Improved bounds on crossing numbers of graphs via semidefinite programming

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Outline

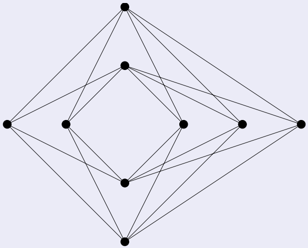
- The (two page) crossing numbers of complete and complete bipartite graphs.
- A maximum cut formulation for the two page crossing number of K_n.
- The Goemans-Williamson maximum cut bound and its implications.
- A nonconvex quadratic programming relaxation of the two page crossing number of $K_{m,n}$.
- A semidefinite programming relaxation of the quadratic program and its implications.

Crossing number of a graph

Definition

The crossing number $\mathrm{cr}(G)$ of a graph $G = (V, E)$ is the minimum number of edge crossings that can be achieved in a drawing of G in the plane.

Example: the complete bipartite graph



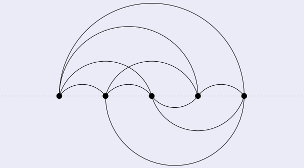
An optimal drawing of $K_{4,5}$ with $\mathrm{cr}(K_{4,5}) = 8$ edge crossings.

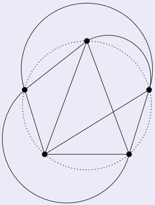
Two-page crossing number of a graph

Definition

In a two-page drawing of $G = (V, E)$ all vertices V must be drawn on a straight line (resp. circle) and all edges either above/below the line (resp. inside/outside the circle). The two-page crossing number $\nu_2(G)$ corresponds to two-page drawings of G.

Example: the complete graph K_5





Equivalent two-page drawings of K_5 with $\nu_2(K_5) = 1$ crossing.

Applications and complexity

- Crossing numbers are of interest for graph visualization, VLSI design, quantum dot cellular automata, ...
- It is NP-hard to compute $\mathrm{cr}(G)$ or $\nu_2(G)$ [Garey-Johnson (1982), Masuda et al. (1987)];
- The (two-page) crossing numbers of K_n and $K_{n,m}$ are not known, ...
- Crossing number of $K_{n,m}$ known as Turán brickyard problem — posed by Paul Turán in the 1940's.

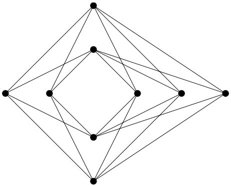
Erdös and Guy (1973):

"Almost all questions that one can ask about crossing numbers remain unsolved."

The Zarankiewicz conjecture

$K_{m,n}$ can be drawn in the plane with at most $Z(m,n)$ edges crossing, where

$$Z(m,n) = \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor.$$



A drawing of $K_{4,5}$ with $Z(4,5) = 8$ crossings.

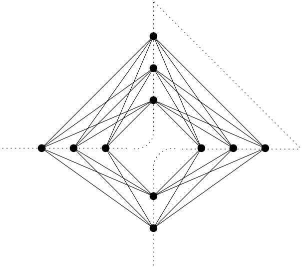
Zarankiewicz conjecture (1954)

$$\mathrm{cr}(K_{m,n}) \stackrel{?}{=} Z(m,n).$$

Known to be true for $\min\{m,n\} \leq 6$ (Kleitman, 1970), and some special cases.

The 2-page Zarankiewicz conjecture

The Zarankiewicz drawing may be mapped to a 2-page drawing:



"Straighten the dotted line".

2-page Zarankiewicz conjecture

$$\nu_2(K_{m,n}) \stackrel{?}{=} Z(m,n).$$

Weaker conjecture since $\mathrm{cr}(G) \le \nu_2(G)$.

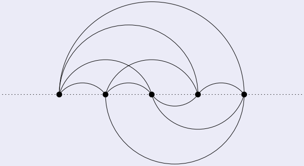
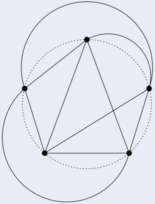
The (2-page) Harary-Hill conjecture

Conjecture (Harary-Hill (1963))

$$\mathrm{cr}(K_n) \stackrel{?}{=} \nu_2(K_n) \stackrel{?}{=} Z(n) := \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

NB: it is only known that $\mathrm{cr}(K_n) \le \nu_2(K_n) \le Z(n)$ in general.

Example: the complete graph K_5

Optimal two-page drawings of K_5 with $Z(5) = 1$ crossing.

Some known results

Theorem (De Klerk, Pasechnik, Schrijver (2007))

One has

$$1 \geq \lim_{n\to\infty} \frac{\mathrm{cr}(K_n)}{Z(n)} \geq 0.8594, \quad 1 \geq \lim_{n\to\infty} \frac{\mathrm{cr}(K_{m,n})}{Z(m,n)} \geq 0.8594 \textit{ if } m \geq 9,$$

Theorem (Pan and Richter (2007), Buchheim and Zheng (2007))

$$\mathrm{cr}(K_n) = Z(n) \quad \textit{if } n \leq 12, \quad \nu_2(K_n) = Z(n) \quad \textit{if } n \leq 14.$$

New results (this talk)

Theorem (De Klerk and Pasechnik (2011))

For the complete graph K_n, one has

$$1 \geq \lim_{n \to \infty} \frac{\nu_2(K_n)}{Z(n)} \geq 0.9253$$

and

$$\nu_2(K_n) = Z(n) \quad \textit{if } n \leq 18 \textit{ or } n \in \{20, 22\}.$$

For the complete bipartite graph $K_{m,n}$, one has

$$\lim_{n \to \infty} \frac{\nu_2(K_{m,n})}{Z(m,n)} = 1 \textit{ if } m \in \{7, 8\}.$$

New results: outline of the proofs

For K_n:

- The problem of computing $\nu_2(K_n)$ has a formulation as a maximum cut problem (Buchheim and Zheng (2007));
- The new results for $\nu_2(K_n)$ follow by computing the Goemans-Williamson maximum cut bound for $n = 899$.
- The Goemans-Williamson bound is computed using semidefinite programming (SDP) software and using algebraic symmetry reduction.

For $K_{m,n}$:

- We will formulate a (nonconvex) quadratic programming (QP) lower bound on $\nu_2(K_{m,n})$.
- Subsequently we compute an SDP lower bound on the QP bound for $m = 7$, again using algebraic symmetry reduction.

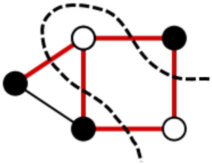
The maximum cut problem for graphs

Definition

For $G = (V, E)$ and a subset $W \subseteq V$, $\text{cut}_W(G)$ denotes the set of edges with precisely one endpoint in W, and is called the edges in the cut W. The weight of a maximum cut is

$$\text{maxcut}(G) := \max_{W \subseteq V} |\text{cut}_W(G)|.$$

Example:



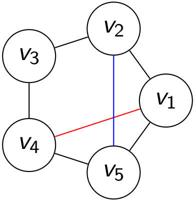
The black vertices yield the maximum cut in the figure, $\text{maxcut}(G) = 5$.

$\nu_2(K_n)$ as a maximum cut problem

Results of Buchheim and Zheng (2007) imply that $\nu_2(K_n)$ may be obtained by solving a maximum cut problem in a suitable graph, $G_n = (V_n, E_n)$, say.

- V_n is the set of chords in the standard n-cycle, thus $|V_n| = \binom{n}{2} - n$;
- Two vertices are adjacent if the corresponding chords cross when drawn inside the cycle, thus $|E_n| = \binom{n}{4}$.

Illustration:



The chords $\{v_1, v_4\}$ and $\{v_2, v_5\}$ form adjacent vertices in the graph G_5.

$\nu_2(K_n)$ as a maximum cut problem (ctd)

Lemma (cf. Buchheim and Zheng (2007))

One has

$$\nu_2(K_n) = |E_n| - \mathrm{maxcut}(G_n).$$

Proof sketch: Given a two page (circle) drawing of K_n, define a cut $W \subset V_n$ as the chords that are drawn inside the circle.

- Using this formulation, Buchheim and Zheng (2007) showed that $\nu_2(K_n) = Z(n)$ for $n \leq 14$, by using a branch and bound method.
- Using the maximum cut software BiqMac (Rendl, Rinaldi, and Wiegele (2010)) we could extend this result for n up to 18, and for $n = 20, 22$.

Further progress possible by considering the Goemans-Williamson maximum cut bound ... (next slide)

The Goemans-Williamson maxcut bound

Goemans-Williamson bound (dual formulation)

For a graph $G = (V, E)$ with Laplacian matrix L define:

$$\mathcal{GW}(G) := \min_{y \in \mathbb{R}^{|V|}} \left\{ \sum_{i \in V} y_i \;\middle|\; \mathrm{Diag}(y) - \frac{1}{4} L \succeq 0 \right\},$$

where

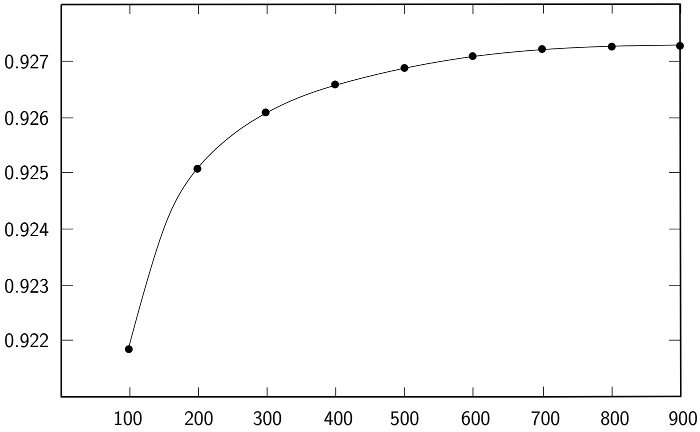
- $A \succeq 0$ means the matrix A is hermitian positive semidefinite;
- $\mathrm{Diag}(y)$ is the diagonal matrix with the vector y as diagonal.

Theorem (Goemans-Williamson (1995))

$$0.878\mathcal{GW}(G) \le \mathrm{maxcut}(G) \le \mathcal{GW}(G).$$

Since the Laplacian of G_n has block-circulant structure, we may compute $\mathcal{GW}(G_n)$ for n up to 1000 via symmetry reduction.

Computational results with SDPT3 solver



The ratio $\frac{\binom{n}{4} - \mathcal{GW}(G_n)}{Z(n)}$ for $n = 99, 199, \ldots, 899$.

Implication for general n

Lemma

For any integer $m > 3$,

$$\lim_{n\to\infty} \frac{\nu_2(K_n)}{Z(n)} \geq \frac{64}{m(m-1)(m-2)(m-3)}\ \nu_2(K_m).$$

As a consequence, setting $m = 899$ and using

$$\nu_2(K_m) \geq \binom{m}{4} - \mathcal{GW}(G_m)$$

we obtain:

Corollary

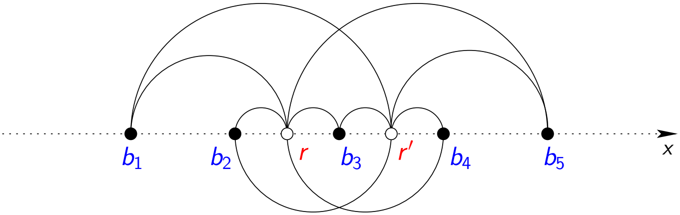
$$\lim_{n\to\infty} \nu_2(K_n)/Z(n) \geq 0.9253.$$

Drawings of $K_{m,n}$

Consider a drawing of $K_{m,n}$ with the n coclique colored red, and the m coclique blue.

Definition

Each red vertex r has a position $p(r) \in \{1, \ldots, m\}$ in the drawing, and a set of incident edges $U(r) \subseteq \{1, \ldots, m\}$ drawn in the upper half plane. We say r is of the type $(p(r), U(r))$. The set of all possible types is denoted by $\text{Types}(m)$, i.e. $|\text{Types}(m)| = m2^m$.



In the figure, r has type $(p(r), U(r)) = (2, \{1, 2, 3, 5\})$.

A quadratic programming relaxation of $\nu_2(K_{m,n})$

We define a $m2^m \times m2^m$ matrix Q with rows/colums indexed by Types(m).

Definition

Let $\sigma, \tau \in$ Types(m). Define $Q_{\tau,\sigma}$ as the number of unavoidable edge crossings in a 2-page drawing of $K_{2,m}$, where the vertices from the 2-coclique have type σ and τ respectively in the drawing.

Lemma

$$\nu_2(K_{m,n}) \geq \frac{n^2}{2}\left(\min_{x\in\Delta} x^T Qx\right) - \frac{m(m-1)n}{4}$$

where $\Delta = \left\{x \in \mathbb{R}^{m2^m} \;\middle|\; \sum_i x_i = 1,\ x_i \geq 0\right\}$ *is the standard simplex.*

This is a nonconvex quadratic program — we again use a semidefinite programming relaxation (next slide).

A semidefinite programming relaxation of $\nu_2(K_{m,n})$ (ctd)

Standard semidefinite programming relaxation:

$$\min_{x\in\Delta} x^T Q x \quad \geq \quad \min\left\{\mathrm{trace}(QX) \,\middle|\, \mathrm{trace}(JX) = 1,\ X \succeq 0,\ X \geq 0\right\},$$

where J is the all-ones matrix and $X \geq 0$ means X is entrywise nonnegative.

- We may again perform symmetry reduction using the structure of Q ...
- ... namely Q is a block matrix with $2m \times 2m$ circulant blocks (after reordering rows/columns).
- The reduced problem has $2m$ linear matrix inequalities involving $(2^{m-1}) \times (2^{m-1})$ matrices.

Computational results and implications

We could compute the SDP bound for $m = 7$ to obtain

$$\nu_2(K_{7,n}) \geq (9/4)n^2 - (21/2)n = Z(7,n) - O(n).$$

Since $\nu_2(K_{8,n}) \geq 8\nu_2(K_{7,n})/6$, we also get $\nu_2(K_{8,n}) \geq 3n^2 - 14n = Z(8,n) - O(n)$.

Corollary

$$\lim_{n\to\infty} \nu_2(K_{m,n})/Z(m,n) = 1 \text{ for } m = 7 \text{ and } 8.$$

In words, the 2-page Zarankiewicz conjecture is true asymptotically for $m = 7$ and 8.

Conclusion and summary

- We demonstrated improved asymptotic lower bounds on $\nu_2(K_n)$, $\nu_2(K_{7,n})$, and $\nu_2(K_{8,n})$.
- The proofs were computer-assisted, and the main tools were semidefinite programming (SDP) relaxations and symmetry reduction.
- The SDP relaxation was too large to solve for $\nu_2(K_{9,n})$ — challenge for SDP community.
- Preprint available at Optimization Online and arXiv.