mathbb{Q}\left(i, \sqrt{a}\right)$ and $G = Gal(K/M) \simeq \mathbb{Z}_2$. And we compute $\eta(J(C'))= 5\mathcal{E} + 4\chi$.
Thus,

$$M_{2n}(a_1(J(C')/\mathbb{Q})) = \frac{3^{2n}+1}{4}\tilde{c}_n.$$

The Sato-Tate conjecture for the twists of the Fermat curve.

Next step is computing the Sato-Tate groups associated to the jacobian of the twists of the Fermat quartic curve.

Theorem *([BK11])*: *Let A/k an abelian variety of dimension at most 3, then $ST_k(A)$ is a maximal compact subgroup of $TL_k(A)$.*

Where $TL_k(A)$ is the Twisted Lefschetz group of A/k, defined as

$$TL_k(A) = \sqcup_{\sigma \in Gal(K/k)} L_{A/k}(\sigma)$$

where

$$L_{A/k}(\sigma) = \left\{ \gamma \in Sp_{2g} : \gamma\beta\gamma^{-1} =^{\sigma} \beta \, \forall \beta \in End_{\bar{k}}(V_l(A)) \right\}.$$

The Sato-Tate conjecture for the twists of the Fermat curve.

In our example we compute:

$$L_{A/k}(1) = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix} : A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} a, b \in \mathbb{R},\ a^2 + b^2 = 1 \right\}$$

$$L_{A/k}(\tau) = \left\{ \begin{pmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{pmatrix} : B = \begin{pmatrix} ai & bi \\ bi & -ai \end{pmatrix} a, b \in \mathbb{R},\ a^2 + b^2 = 1 \right\}$$

$$L_{A/k}(\sigma) = \left\{ \begin{pmatrix} -A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix} : A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} a, b \in \mathbb{R},\ a^2 + b^2 = 1 \right\}$$

$$L_{A/k}(\sigma\tau) = \left\{ \begin{pmatrix} -B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{pmatrix} : B = \begin{pmatrix} ai & bi \\ bi & -ai \end{pmatrix} a, b \in \mathbb{R},\ a^2 + b^2 = 1 \right\}.$$

Where $Gal\left(\mathbb{Q}\left(i, \sqrt{a}\right)/\mathbb{Q}\right) = \langle \tau, \sigma \rangle$ and

$$\tau : \begin{array}{c} i \longrightarrow -i \\ \sqrt{a} \longrightarrow \sqrt{a} \end{array} \text{ and } \sigma : \begin{array}{c} i \longrightarrow i \\ \sqrt{a} \longrightarrow -\sqrt{a} \end{array}.$$

The Sato-Tate conjecture for the twists of the Fermat curve.

So, for each connected component on the Sato-Tate group of $J(C')$ we get the distribution of the random variable a_1 given by:

- $L_{A/k}(1)$: $\mathbb{E}[a_1^n] = 6^n \int_0^\pi \frac{cos^n\theta}{\pi} d\theta = \begin{cases} 0 \text{ if } n \text{ odd} \\ 3^n \tilde{c}_{n/2} \text{ if } n \text{ even} \end{cases}$
- $L_{A/k}(\tau)$: $\mathbb{E}[a_1^n] = 0$
- $L_{A/k}(\sigma)$: $\mathbb{E}[a_1^n] = 2^n \int_0^\pi \frac{cos^n\theta}{\pi} d\theta = \begin{cases} 0 \text{ if } n \text{ odd} \\ \tilde{c}_{n/2} \text{ if } n \text{ even} \end{cases}$
- $L_{A/k}(\sigma\tau)$: $\mathbb{E}[a_1^n] = 0$

Then, we get

$$M_{2n}(a_1(J(C')/\mathbb{Q})) = \frac{3^{2n}+1}{4}\tilde{c}_n.$$

Bibliography

- [BK11]: Banaszak G., Kedlaya K.: "An algebraic Sato-Tate group and Sato-Tate conjecture", preprint.
- [Fit11]: Fité F., e-mail correspondence Sep11-Nov11.
- [FKRS11]: Fité F., Kedlaya K., Rotger V., Sutherland A.: "Sato-Tate distribution and Galois endomorphism modules in genus 2", preprint.
- [KS09]: Kedlaya K., Sutherland A.: "Hypereliptic curves, L-polynomials and random matrices", Contemporary Mathematics (AMS) v.487, 2009.
- [Lor10]: Lorenzo E.: "Torcimientos de cuárticas planas lisas", master thesis, 2010.