Spatial Patterns

Higher Order Models in Physics and Mechanics

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Outline

This book presents a study of the equation

$$\frac{d^4u}{dx^4} + q\frac{d^2u}{dx^2} + f(u) = 0, \tag{CE}$$

in which q is an eigenvalue parameter and f a given function. We shall call it the *Canonical Equation*. Our main concern will be the description of the set $\mathcal{B}$ of bounded solutions on $\mathbf{R}$ for different values of q.

In Chapter 1 we show how this equation plays a central role in several fourth order model equations and second order variational problems arising in studies of pattern formation in physics, material science, mechanics and biology. There we also give a sketch of a few mathematical methods which have proved fruitful in the analysis of this equation.

Beyond Chapter 1, the book is divided into two parts: in Part I we focus on an important example of the Canonical Equation, in which the nonlinearity f is given by

$$f(u) = u^3 - u.$$

The corresponding potential function F, defined as the primitive of f, is symmetric and has two wells, one located at $u = -1$ and the other at $u = +1$. This particular equation arises in theories of phase transitions and derives from the *Swift-Hohenberg equation* and the *Extended Fisher-Kolmogorov equation*. In Part II we branch out to equations with more general source functions f. In particular, here we discuss examples of functions which give rise to single well potentials. Such potentials also arise in the context of the Swift-Hohenberg equation.

In Part I a pivotal role is played by the spectrum of the linearisation at the bottom of the wells, i.e. at $u = \pm 1$. Its character is different in each of the following three intervals:

$$q \le -\sqrt{8}, \qquad -\sqrt{8} < q < \sqrt{8} \quad \text{and} \quad q > \sqrt{8}.$$

For $q \le -\sqrt{8}$ the uniform solutions $u = -1$ and $u = +1$ are saddles, for $-\sqrt{8} < q < \sqrt{8}$ they are saddle-foci, and for $q > \sqrt{8}$ they are centers.

In Chapter 2 we discuss the set of bounded solutions when $q \le -\sqrt{8}$ and find that it is essentially the same as that of the classical second order Fisher-Kolmogorov equation.

$$\frac{d^2u}{dx^2} + u - u^3 = 0. \tag{FK}$$

When q exceeds the threshold value $-\sqrt{8}$ the set of bounded solutions becomes extremely rich. Before exploring the existence and structure of different types of bounded solutions in this regime, we develop in Chapter 3 the topological shooting method which will be used throughout the book, and obtain a series of general results about solution graphs, such as bounds, and properties of critical points. Having prepared the ground work we discuss in Chapters 4, 5 and 6, successively, periodic solutions, kinks and pulses, and chaotic solutions.

In Chapter 7 we present a variational approach to the Canonical Equation viewing it as the Euler-Lagrange equation of the functional

$$J(u) = \int \left\{ \frac{1}{2}(u'')^2 - \frac{q}{2}(u')^2 + F(u) \right\} dx,$$

where primes denote differentiation. Here we also present a symmetry argument which makes it possible to characterise the qualitative properties of global minimisers.

Part II is made up of Chapters 8, 9 and 10. In Chapter 8 we still consider functions f of which the potential F possesses two wells. However, here F is no longer symmetric. In Chapters 9 and 10 we discuss in some detail two important examples, one from physics and one from mechanics. In Chapter 9 we investigate stationary solutions of the Swift-Hohenberg equation. This leads to a study of the equation

$$\frac{d^4u}{dx^4} + 2\frac{d^2u}{dx^2} + (1-\kappa)u + u^3 = 0,$$

in which κ is the eigenvalue parameter. Here the nonlinearity is associated with a single well potential if $\kappa < 1$, and with a double well potential if $\kappa \ge 1$. In both cases the well is symmetric. In Chapter 10 we analyse an equation suggested by the study of travelling waves in supported beams and suspension bridges in which the nonlinearity leads to a single well potential which is not symmetric.

Numerical computations have proved invaluable in gaining insight and inspiration in the study of the very complex structure of the set of bounded solutions. It is probably no exaggeration to say that without these computations the detailed results presented in this book would not yet have been discovered. Feeling that we should not withhold these insights from the reader, we have included a generous amount of numerically obtained solution graphs and bifurcation curves throughout the text.

The numerical computations of solution graphs have been made with Maple and the package Phaseplane, developed by G.B. Ermentrout [Er]. These graphs were subsequently used as input in AUTO97 to compute the corresponding bifurcation curves. AUTO97 is continuation and bifurcation software created by E. Doedel, A.R. Champneys, T.F. Fairgrieve, Y.A. Kutznetsov, B. Sandstede & X. Wang, [DCFKSW].