Generalizing canonical extension to the categorical setting

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Outline

Aim: generalize techniques used in the algebraic study of propositional logics to the setting of first order logics.

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1. Propositional logic and algebra
2. Duality theory and canonical extension
3. ‘Algebraic’ study of first order logics
4. Canonical extension in the categorical setting

Propositional logic and algebra

Logic	Algebra
connectives, e.g., $\wedge$, $\vee$, $\neg$, $\bot$, $\top$	operations
propositional variables $P = \{p_0, p_1, \ldots\}$	generations
formulas, e.g., $p_0 \wedge \neg p_1$	terms
notion of derivability, e.g., $p_0 \wedge p_1 \vdash p_0$	inequational theory

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We associate:

$$\textbf{logic } \mathcal{L} \qquad \leftrightsquigarrow \qquad \textbf{category of algebras } \mathcal{V}_{\mathcal{L}}$$

s.t. for all formulas ϕ, ψ in $\mathcal{L}$,

$$\phi \vdash \psi \text{ in } \mathcal{L} \qquad \Leftrightarrow \qquad \phi \vDash \psi \text{ in } \mathcal{V}_{\mathcal{L}}$$

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For example: CPL $\leftrightsquigarrow$ Boolean algebras

IPL $\leftrightsquigarrow$ Heyting algebras

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Boolean space: closed subspace of Cantor space 2^X.

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Boolean space: closed subspace of Cantor space 2^X.

Example: points of the dual space of $F_{BA}(p, q)$ correspond to rows of the truth table:

p	q
$\perp$	$\perp$
$\perp$	$\top$
$\top$	$\perp$
$\top$	$\top$

Duality and canonical extension

Canonical extension: algebraic description of topological duality

For a Boolean algebra B we have

- topological perspective: study (X_B, τ_B)
  (elements of B correspond to a basis)
- canonical extension: study $B \hookrightarrow \mathcal{P}(X_B) = B^\delta$

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Every Boolean algebra has a unique canonical extension and this assignment is functorial:

$$(\_)^\delta \colon \mathbf{BA} \to \mathbf{CABA}$$

Similar situation for Heyting algebras.

Interpolation in propositional logic

Let $\mathbb{T}$ be a theory in IPC.

Question: does $\mathbb{T}$ have the **interpolation property**, i.e.,

for all formulas $\phi(p,q)$ and $\psi(p,r)$ with $\phi(p,q) \vdash_{\mathbb{T}} \psi(p,r)$, there exists a formula $\theta(p)$ s.t.

$$\phi(p,q) \vdash_{\mathbb{T}} \theta(p) \quad \text{and} \quad \theta(p) \vdash_{\mathbb{T}} \psi(p,r).$$

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Question: are embeddings stable under pushout in $\mathcal{V}_{\mathbb{T}}$?

Interpolation in first order logic

Let $\mathbb{T}$ be a theory in intuitionic first order logic.

Question: does $\mathbb{T}$ have the **interpolation property**, i.e.,

for all sentences ϕ, ψ with $\phi \vdash_{\mathbb{T}} \psi$, there exists a sentence θ s.t.

1. $\phi \vdash_{\mathbb{T}} \theta$ and $\theta \vdash_{\mathbb{T}} \psi$;
2. every relation and function symbol which occurs in θ occurs in both ϕ and ψ.

Open problem for some first order intuitionistic theories, e.g.,

$\mathbb{T} = \text{IFOL} + (\phi \to \psi) \vee (\psi \to \phi)$.

Algebraic semantics for intuitionistic logic

We start from

Signature:	$\Sigma = (f_0, \ldots, f_{k-1}, R_0, \ldots, R_{l-1})$
Set of var's:	$X = \{x_0, x_1, \ldots\}$
Equality:	$=$
Connectives:	$\wedge, \vee, \rightarrow, \top, \bot, \exists, \forall$
Derivability notion:	$\vdash$ (given by axioms and rules)

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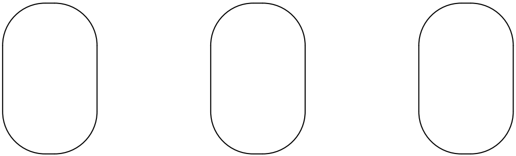
Problem: how to define 'algebraic' semantics for first order logic?

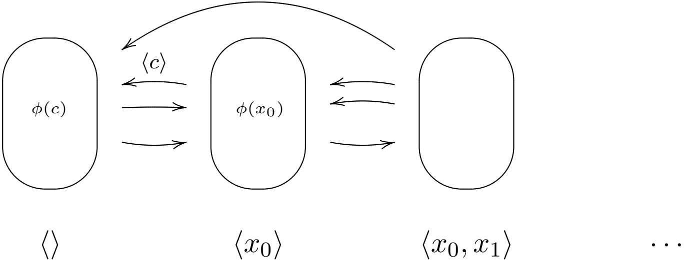
- formulas with free variables, e.g., $\forall x.R(x, y)$;
- quantifiers $\exists x$, $\forall x$.

Algebraic semantics for intuitionistic logic



$\langle\rangle$ $\quad\quad$ $\langle x_0 \rangle$ $\quad\quad$ $\langle x_0, x_1 \rangle$ $\quad\quad$ $\cdots$

Algebraic semantics for intuitionistic logic



Substitutions:

$$\begin{array}{ccc} x_0 & \mapsto & c \\ \phi(x_0) & \mapsto & \phi(c) \end{array}$$

Algebraic semantics for intuitionistic logic

Existential quantification: related to the inclusion map

$$\begin{tikzcd} \phi(x_0) \arrow[r, bend left, "i"] & \psi(x_0, x_1) \arrow[l, bend left, "\exists_{x_1}"] \end{tikzcd}$$

$$\langle x_0 \rangle \xleftarrow{\langle x_0 \rangle} \langle x_0, x_1 \rangle$$

$$\frac{\exists_{x_1}(\psi(x_0, x_1)) \quad \vdash \quad \phi(x_0)}{\psi(x_0, x_1) \quad \vdash \quad \phi(x_0)}$$

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$$\langle x_0 \rangle \xleftarrow{\langle x_0 \rangle} \langle x_0, x_1 \rangle$$

$$\begin{array}{ccc} \exists_{x_1}(\psi(x_0, x_1)) & \vdash_{x_0} & \phi(x_0) \\ \hline\hline \psi(x_0, x_1) & \vdash_{x_0, x_1} & i(\phi(x_0)) \end{array}$$

Algebraic semantics for intuitionistic logic

An **intuitionistic hyperdoctrine** is a functor $P \colon \mathbf{B}^{op} \to \mathbf{HA}$ s.t.

1. $\mathbf{B}$ is a category with finite limits;
2. for all $A \xrightarrow{\alpha} B \in \mathbf{B}$, $P(\alpha)$ has a left adjoint $\exists_\alpha$ satisfying Beck-Chevalley and Frobenius and a right adjoint $\forall_\alpha$.

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Example: powerset functor

$$\begin{array}{rccc} \mathcal{P}\colon & \mathbf{Set}^{\mathrm{op}} & \to & \mathbf{HA} \\ & A & \mapsto & \mathcal{P}(A) \\ & A \xrightarrow{f} B & \mapsto & \mathcal{P}(B) \xrightarrow{f^{-1}} \mathcal{P}(A). \end{array}$$

Intuitionistic hyperdoctrines and categories

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A **Heyting category** is a category $\mathbf{C}$ satisfying

1. $\mathbf{C}$ has finite limits;
2. $\mathbf{C}$ has stable finite unions;
3. pullback morphisms in $\mathbf{C}$ have stable left and right adjoints.

Canonical extension of intuitionistic hyperdoctrines

Recall: canonical extension of Heyting algebras is a functor

$$\mathbf{HA} \xrightarrow{(\_)^{\delta}} \mathbf{HA}^{+}$$

Definition

For an intuitionistic hyperdoctrine $P\colon \mathbf{B}^{\mathrm{op}} \to \mathbf{HA}$ we define:

$$P^{\delta}\colon \mathbf{B}^{\mathrm{op}} \xrightarrow{P} \mathbf{HA} \xrightarrow{(\_)^{\delta}} \mathbf{HA}.$$

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$$P^\delta\colon \mathbf{B}^{\mathrm{op}} \xrightarrow{P} \mathbf{HA} \xrightarrow{(\_)^\delta} \mathbf{HA}.$$

Proposition

For an int. hyperdoctrine P, P^δ is again an int. hyperdoctrine.

Proof: check that, for all $A \xrightarrow{\alpha} B$ in $\mathbf{B}$, $P^\delta(\alpha)$ has a left adjoint satisfying BC and Frobenius and a right adjoint.

Canonical extension of Heyting categories

Proposition: there is a 2-categorical adjunction

$$\mathcal{A}\colon \mathbf{IHyp} \leftrightarrows \mathbf{HCat}\colon \mathcal{S},$$

where $\mathcal{A} \dashv \mathcal{S}$ and $\mathcal{A}(\mathcal{S}(\mathbf{C})) \simeq \mathbf{C}$.

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Definition

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Proposition

For a Heyting algebra $\mathbf{L}$, $\mathcal{A}(\mathcal{S}(\mathbf{L})^{\delta}) \simeq \mathbf{L}^{\delta}$.

Canonical extension of Heyting categories

Properties of $\mathbf{C}^\delta = \mathcal{A}(\mathcal{S}^\delta_{\mathbf{C}})$:

1. subobject lattices are in $\mathbf{HA}^+$
2. pullback morphisms are complete lattice homomorphisms

$\mathbf{HCat}^+ =$ Heyting categories satisfying (1) and (2).

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Universal characterization:

$$\begin{tikzcd} \mathbf{C} \arrow[r, "M_0"] \arrow[dr, "M"'] & \mathbf{C}^\delta \arrow[d, dotted, "\tilde{M}"] \\ & \mathbf{E} \end{tikzcd}$$

where $\mathbf{C} \in \mathbf{HCat}$, $\mathbf{E}, \mathbf{C}^\delta \in \mathbf{HCat}^+$, M a Heyting functor satisfying some extra conditions (related to Esakia Lemma).

Results and future work

- Canonical extension of coherent categories allows an alternative description of Makkai's topos of types:

$$T(\mathbf{C}) \simeq Sh(\mathcal{S}(\mathbf{C})^{\delta})$$

- Study Makkai's results from this perspective

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- Study Makkai's results from this perspective

- Apply our construction in the study of first order logics

- In particular: study interpolation problems for first order logics, e.g. for IFOL + $(\phi \to \psi) \vee (\psi \to \phi)$