IC Seminar 'Functorial compactification of moduli of abelian varieties'
Organizers: Jochen Heinloth and Robin de Jong

In 2002 V. Alexeev gave a functorial compactification of the moduli space $\mathscr{A}_g$ of principally polarized abelian varieties of dimension g, motivated by, among many others, ideas from the early seventies due to D. Mumford. The boundary parametrizes pairs $(G \curvearrowright P, \Theta)$ where G is a semiabelian variety of dimension g, (P, Θ) is a polarised projective variety (satisfying some additional properties), and G acts on P with finitely many orbits. The compactification given by Alexeev has several irreducible components, and one wants to give a functorial description of the 'main component' containing $\mathscr{A}_g$. A solution to this problem has been given by M. Olsson in 2007, using log structures. The aim of this IC Seminar is to understand these developments. We wish to proceed as much as possible by examples.

Lecture Ia - Introduction, statement of main results
(September 17) Moduli of (principally) polarized abelian varieties, semistable reduction, compactifications. Constructions of Tate-Mumford. Search for a modular compactification. Sketch of solutions proposed by Alexeev, Olsson. Abelic pairs, semiabelic pairs. In the principally polarized case, the groupoids of abelic pairs and ppav are equivalent.

Lecture Ib - Algebraic stacks
(September 17) Moduli problems and moduli stacks. Representability of the diagonal. A (working) definition of algebraic stack, e.g. as in Faltings-Chai. Coarse moduli spaces. Artin's criterion for algebraicity. Keel-Mori theorem: existence of coarse moduli spaces. Valuative criterion of properness. Example: stable curves.

Lecture IIa - Linearization of torus actions
(October 22) Chapter 4 of [Al]. This section discusses the linearization problem in the presence of a torus action. It seems useful to review Mumford's constructions from [Mu], since the present Chapter in some sense 'inverts' those constructions. Main result is Thm. 4.3.1. From 4.1 only state without proof the main result Thm. 4.1.22; it is used in the proof of Thm. 4.3.1.

Lecture IIb - Linearization of semiabelian group actions
(October 22) [Al] §§5.2–5.4. Chapter 5 of [Al] starts by generalising step by step the constructions from Chapter 4, allowing more general semiabelian varieties. The combinatorial description of stable semiabelic pairs used here can become quite complicated; perhaps there is a way to only sketch ideas here. For example, it seems simpler to assume that the reference map $\rho\colon |\tilde{\Delta}| \to X_{\mathbb{R}}$ is injective. The sections require definitions and results from 1.1.16–1.1.29, §2.1, §2.2 and §5.1. The main results are: Thm. 5.3.8, 5.4.1, 5.4.3.

Lecture IIIa - Alexeev's construction
(November 5) [Al] §§5.7 – 5.10. Main result Thm. 5.10.1: Alexeev's moduli problem defines a proper Artin stack with finite stabilizers, and has a coarse moduli space. Do not prove properness and finiteness of automorphisms; these will be discussed later when dealing with Olsson's result. §5.8 and 5.9 discuss local charts, in §5.10 the application of Artin's criteria is sketched. Again [Mu] should be of great use.

Lecture IIIb - Introduction to log geometry
(November 5) Pre-log structures, log structures, associated log structure of a pre-log struc-

ture, morphisms of (pre-)log structures, charts, log smoothness, étale local structure of log smoothness, link with 'toroidal singularities', saturated fine log schemes. The moduli space of stable curves is the moduli space of log smooth curves. Suggestions for literature: §1.3 of [Ol], §§2–4 of [ACGHOSS], Ogus's book project [Og].

Lecture IVa - Olsson's standard family and moduli problem
(November 12) [Ol] §3.1, §3.6. Note that §3.1 gives no examples, but useful ones can be found in 5.12, 5.13 of the published version. In the Essen seminar it was suggested to take $X = \mathbb{Z}$ for the paving S given by the quadratic function $a(n) = n^2$, and $X = \mathbb{Z}^2$ for the pavings S_t given by the quadratic functions $a_t(n, m) = n^2 + tnm + m^2$, with $t \in \{-1, 0, 1\}$. The material can probably be presented by means of (these) examples. §3.6 formulates Olsson's moduli problem, $\mathscr{K}_g$.

Lecture IVb - Deformations and versal families
(November 12) [Ol] §§3.2–3.5 (until 3.5.15). Deformations and automorphisms. Deformation theory. Versal families. It is left to the speaker to decide either for a treatment specifically related to Olsson's moduli problem, or to discuss the more general aspects of the deformation theory that is used here. A key point is Prop. 3.3.3 which refers back to Thm. 4.3.1 of [Al]. It would be good to discuss the stack $\mathscr{W}_3$ from 4.5.14 as it will play a role in Lecture Vb.

Lecture Va - Properness
(December 3) [Ol] §3.7. The main point of the constructions discussed in this seminar is to obtain a *proper* moduli problem. We therefore try to understand §3.7, which deals with precisely this point, in detail.

Lecture Vb - Comparison with Alexeev's compactification
(December 3) [Ol] §3.9. In this final talk we wish to connect Olsson's construction with Alexeev's. Section 3.9 concludes the proof of Olsson's main result (see Section 3.6), part of which deals with this comparison. The proof uses 3.5.16–20 and §3.8.

References

[ACGHOSS] D. Abramovich, Q. Chen, D. Gillam, Y. Huang, M. Olsson, M. Satriano, S. Sun, *Logarithmic geometry and moduli.* `http://www.math.arxiv:1006.5870`.
[Al] V. Alexeev, *Complete moduli in the presence of semiabelian group action.* Ann. of Math. 155 (2002), 611–708.
[Mu] D. Mumford, *An analytic construction of degenerating abelian varieties over complete rings.* Compositio Math. 24 (1972), 239–272.
[Og] A. Ogus, *Lectures on logarithmic algebraic geometry.*
`http://math.berkeley.edu/~ogus/preprints/log_book/logbook.pdf`.
[Ol] M. Olsson, *Compactifying moduli spaces for abelian varieties.*
`http://math.berkeley.edu/~molsson/mono020807.pdf`, or Springer Lecture Notes in Mathematics 1958. Unless mentioned otherwise, the section numbers above refer to the *online* version (the published version contains an additional introductory chapter).