On the large time behavior of solutions of fourth order parabolic equations and ε-entropy of their attractors

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Abstract

We study the large time behavior of solutions of a class of fourth order parabolic equations defined on unbounded domains. Specific examples of the equations we study are the *Swift-Hohenberg equation* and the *Extended Fisher-Kolmogorov equation*. We establish the existence of a global attractor in a local topology. Since the dynamics is infinite dimensional, we use the Kolmogorov ε-entropy as a measure, deriving a sharp upper and lower bound.

Résumé

Sur le comportement en temps grand des solutions d'équations paraboliques d'ordre quattre, et l'entropie de leurs attracteurs

Nous étudions le comportement pour des grandes valuers du temps des solutions d'une classe déquations parabolique d'ordre quattre defini sûr des domaines non bornés. Les examples specifiques que nous considerons sont l'équation de *Swift-Hohenberg* et une généralisation de l'équation de *Fisher-Kolmogorov*. Nous démontrons l'existence d'une attracteur globale dans une topologie locale, et obtenons des limites supérieure et inférieure de l'entropie de Kolmogorov.

1 Introduction

In this paper we give a description of the large time behavior of solutions of a family of well-known fourth order model equations of parabolic type of the form

$$u_t + \Delta^2 u + q\Delta u + f(u) = g \qquad \text{in} \qquad \mathbf{R}^3, \tag{1.1}$$

where $q \in \mathbf{R}$ and $f(u)$ and $g = g(x)$ are given functions. Typical examples include the *Extended Fisher-Kolmogorov* equation

$$u_t + \gamma\Delta^2 u - \Delta u + u^3 - u = 0 \qquad \text{in} \qquad \mathbf{R}^3 \qquad (\gamma > 0), \tag{1.2}$$

which arises in the study of bi-stable systems [1] and in statistical mechanics, and the *Swift-Hohenberg* equation

$$u_t = \alpha u - \left(1 + \partial_x^2\right)^2 u - u^3 \qquad \text{in} \qquad \mathbf{R} \qquad (\alpha > 0), \tag{1.3}$$

which is used as a model equation in many studies of pattern formation [7]. Both equations can be brought into the form of equation (1.1) by suitable transformations of t, x and u.

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For one-dimensional bounded domains the large time behavior of solutions of equation (1.1) and the selection of patterns has been discussed in [5], in relation to the length of the domain.

In this paper we characterize the large time behavior in terms of the global attractor, which is the maximal compact invariant set which attracts all bounded sets in phase space. At present, the question of existence of global attractors and estimation of their dimension is well established for systems defined in bounded domains. Global attractors are then finite dimensional and their analysis can be reduced to the study of finite dimensional dynamical systems. For systems defined on unbounded domains the dynamics is necessarily infinite dimensional [2], and classical tools like the (Hausdorff, fractal) dimension are no longer effective. Here the concept of Kolmogorov ε-entropy plays an important role. Recently this concept was applied to discuss dynamics described by second order equations in unbounded domains. In a series of papers in which the maximum principle was an important ingredient, the existence of the attractor was established and estimates of its ε-entropy were obtained [2], [3]. However, for higher order equations such as (1.1) we encounter serious difficulties due to the lack of compactness, and the interplay between different topologies will play a crucial role.

We prove the existence of a global attractor of solutions of (1.1) in a phase space endowed with a local topology, and present sharp estimates for the Kolmogorov ε-entropy. For (1.1) the local topology is appropriate because of the plethora of bounded stationary solutions such as periodic solutions (cf. e.g. [6]), homoclinic and heteroclinic orbits and chaotic orbits, which were found for (1.2) and (1.3) in one spatial dimension. A strong topology would exclude such stationary solutions and lead to much simpler dynamics.

2 Preliminaries

We shall assume that the nonlinearity $f \in C^2(\mathbf{R})$ in equation (1.1) satisfies the following structure hypotheses:

$$\begin{cases} \mathbf{H1}: & f(s)\cdot s \geq -c_1 + c_2|s|^{2+\delta} \qquad \text{for } |s| \gg 1,\ s\in\mathbf{R}, \\ \mathbf{H2}: & |f(s)\cdot s| \leq c_3|s| \qquad \text{for } |s| \ll 1. \end{cases} \tag{2.1}$$

in which c_1, c_2, c_3 and δ are positive constants.

We shall be using function spaces which are typically designed for unbounded domains:

$$W_b^{\ell,p}(\mathbf{R}^3) = \{u \in \mathcal{D}^1(\mathbf{R}^3) : \|u\|_{b,\ell,p} \overset{\text{def}}{=} \sup_{x_0\in\mathbf{R}^3} \|u, B^1_{x_0}\|_{\ell,p} < \infty\}, \tag{2.2}$$

where $B^1_{x_0}$ is the unit ball centered at x_0. In particular we frequently use the space $\Phi_b \overset{\text{def}}{=} W_b^{4,2}(\mathbf{R}^3)$. In this paper we assume that

$$\mathbf{H3}: \qquad g \in L^2_b(\mathbf{R}^3). \tag{2.3}$$

We also introduce the weighted Sobolev spaces with weight $\varphi(x)$ such that $\varphi(x) \to 0$ as $|x| \to \infty$:

$$\begin{aligned} L^p_\phi(\mathbf{R}^3) &= \{u \in \mathcal{D}'(\mathbf{R}^3) : \|u\|^p_{\phi,p} \overset{\text{def}}{=} \int_{\mathbf{R}^3} \phi(x)|u(x)|^p\,dx < \infty\}, \\ W^{\ell,p}_\phi(\mathbf{R}^3) &= \{u \in \mathcal{D}'(\mathbf{R}^3) : D^\alpha u \in L^p_\phi(\mathbf{R}^3) \quad \text{for all} \quad |\alpha| \leq \ell\}. \end{aligned} \tag{2.4}$$

For the estimation of the lower bound on the Kolmogorov entropy of the attractor, we use the the properties of the space B_{θ,ξ_0}, defined by

$$B_{\theta,\xi_0} \overset{\text{def}}{=} \{u \in L^\infty(\mathbf{R}^3) : \text{supp}(\hat{u}) \subset [-\theta,\theta]^3 + \xi_0, \quad \xi_0 \in \mathbf{R}^3\}, \tag{2.5}$$

where $\hat{u}$ is the Fourier Transform of u. $B(0,\rho_0,B_{\theta,\xi_0})$ will denote the ball of radius ρ_0 about the origin in the space B_{θ,ξ_0}.

3 Existence and uniqueness

The existence of a semigroup for the Cauchy Problem for (1.1) follows from the a-priori estimate:

Theorem 3.1 *Let $u_0 \in \Phi_b$ and let (f, g) in equation(1.1) satisfy* **H1-3**. *Then equation (1.1) possesses a unique global solution $u(t)$ for $0 \leq t < \infty$, which satisfies the following estimate:*

$$\|u(t)\|_{\Phi_b} \leq Q(\|u(0)\|_{\Phi_b})e^{-\alpha t} + Q(\|g\|_{L_b^2}) := Q(u_0, g) \tag{3.1}$$

in which α is a positive constant and Q a generic function of its argument(s).

The proof of Theorem 3.1 is based on successively multiplying equation (1.1) by $u(t)\phi_{\varepsilon,x_0}(x)$, $\partial_{x_i}(\phi_{\varepsilon,x_0}\partial_{x_i}u)$ and $\phi_{\varepsilon,x_0}\partial_t u$, integration by parts, and using assumptions **H1-3**. This leads to an estimate for the L^∞-norm of the nonlinearity. After that, the estimate (3.1) can be obtained exactly as for the linear case. Here $\phi_{\varepsilon,x_0}(x)$ is a smooth weight function such that $\phi_{\varepsilon,x_0}(x) \sim e^{-\varepsilon|x-x_0|}$ as $|x| \to \infty$ where $x_0 \in \mathbf{R}^3$ and $\varepsilon > 0$. We note that for every two solutions $u_1(t)$ and $u_2(t)$ of (1.1) we have

$$\|u_1(t) - u_2(t)\|_{W_\phi^{1,2}} \leq Ce^{Kt}\|u_1(0) - u_2(0)\|_{L_\phi^2}, \tag{3.2}$$

where the constants C and K are independent of the choice of u_1 and u_2. Obviously, this estimate implies uniqueness of the solution.

The existence of a global (in time) solution of (1.1) is a standard consequence of (3.1). Therefore, equation (1.1) generates a semigroup $S_t : \Phi_b \to \Phi_b,\ S_t u_0 = u(t)$, and moreover $S_t : L_b^2(\mathbf{R}^3) \to W_b^{4,2}(\mathbf{R}^3),\ t > 0$.

4 Existence of a global attractor

Definition 4.1 *A set $\mathcal{A} \subset \Phi_b$ is called a (locally compact) attractor of the semigroup S_t if (1) $\mathcal{A}$ is bounded in Φ_b and compact in $\Phi_{\text{loc}} := W_{\text{loc}}^{4,2}(\mathbf{R}^3)$; (2) $S_t\mathcal{A} = \mathcal{A}$ for all $t > 0$; (3) $\mathcal{A}$ is an attracting set of a bounded set in the local topology of Φ_{loc}.*

In the sequel we call an attractor as defined above a $(\Phi_b, \Phi_{\text{loc}})$-attractor for the semigroup $S_t : \Phi_b \to \Phi_b$.

Theorem 4.1 *Let the nonlinearity f, and g in equation (1.1) satisfy the hypotheses* **H1-3**. *Then the semigroup S_t defined above possesses a $(\Phi_b, \Phi_{\text{loc}})$-attractor.*

The proof of Theorem 4.1 is based on the estimate (3.1), standard regularity theory and compactness of the embedding $W_b^{4+\delta,2}(\mathbf{R}^3) \subset\subset W_{\text{loc}}^{4,2}(\mathbf{R}^3)$ for any $\delta > 0$.

Remark We have proved that equation (1.1) has a $(\Phi_b, \Phi_{\text{loc}})$-global attractor $\mathcal{A}$. When g is constant it is easy to see that in general $\mathcal{A}$ cannot be compact in Φ_b.

Indeed, if $\mathcal{A}$ is compact in Φ_b, and $u_0 \in \mathcal{A}$, one can show that translation invariance implies that u_0 must be almost periodic. Since equilibria belong to $\mathcal{A}$, it follows that all equilibria must be almost periodic. It is well known however [6], that equation (1.1) for $q < 0$ and $f(u) = u^3 - u$, as well as equation (1.2), have kinks and pulse type stationary solutions. Plainly, they are not almost periodic.

5 Upper bound of the Kolmogorov entropy of the attractor

We prove the following upper bound for the Kolmogorov ε-entropy of $\mathcal{A}$:

Theorem 5.1 *Let the nonlinearity f, and g in equation (1.1) satisfy the hypotheses* **H1-3**. *Then*

$$\mathcal{H}_\varepsilon(\mathcal{A}\big|_{B^R_{x_0}}, W^{4,2}(B^R_{x_0})) \leq C\left(R + \ln\frac{R_0}{\varepsilon}\right)^3 \ln\frac{R_0}{\varepsilon}, \tag{5.1}$$

where the constants C and R_0 do not depend on ε, R and x_0.

Proof Let $\phi = \phi_{R,x_0}(x)$ be the weight function introduced in Section 2, and let $\phi_{R,x_0}(x) \equiv 1$ in $B^R_{x_0}$. Then obviously

$$\mathcal{H}_\varepsilon(\mathcal{A}\big|_{B^R_{x_0}}, W^{4,2}(B^R_{x_0})) \leq \mathcal{H}_\varepsilon(\mathcal{A}, W^{4,2}_\phi(\mathbf{R}^3)), \tag{5.2}$$

so that it suffices to prove (5.1) for $\mathcal{H}_\varepsilon(\mathcal{A}, W^{4,2}_\phi(\mathbf{R}^3))$.

Let $S \stackrel{\text{def}}{=} S_1$. Then it follows from (3.2) that for $u_1, u_2 \in \mathcal{A}$, we have

$$\|Su_1 - Su_2\|_{W^{4,2}_\phi(\mathbf{R}^3)} \leq L\|u_1 - u_2\|_{L^2_\phi(\mathbf{R}^3)}, \tag{5.3}$$

and hence that

$$\mathcal{H}_\varepsilon(\mathcal{A}, W^{4,2}_\phi(\mathbf{R}^3)) \leq \mathcal{H}_{\varepsilon/L}(\mathcal{A}, L^2_\phi(\mathbf{R}^3)). \tag{5.4}$$

Thus, it will be enough to estimate the right hand side of (5.4), i.e., to prove that

$$\mathcal{H}_{\varepsilon/L}(\mathcal{A}, L^2_\phi(\mathbf{R}^3)) \leq C\left(R + \ln\frac{R_0}{\varepsilon}\right)^3 \ln\frac{R_0}{\varepsilon}, \tag{5.5}$$

where C and R_0 do not depend on ε, R and x_0.

In order to prove (5.5) we use the following recurrence formula:

$$\mathcal{H}_{\varepsilon/2}(\mathcal{A}, L^2_\phi(\mathbf{R}^3)) \leq C\left(R + \ln\frac{R_0}{\varepsilon}\right)^3 + \mathcal{H}_\varepsilon(\mathcal{A}, L^2_\phi(\mathbf{R}^3)). \tag{5.6}$$

This relation can be established following [3] and using (3.2).

Thus, assume that (5.6) holds. Then, plainly, $\mathcal{H}_{R_0}(\mathcal{A}, L^2_\phi(\mathbf{R}^3)) = 0$ for any $R > 0$ and x_0 (recall that $\|\mathcal{A}\|_{L^2_\phi(\mathbf{R}^3)} \leq R_0$). Now we iterate (5.6) k times to obtain, since here the logarithm has base 2,

$$\mathcal{H}_{R_0/2^k}(\mathcal{A}, L^2_\phi(\mathbf{R}^3)) \leq C\sum_{j=0}^{k-1}(R+j)^3 \leq C(R+k)^3 k, \tag{5.7}$$

where C does not depend on k. When we now choose k such that $\varepsilon \sim R_0/2^k$, we obtain from (5.7) that

$$\mathcal{H}_\varepsilon(\mathcal{A}, L^2_\phi(\mathbf{R}^3)) \leq C\left(R + \ln\frac{R_0}{\varepsilon}\right)^3 \ln\frac{R_0}{\varepsilon}, \tag{5.8}$$

as required. □

6 Lower bound of the Kolmogorov entropy of the attractor

We proceed in a more or less standard manner i.e., we aim at constructing the unstable manifold $\mathcal{M}^+(z_0)$ at some equilibrium point z_0. Because $\mathcal{M}^+(z_0) \subset \mathcal{A}$, we know that

$$\mathcal{H}_\varepsilon(\mathcal{A}\big|_{B^R_{x_0}}, W^{4,2}(B^R_{x_0})) \geq \mathcal{H}_\varepsilon(\mathcal{M}^+(z_0), W^{4,2}(B^R_{x_0})), \tag{6.1}$$

so that it suffices to compute the right hand side in (6.1).

To this end, we assume that $g = 0$ and $f(0) = 0$, so that $u = 0$ is an equilibrium solution. Linearizing equation (1.1) about $u = 0$, we obtain the equation

$$w_t + \Delta^2 w + q\Delta w + f'(0)w = 0 \qquad \text{in} \qquad \mathbf{R}^3. \tag{6.2}$$

We now assume that $u = 0$ is exponentially unstable, i.e.,

$$\mathbf{H4}: \qquad \sigma(-\Delta^2 - q\Delta - f'(0), L^2(\mathbf{R}^3)) \cap \{\mathrm{Re}\lambda > 0\} \neq \varnothing, \tag{6.3}$$

where $\sigma(L, W)$ denotes the spectrum of the linear operator L in the space W. The characteristic equation corresponding to equation (6.2) has been studied in [5] (for the Swift-Hohenberg equation). From this analysis conditions on q can readily be identified for which **H4** is satisfied.

We use the following lemma from [3]:

Lemma 6.1 *Assume that $g = 0$ and $f(0) = 0$, and that* **H1-H4** *are satisfied. Then there exist constants $\theta > 0$, $\xi_0 \in \mathbf{R}^3$ and $\rho_0 > 0$, and a C^1-map $\mathbf{V} : B(0, \rho_0, B_{\theta,\xi_0}) \to \mathcal{M}^+ \subset \mathcal{A}$ such that for $u_1, u_2 \in B(0, \rho_0, B_{\theta,\xi_0})$ we have*

$$\|\mathbf{V}(u_1) - \mathbf{V}(u_2)\|_{W^{4,2}(B_0^R)} \geq \|u_1 - u_2\|_{L^\infty(B_0^R)} - C\rho_0^2, \tag{6.4}$$

where C is a constant which is independent of R.

Let $\varepsilon > 0$ be small enough, and $\rho \overset{\text{def}}{=} (\varepsilon/2C)^{1/2} \leq \rho_0$. We choose $u_1, u_2 \in B(0, \rho_0, B_{\theta,\xi_0})$ such that $\|u_1 - u_2\|_{L^\infty(B_0^R)} > \varepsilon$ Then we conclude from (6.4) that

$$\|\mathbf{V}(u_1) - \mathbf{V}(u_2)\|_{W^{4,2}(B_0^R)} \geq \varepsilon/2. \tag{6.5}$$

Combining (6.4) and (6.5) we obtain

$$\mathcal{H}_{\varepsilon/4}(\mathcal{A}\big|_{B_{x_0}^R}, W^{4,2}(B_{x_0}^R)) \geq \mathcal{H}_\varepsilon(B(0, (\varepsilon/2C)^{1/2}, B_{\theta,\xi_0}), L^\infty(\mathbf{R}^3)). \tag{6.6}$$

By a scaling argument we then have

$$\mathcal{H}_\varepsilon(B(0, (\varepsilon/2C)^{1/2}, B_{\theta,\xi_0}), L^\infty(\mathbf{R}^3)) = \mathcal{H}_{(2C\varepsilon)^{1/2}}(B(0, 1, B_{\theta,\xi_0}), L^\infty(\mathbf{R}^3)) \geq C_* R^3 \ln \frac{\widetilde{R}_0}{\varepsilon}, \tag{6.7}$$

where the last estimate is due to Kolmogorov [4]. Here the positive constants C_* and $\widetilde{R}_0$ are independent of R and ε. Thus we have proved:

Theorem 6.1 *Assume that $g = 0$ and $f(0) = 0$, and that* **H1-H4** *are satisfied. Then, the entropy of the restrictions $\mathcal{A}\big|_{B_0^R}$ satisfies the following lower bound:*

$$\mathcal{H}_\varepsilon(\mathcal{A}\big|_{B_0^R}, L^\infty(B_0^R)) \geq C_* R^3 \ln \frac{\widetilde{R}_0}{\varepsilon}, \tag{6.8}$$

where the positive constants C_* and $\widetilde{R}_0$ are independent of R and ε.

Remark The estimate (6.8) shows that the restrictions $\mathcal{A}\big|_{B_{x_0}^R}$ have usually infinite fractal dimension. Moreover, estimates (5.1) and (6.8) confirm the fact that the upper bounds (5.1) are in a sense sharp for every fixed values of R, ε and x_0.

We emphasize that in the case $g = \text{const}$, we have an additional structure on $\mathcal{A}$, namely we have an action of a $(1 + 3)$ - parametric semigroup $\{\mathbb{S}_{(t,h)}, t \geq 0, h \in \mathbf{R}^3\}$ on $\mathcal{A}$, that is

$$\mathbb{S}_{(t,h)}\mathcal{A} = \mathcal{A}, \quad \mathbb{S}_{(t,h)} := S_t \circ T_h, \quad \mathbb{S}_{(t,h)} : \Phi_b \to \Phi_b \quad \text{with} \quad T_h \circ S_t = S_t \circ T_h. \tag{6.9}$$

Here $\{T_h, h \in \mathbf{R}^3\}$ is a group of translation, $(T_h u)(x) := u(x + h)$. In a forthcoming paper, we will study dynamical properties of $\mathbb{S}_{(t,h)}$ in order to describe spatio-temporal complexity (for some values of q in (1.1)) of the attractor.

References

[1] Dee, G.T. & W. van Saarloos, *Bistable systems with propagating fronts leading to pattern formation*, Phys. Rev. Lett. **60** (1988) 2641-2644.

[2] Efendiev, M.A. & S. Zelik,*The attractor for a nonlinear reaction-diffusion system in an unbounded domain*, Comm. Pure Appl. Math. **LIV** (2001) 625-688

[3] Efendiev, M.A. & S. Zelik, *Upper and lower bounds for the Kolmogorov entropy of the attractor for an RDE in an unbounded domain*, J. Dyn. Diff. Equ. **14** (2002) 369-404.

[4] Kolmogorov, A.N. & V.M. Tikhomirov, *$\mathcal{E}$-entropy and ε-capacity of sets in functional spaces* In: *Selected works of A.N. Kolmogorov*, Vol. III, Kluwer, Dordrecht, 1993.

[5] Peletier, L.A. & V. Rottschäfer, *Pattern selection of solutions of the Swift-Hohenberg equation*, Physica D **194** (2004) 95-126.

[6] Peletier, L.A. & W.C. Troy, *Spatial Patterns: Higher order models in Physics and Mechanics*, Birkhäuser, Boston, 2001.

[7] Swift, J.B. & P.C. Hohenberg, *Hydrodynamic fluctuations at the convective instability*, Phys. Rev. A **15** (1977) 319-328.