A new class of entropy solutions of the Buckley-Leverett equation

C.J. van Duijn*, L.A. Peletier† & I.S. Pop*

Abstract

We discuss an extension of the Buckley-Leverett (BL) equation describing two-phase flow in porous media. This extension includes a third order mixed derivatives term and models the dynamic effects in the pressure difference between the two phases. We derive existence conditions for traveling wave solutions of the extended model. This leads to admissible shocks for the original BL equation, which violate the Oleinik entropy condition and are therefore called nonclassical. In this way we obtain non-monotone weak solutions of the initial-boundary value problem for the BL equation consisting of constant states separated by shocks, confirming results obtained experimentally.

1 Introduction

We consider the first order initial-boundary value problem

\[
\begin{align*}
\text{(BL)} \quad \left\{ \begin{array}{ll}
\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 & \text{in } Q = \{(x,t) : x > 0, t > 0\}, \\
u(x,0) = 0 & x > 0, \\
u(0,t) = u_B & t > 0,
\end{array} \right.
\end{align*}
\]

where \(u_B\) is a constant such that \(0 \leq u_B \leq 1\). The nonlinearity \(f : \mathbb{R} \to \mathbb{R}\) is given by

\[
f(u) = \frac{u^2}{u^2 + M(1-u)^2}, \quad \text{if } 0 \leq u \leq 1,
\]

whilst \(f(u) = 0\) if \(u < 0\) and \(f(u) = 1\) if \(u > 1\). Here, \(M > 0\) is a fixed constant, The function \(f(u)\) is shown in Figure 1.

Equation (1.1), with the given flux function \(f\), arises in two-phase flow in porous media and Problem (BL) models oil recovery by water-drive in one-dimensional horizontal

*Department of Mathematics and Computer Science, Technische Universiteit Eindhoven, PO Box 513, 5600 MB Eindhoven, The Netherlands \{c.j.v.duijn, i.pop\}@tue.nl.
†Mathematical Institute, Leiden University, PO Box 9512, 2300 RA Leiden, The Netherlands peletier@math.leidenuniv.nl, and Centrum voor Wiskunde en Informatica, PB 94079, 1090 GB Amsterdam, The Netherlands.
flow. In this context, \( u : \overrightarrow{Q} \to [0, 1] \) denotes water saturation, \( f \) the water fractional flow function and \( M \) the water/oil viscosity ratio. In petroleum engineering, equation (1.1) is known as the Buckley-Leverett equation [4]. It is a prototype for first-order conservation laws with convex-concave flux functions.

It is well known that first-order equations such as (1.1) may have solutions with discontinuities, or shocks. The value \((u_\ell)\) to the left of the shock, the value \((u_r)\) to the right, and the speed \( s \) of the shock with trace \( x = x(t) \) are related through the Rankine-Hugoniot condition,

\[
\text{(RH)} \quad \frac{dx}{dt} = s = \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r}.
\]

We will denote shocks by their values to the left and to the right: \( \{u_\ell, u_r\} \).

If a function \( u \) is such that equation (1.1) is satisfied away from the shock curve, and the Rankine-Hugoniot condition is satisfied across the curve, then \( u \) satisfies the identity

\[
\int_Q \left\{ u \frac{\partial \varphi}{\partial t} + f(u) \frac{\partial \varphi}{\partial x} \right\} = 0 \quad \text{for all} \quad \varphi \in C_0^\infty(Q).
\]

Functions \( u \in L^\infty(Q) \), which satisfy (1.4) are called weak solutions of equation (1.1). Clearly, for any \( u_B \in [0, 1] \), a weak solution of Problem (BL) is given by the shock wave

\[
u(x, t) = S(x, t) \overset{\text{def}}{=} \begin{cases} u_B & \text{for} \quad x < st \\ 0 & \text{for} \quad x > st \end{cases} \quad \text{where} \quad s = \frac{f(u_B)}{u_B}.
\]

Experiments of two-phase flow in porous media reveal complex infiltration profiles, which may involve overshoot, i.e. profiles may not be monotone [12]. Our main objective is to understand the shape of these profiles and to determine how the shape depends on the boundary value \( u_B \) and the flux function \( f(u) \).

Equation (1.1) usually arises as the limit of a family of extended equations of the form

\[
\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = A_\varepsilon(u), \quad \varepsilon > 0,
\]

in which \( A_\varepsilon(u) \) is a singular regularisation term involving higher order derivatives. It is often referred to as a viscosity term. Weak solutions of Problem (BL) are called admissible.
when they can be constructed as limits, as $\varepsilon \to 0$, of solutions $u_\varepsilon$ of equation (1.6), i.e., for which $A_\varepsilon(u_\varepsilon) \to 0$ as $\varepsilon \to 0$ in some weak sense. We return to this limit in Section 6. This raises the question which of the shock waves $S(x,t)$ defined in (1.5) are admissible. We shall see that this depends on the operator $A_\varepsilon$. To obtain criteria for admissibility we shall use families of traveling wave solutions.

A classical viscosity term is

$$A_\varepsilon(u) = \varepsilon \frac{\partial^2 u}{\partial x^2},$$

and with this term, equation (1.6) becomes

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}. \quad (1.7)$$

Seeking a traveling wave solution, we put

$$u = u(\eta) \quad \text{with} \quad \eta = \frac{x - st}{\varepsilon}, \quad (1.8)$$

and we find that $u(\eta)$ satisfies the following two-point boundary value problem:

$$\begin{aligned}
- s u' + (f(u))' &= u'' \quad \text{in} \quad \mathbb{R} \quad (1.9a) \\
u(-\infty) &= u_\ell, \quad u(\infty) = u_r, \quad (1.9b)
\end{aligned}$$

where primes denote differentiation with respect to $\eta$. An elementary analysis shows that Problem (1.9) has a solution if and only if $f$ and the limiting values $u_\ell$ and $u_r$ satisfy (i) the Rankine-Hugoniot condition (1.3), and (ii) the Oleinik entropy condition [23]:

$$f(u_\ell) - f(u) \geq \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r} \quad \text{for} \quad u \quad \text{between} \quad u_\ell \quad \text{and} \quad u_r. \quad (1.10)$$

Shocks $\{u_\ell, u_r\}$ which satisfy (E) are called classical shocks.

Note that in the limit as $\varepsilon \to 0^+$, traveling waves converge to the shock $\{u_\ell, u_r\}$.

Applying (RH) and (E) to the flux function (1.2) we find that the function $S(x,t)$ defined in (1.5) is an admissible shock wave if and only if

$$s = \frac{f(u_B)}{u_B} \quad (\text{RH}) \quad \text{and} \quad u_B \leq \alpha \quad (\text{E}), \quad (1.11)$$

where $\alpha$ is the unique root of the equation

$$f'(u) = \frac{f(u)}{u}. \quad (1.12)$$

It is found to be given by

$$\alpha = \sqrt{\frac{M}{M + 1}} \in (0,1).$$

If $u_B > \alpha$, then the weak solution is composed of a rarefaction wave in the region where $u > \alpha$, and a shock which spans the range $0 < u < \alpha$. Thus, for any $u_B \in (0,1]$ the weak
solution \( u(x, t) \) is, at any given time \( t \), a non-increasing function of \( x \), in contrast to the experimental data for infiltration in porous media.

For gaining a better understanding of the data, it is natural to go back to the origins of equation (1.1). With \( S_i \) \((i = o, w)\) being the saturations of the two phases, oil and water, conservation of mass yields the equation

\[
\phi \frac{\partial S_i}{\partial t} + \frac{\partial q_i}{\partial x} = 0, \quad i = o, w
\]

where \( q_i \) denotes the specific discharge of oil/water and \( \phi \) the porosity of the medium. By Darcy’s law \( q_i \) is proportional to the the gradient of the phase pressure \( P_i \):

\[
q_i = -k_{ri}(S_i) \frac{\partial P_i}{\partial x},
\]

where \( k \) denotes the absolute permeability, and \( k_{ri} \) and \( \mu_i \) the relative permeability and the viscosity of water, respectively oil. The capillary pressure \( P_c \) expresses the difference in the pressures of the two phases:

\[
P_c = P_o - P_w,
\]

This quantity is commonly found to depend on one phase saturation, say \( S_w \). In addition to this, studies like [22] and [24] show that \( P_c \) does not only depend on \( S_w \), but also involves hysteretic and dynamic effects. Hassanizadeh and Gray [16] and [17] have defined the dynamic capillary pressure as

\[
P_c = p_c(S_w) - \phi \frac{\partial S_w}{\partial t},
\]

where \( p_c(S_w) \) is the static capillary pressure and \( \tau \) a positive constant. Assuming that the medium is completely saturated,

\[
S_w + S_o = 1,
\]

and we obtain, upon combining equations (1.12)-(1.15), the single equation for the water saturation \( u = S_w \):

\[
\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = - \frac{\partial}{\partial x} \left\{ H(u) \frac{\partial}{\partial x} \left( J(u) - \tau \frac{\partial u}{\partial t} \right) \right\},
\]

in which the functions \( f, H \) and \( J \) are related to \( k_{ri} \) and \( p_c \). Restricting – for simplicity – to linear terms on the right-hand side of (1.16), we obtain, after a suitable scaling, the pseudo-parabolic equation

\[
\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \epsilon^2 \frac{\partial^2 u}{\partial x^2} + \epsilon^2 \tau \frac{\partial^3 u}{\partial x^2 \partial t}.
\]

Thus, in addition to the classical second order term \( \epsilon u_{xx} \), we find a third order term \( \epsilon^2 \tau u_{xxt} \), its relative importance being determined by the parameter \( \tau \). We shall show that the value of \( \tau \) is critical in determining the type of profile the solution of Problem (BL) will have.
We derive admissibility conditions of shocks by analyzing the existence and the nonexistence of traveling wave solutions of equation (1.17) with appropriate limit values \( u_\ell \) and \( u_r \). The parameter \( \tau \) will be seen to serve as a bifurcation parameter: for small values of \( \tau > 0 \) the situation will be much like in the classical case (E), but when \( \tau \) exceeds a critical value \( \tau_* > 0 \) the situation changes abruptly and new types of shock waves become admissible. In the following three theorems we give conditions for the existence and nonexistence of traveling wave solutions of equation (1.17) which form the basis of classical and nonclassical admissibility conditions for equation (1.1). In the next section, they will be used to construct admissible weak solutions of Problem (BL).

Substituting (1.8) into (1.17) we obtain the equation

\[
-su' + (f(u))' = u'' - s\tau u''' \quad \text{in} \quad R.
\]

When we integrate this equation over \((\eta, \infty)\), we obtain the second order boundary value problem

\[
(TW) \begin{cases} 
-s(u - u_r) + \{f(u) - f(u_r)\} = u' - s\tau u'' \quad \text{in} \quad R, \\
(1.18a) 
\end{cases}
\]

where \( s = s(u_\ell, u_r) \) is given by the Rankine-Hugoniot condition (1.3).

We consider two cases:

\[\begin{array}{ll}
(I) & u_r = 0, \quad u_\ell > 0 \quad \text{and} \quad (II) \quad u_r > u_\ell > 0. \\
\end{array}\]

**Case I: \( u_r = 0 \)** We first establish an upper bound for \( u_\ell \).

**Proposition 1.1** Let \( u \) be a solution of Problem (TW) such that \( u_r = 0 \). Then, \( u_\ell < \beta \), where \( \beta \) is the value of \( u \) for which the equal area rule holds:

\[
\int_0^\beta \left\{ f(u) - \frac{f(\beta)}{\beta} u \right\} du = 0. \quad (1.20)
\]

In Figure 2 we indicate the different critical values of \( u \) in a graph of \( f(u) \) when \( M = 2 \).

**Proof.** When we put \( u_r = 0 \) into (1.18a), multiply by \( u' \) and integrate over \( R \), we obtain the inequality

\[
\int_0^{u_\ell} \left\{ f(u) - \frac{f(u_\ell)}{u_\ell} u \right\} du = -\int_R (u')^2(\eta) d\eta < 0,
\]

from which it readily follows that \( u_\ell < \beta \).

Next, we turn to the questions of existence and uniqueness. Note that if \( u_\ell \in (\alpha, \beta) \), then

\[
\begin{align*}
\begin{array}{ll}
s = s(u_\ell, 0) = \frac{f(u_\ell)}{u_\ell} > f'(u_\ell) \geq f'(0) & \quad \text{for} \quad u_\ell > \alpha,
\end{array}
\end{align*}
\]

and traveling waves – if they exist – lead to an admissibility condition for fast undercompressive waves. For convenience we write \( s(u_r, 0) = s(u_\ell) \).

In the theorems below we first show that for each \( \tau > 0 \), there exists a unique value of \( u_\ell \geq \alpha \) – denoted by \( \overline{u}(\tau) \) – for which there exists a solution of Problem (TW) such that \( u_r = 0 \).
Theorem 1.1 Let $M > 0$ be given. Then there exists a constant $\tau_* > 0$ such that:

(a) for every $0 \leq \tau \leq \tau_*$, Problem (TW) has a unique solution with $u_\ell = \alpha$ and $u_r = 0$.

(b) for each $\tau > \tau_*$ there exists a unique constant $\overline{u}(\tau) \in (\alpha, \beta)$ such that Problem (TW) has a unique solution with $u_\ell = \overline{u}(\tau)$ and $u_r = 0$.

(c) the function $\overline{u} : [0, \infty) \to [\alpha, \beta]$ defined by

$$
\overline{u}(\tau) = \begin{cases} 
\alpha & \text{for } 0 \leq \tau \leq \tau_* \\
\overline{u}(\tau) & \text{for } \tau > \tau_*
\end{cases}
$$

is continuous, strictly increasing for $\tau \geq \tau_*$, and $\overline{u}(\infty) = \beta$.

The solutions in Parts (a) and (b) are strictly decreasing.

We shall refer to $\overline{u} = \overline{u}(\tau)$ as the plateau value of $u$. In the sequel we shall often denote the speed $s(\overline{u})$ of the shock $\{\overline{u}, 0\}$ by $\overline{s}$.

Next, suppose that $u_\ell \neq \overline{u}(\tau)$. To deal with this case we need to introduce another critical value of $u$, which we denote by $\underline{u}(\tau)$.

- For $\tau \in [0, \tau_*]$ we put $\underline{u}(\tau) = \alpha$, and
- For $\tau > \tau_*$ we define $\underline{u}(\tau)$ as the unique zero in the interval $(0, \overline{u}(\tau))$ of the equation

$$
f(r) - \frac{f(\overline{u})}{\overline{u}}r = 0, \quad 0 < r < \overline{u}.
$$

Plainly, if $\tau > \tau_*$, then

$$
0 < \underline{u}(\tau) < \alpha < \overline{u}(\tau) < \beta \quad \text{for} \quad \tau > \tau_*.
$$

In Figure 3, we show graphs of the functions $\overline{u}(\tau)$ and $\underline{u}(\tau)$. Both graphs have been computed numerically for $M = 2$. The function $\overline{u}(\tau)$ was obtained by applying a shooting technique to a first order problem derived from (1.18) in which $u$ is the independent variable (see Remark 4.2), and $\underline{u}(\tau)$ then follows from a simple construction involving $f(u)$.

The following theorem states that if $u_r = 0$ and $u_\ell \in (0, \overline{u})$, then traveling waves exist if and only if $u_\ell < \underline{u}(\tau)$.  

Figure 2: Critical values of $u$ when $M = 2$: $\alpha = 0.816...$ and $\beta = 1.147...$
Figure 3: The functions $\bar{u}(\tau)$ and $\underline{u}(\tau)$ computed for $M = 2$

**Theorem 1.2** Let $M > 0$ and $\tau > 0$ be given, and let $u = u(\tau)$ and $\bar{u} = \bar{u}(\tau)$.

(a) For any $u_\ell \in (0, u)$, there exists a unique solution of Problem (TW) such that $u_\tau = 0$. We have $s(u_\ell) < \bar{u}$.

(b) Let $\tau > \tau_\star$. Then for any $u_\ell \in (\underline{u}, \bar{u})$, there exists no solution of Problem (TW) such that $u_\tau = 0$.

The solution in Part (a) may exhibit a damped oscillation as it tends to $u_\ell$.

**Case II: $u_\tau > 0$** The results of Case I raise the question as to how to deal with solutions of Problem (BL) when $u_B \in (\underline{u}, \bar{u})$ and by Theorem 1.2 there is no traveling wave solution with $u_\tau = 0$. In this situation we use two traveling waves in succession: one from $u_B$ to the plateau value $\bar{u}$, and one from $\bar{u}$ down to $u = 0$. The existence of the latter has been established in Theorem 1.1. In the next theorem we deal with the former, in which $u_\tau = \bar{u}$.

**Theorem 1.3** Let $M > 0$ and $\tau > \tau_\star$ be given, and let $u = u(\tau)$ and $\bar{u} = \bar{u}(\tau)$.

(a) For any $u_\ell \in (\underline{u}, \bar{u})$, there exists a unique solution of Problem (TW) such that $u_\tau = \bar{u}$. We have $s(u_\ell, \bar{u}) < \bar{u}$.

(b) For any $u_\ell \in (0, u)$, there exists no solution of Problem (TW) such that $u_\tau = \bar{u}$.

The solution in Part (a) may exhibit a damped oscillation as it tends to $u_\ell$.

In Section 2 we show how these theorems can be used to construct weak solutions of Problem (BL), i.e. weak solutions, which are admissible within the context of the regularization proposed in equation (1.17), and which involve shocks which may be either classical or nonclassical. In Section 3 we solve the Cauchy Problem for equation (1.17) numerically, starting from a smoothed step function, i.e., $u(x, 0) = u_B \tilde{H}(-x)$, where $\tilde{H}(x)$ is a regularized Heaviside function, and $M = 2$. We find that for different values of the parameters $u_B$, $\tau$ and $\varepsilon$ the solution converges to solutions constructed in Section 2 as $t \to \infty$. In Sections 4 and 5 we prove Theorems 1.1, 1.2 and 1.3. The proofs rely on phase plane arguments. We conclude this paper with a discussion of the dissipation of the entropy function $u^2/2$ when $u$ is the solution of the Cauchy Problem for equation (1.17) (cf. Section 6).
Nonclassical entropy solutions of equations (1.1), such as those discussed in this paper, have been found in the context of different regularizations $\mathcal{A}_\varepsilon(u)$ and flux functions $f(u)$. We mention in particular the seminal paper [18] in which, after scaling, $\mathcal{A}_\varepsilon(u) = \varepsilon u_{xx} + \varepsilon^2 \delta u_{xxx}$, where $\delta$ is a constant, and $f(u) = u^3$. Here, for $\delta > 0$ an explicit function $\varphi(u; \delta)$ is derived such that the shock $\{u_\ell, u_r\}$ is admissible if and only if $u_r = \varphi(u_\ell; \delta)$. Properties of this kinetic function $\varphi$, such as monotonicity with respect to $u_\ell$, have been studied in a series of papers (see [5] and the references cited there). In this paper we usually fix $u_r$ and focus on the relation between $u_\ell$ and the parameter $\tau$. We also mention [6] and [7] in which the viscosity term was motivated by thin film flow: $\mathcal{A}_\varepsilon(u) = -(u^3 u_{xxx})_x$ and the flux function is $f(u) = u^2 - u^3$, and a second order model for highway traffic flow which may lead to wave phenomena, that was recently discussed in [25].

Going back to the Hassanizadeh-Gray model we mention a numerical study of traveling waves of the original, fully nonlinear, equations of this model in [13, 14]. It will be interesting to determine to what extent such results as derived in this paper for the simplified equation (1.17), continue to hold for the full equation (1.16) when realistic functions $H(u)$ and $J(u)$ are used. Such equations may be degenerate at $u = 0$ as well as at $u = 1$, and singular behavior, as in the Porous Media Equation [1], [21] may be expected.

2 Entropy solutions of Problem (BL)

In this section we give a classification of admissible solutions of Problem (BL) based on the "extended viscosity model" (1.17), using the results about traveling wave solutions formulated in Theorems 1.1, 1.2 and 1.3. Before doing that we make a few preliminary observations, and we recall the construction based on the classical model (1.7).

Because equation (1.1) is a first order partial differential equation, and $u_B$ is a constant, any solution of Problem (BL) only depends on the combination $x/t$, with shocks, constant states and rarefaction waves as building blocks [23]. The latter are continuous solutions of the form

$$u(x,t) = r(\zeta) \quad \text{with} \quad \zeta = \frac{x}{t}. \quad (2.1)$$

After substitution into (1.1) this yields the equation

$$\frac{dr}{d\zeta} \left(-\zeta + \frac{df}{du}(r(\zeta))\right) = 0. \quad (2.2)$$

Hence, the function $r(\zeta)$ satisfies:

$$\text{either} \quad r = \text{constant} \quad \text{or} \quad \frac{df}{du}(r(\zeta)) = \zeta.$$

When solving Problem (BL), we will combine solutions of equation (2.2) with admissible shocks, i.e. shocks $\{u_\ell, u_r\}$ in which $u_\ell$ and $u_r$ are such that equation (1.6), with the a priori selected and physically relevant viscous extension $\mathcal{A}_\varepsilon$ has a traveling wave solution $u(\eta)$ such that $u(\eta) \rightarrow u_\ell$ as $\eta \rightarrow -\infty$ and $u(\eta) \rightarrow u_r$ as $\eta \rightarrow +\infty$. Although in the physical context in which the viscous extension employed in equation (1.17) was derived, $0 \leq u_B \leq 1$, we shall drop this restriction. It will be convenient to first assume that $0 \leq u_B \leq \beta$. At the end of this section we discuss the case that $u_B > \beta$. 


All solution graphs shown in this section and the next, are numerically obtained solutions of equation (1.17). They are expressed in terms of the independent variable $\zeta$ and $t$, i.e.

$$u(x, t) = w(\zeta, t),$$
and considered for fixed $\varepsilon > 0$ ($= 1$) and for large times $t$. We return to the computational aspects in Section 3.

Before discussing the implications of the viscous extension in (1.17), we recall the construction of classical entropy solutions of Problem (BL). It uses (RH), and the entropy condition (E) which was derived for the diffusive viscous extension used in (1.8). We distinguish two cases:

(a) $0 \leq u_B \leq \alpha$ and (b) $\alpha < u_B \leq \beta$.

Case (a): $0 \leq u_B \leq \alpha$. This case was discussed in the Introduction, where we found that the entropy solution is given by the shock $\{u_B, 0\}$.

Case (b): $\alpha < u_B \leq \beta$. In the Introduction we saw that in this case, the shock $\{u_B, 0\}$ is no longer a classical entropy solution. Instead, in this case the entropy solution is a composition of three functions:

$$u(x, t) = v(\zeta) = \begin{cases} 
  u_B & \text{for } 0 \leq \zeta \leq \zeta_B \\
  r(\zeta) & \text{for } \zeta_B \leq \zeta \leq \zeta_* \\
  0 & \text{for } \zeta_* \leq \zeta < \infty,
\end{cases} \quad (2.3)$$

where $\zeta_B$ and $\zeta_*$ are determined by

$$\zeta_B = \frac{df}{du}(u_B) \quad \text{and} \quad \zeta_* = \frac{df}{du}(\alpha) = \left( \frac{f(\alpha)}{\alpha} \right) = s(\alpha),$$

and $r : [\zeta_B, \zeta_*] \to [\alpha, u_B]$ by the relation

$$\frac{df}{du}(r(\zeta)) = \zeta \quad \text{for} \quad \zeta_B \leq \zeta \leq \zeta_* \quad (2.4)$$

Since $f''(u) < 0$ for $u \in [\alpha, u_B]$, equation (2.4) has a unique solution, and hence $r(\zeta)$ is well defined. Note that if $u_B \geq 1$, then $\zeta_B = 0$, because $f'(u) = 0$ if $u \geq 1$.

Solutions corresponding to Case (b) are shown in Figure 4.

We now turn to the pseudo-parabolic equation (1.17) that arises in the context of the two-phase flow model of Hassanizadeh and Gray [16], [17]. For this problem, we define a class of non-classical entropy solutions in which shocks are admissible if Problem (TW) has a travelling wave solution with the required limit conditions.

For given $M > 0$ and $\tau > 0$, the relative values of $u_B$ and $\overline{u}(\tau)$ and $\underline{u}(\tau)$ are now important for the type of solution we are going to get. It is easiest to represent them in the $(u_B, \tau)$-plane. Specifically, we distinguish three regions in this plane:

$$A = \{(u_B, \tau) : \tau > 0, \quad \overline{u}(\tau) \leq u_B < \beta\},$$

$$B = \{(u_B, \tau) : \tau > \tau_*, \quad \underline{u}(\tau) < u_B < \overline{u}(\tau)\},$$

$$C = \{(u_B, \tau) : \tau > 0, \quad 0 < u_B < \underline{u}(\tau)\},$$
$\zeta = \tan(\Phi)$

Figure 4: Case (b): Solution graph (left) and flux function with transitions from $u_B$ to $\alpha$ and from $\alpha$ to 0 (right)

Figure 5: The regions $A$, $B$ and $C$ in the $(u_B, \tau)$-plane

These three regions are shown in Figure 5.

**Case I:** $(u_B, \tau) \in A$

If $0 \leq \tau \leq \tau_s$, i.e., $(u_B, \tau) \in A_1$, the construction is as in the classical case described above. After a plateau, where $u = u_B$, and $0 \leq \zeta = x/t \leq \zeta_B$, we find a rarefaction wave $r(\zeta)$ from $u_B$ down to $\alpha$ followed by a classical shock connecting $\alpha$ to the initial state $u = 0$.

If $\tau > \tau_s$, i.e., $(u_B, \tau) \in A_2$, the solution starts out as before, with a plateau where $u = u_B$ and $0 \leq \zeta \leq \zeta_B$, and a rarefaction wave $r(\zeta)$ which now takes $u$ down from $u_B$ to $\alpha$. This takes place over the interval $\zeta_B \leq \zeta \leq \zeta_B$. By (2.2),

$$\zeta = \frac{df}{du}(\bar{\tau}(\tau)).$$

Subsequently, $u$ drops down to the initial state $u = 0$ through a shock, $\{\bar{\tau}, 0\}$, which is admissible by Theorem 1.1. By (RH) the shock moves with speed

$$s = \bar{s} = \frac{f(\bar{\tau})}{\bar{\tau}} > \frac{df}{du}(\bar{\tau}) = \zeta.$$
because $f$ is concave on $(\alpha, \infty)$. Therefore, the shock outruns the rarefaction wave and a second plateau develops between the rarefaction wave and the shock in which $u = \overline{u}$. Summarizing, we find that the (non-classical) entropy solution has the form:

$$u(x,t) = v(\zeta) = \begin{cases} 
  u_B & \text{for } 0 \leq \zeta \leq \zeta_B \\
  r(\zeta) & \text{for } \zeta_B \leq \zeta \leq \overline{\zeta} \\
  \overline{u}(\tau) & \text{for } \overline{\zeta} \leq \zeta \leq \overline{s} \\
  0 & \text{for } \overline{s} \leq \zeta < \infty.
\end{cases} \quad (2.6)$$

A graph of $v(\zeta)$ is given in Figure 6.

Figure 6: Case I: Solution graph (left) and flux function (right), with transitions from $u_B = 1$ to $\overline{u}(\tau)$ and from $\overline{u}(\tau)$ to 0

Note that if $u_B \geq 1$, then $\zeta_B = 0$. At this point $v$ shocks to the maximum of $\overline{u}(\tau)$ and 1. If $\overline{u}(\tau) \geq 1$, then the rarefaction wave disappears and for $\zeta > 0$ the solution is continued by the shock $\{\overline{u}(\tau), 0\}$.

Case II: $(u_B, \tau) \in \mathcal{B}$

It follows from Theorem 1.2 that there are no traveling wave solutions with $u_l = u_B$ and $u_r = 0$, so that the shock $\{u_B, 0\}$ is now not admissible. However, in Theorem 1.3 we have shown that there does exist a traveling wave solution, and hence an admissible shock, with $u_l = u_B$ and $u_r = \overline{u}(\tau)$, and speed $s = s(u_B, \overline{u}(\tau))$. This shock is then followed by a second shock from $u = \overline{u}(\tau)$ down to $u = 0$, which is admissible because by Theorem 1.1, there does exist a traveling wave solution which connects $\overline{u}$ and $u = 0$ with speed $\overline{s} > s(u_B, \overline{u}(\tau))$. Thus

$$u(x,t) = v(\zeta) = \begin{cases} 
  u_B & \text{for } 0 \leq \zeta \leq s(u_B, \overline{u}) \\
  \overline{u}(\tau) & \text{for } s(u_B, \overline{u}) \leq \zeta \leq \overline{s} \\
  0 & \text{for } \overline{s} \leq \zeta < \infty.
\end{cases} \quad (2.7)$$

An example of this type of solution is shown in Figure 7. The undershoot in the solution graph is due to oscillations which are also present in the traveling waves.
Figure 7: Case II: Solution graph (left) and flux function (right), with transitions from $u_B = 0.75$ to $\overline{u}(\tau)$ and from $\overline{u}(\tau)$ to 0

Remark 2.1 It is readily seen that 

$$s(u_B, \overline{u}(\tau)) \nearrow s \text{ as } u_B \searrow u,$$

while the plateau level $\overline{u}$ remains the same. Thus, in this limit, the plateau

$$\{ \left( u, \frac{x}{t} \right) : u = \overline{u}(\tau), s(u_B, \overline{u}(\tau)) < \frac{x}{t} < \overline{s} \}$$

becomes thinner and thinner and eventually disappears when $u_B = u$.

Remark 2.2 If $u_B = 1$ and $\overline{u}(\tau) > 1$, then the first shock degenerates in the sense that

$$s(u_B, \overline{u}(\tau)) = 0 \quad \text{and} \quad u(x, t) = \overline{u}(\tau) \quad \text{for all} \quad 0 < \frac{x}{t} < \overline{s}.$$

Case III: $(u_B, \tau) \in \mathcal{C}$

We have seen in Theorem 1.2 that in this case there exists a traveling wave solution with $u_\ell = u_B$ and $u_r = 0$. It may exhibit oscillatory behavior near $u = u_\ell$, and it leads to the classical entropy shock solution $\{u_B, 0\}$. An example of such a solution is shown in Figure 8. Note the overshoot in the solution graph, reflecting oscillations also present in the traveling waves.

We conclude with a remark about the case when $u_B > \beta$. It is readily verified that for such values of $u_B$ the situation is completely analogous to the one for $(u_B, \tau) \in \mathcal{A}$.

3 Numerical experiments for large times

In this section we report on numerical experiments carried out for the initial-value problem for equation (1.17) in the domain $S = \mathbb{R} \times \mathbb{R}^+$:

$$\begin{cases}
\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2} + \varepsilon^2 \tau \frac{\partial^3 u}{\partial x^2 \partial t} \\
u(x, 0) = u_B \hat{H}(-x)
\end{cases} \quad \text{in } S, \quad \text{(3.1a)}$$

$$u(x, 0) = u_B \hat{H}(-x) \quad \text{for } x \in \mathbb{R}, \quad \text{(3.1b)}$$
Here $\tilde{H}(x)$ is a smooth monotone approximation of the Heaviside function $H$. Following [11], starting initially with $u_B H(-x)$ would cause a jump discontinuity in $u$ at $x = 0$, which persists for all $t > 0$. This would require an appropriate numerical approach to ensure the continuity in flux and pressure. By the above choice we avoid this unnecessary complication.

For solving (3.1) numerically we use the forward Euler scheme. The terms involving $\partial_{xx}$ are discretized by finite differences. To deal with the first order term we apply a minmod slope limiter method that is based on first order upwinding and Richtmyer’s scheme. Even though the scheme is explicit, we have to solve a linear system for each time step. This is due to the time derivative in the last term in (3.1a). Details can be found in [9] (see also [8], Chapter 3), where alternative schemes are discussed.

Important parameters in this problem are $M$, $\varepsilon$, $\tau > 0$, and $u_B \in (0, 1]$. The scaling

$$x \rightarrow \frac{x}{\varepsilon}, \quad t \rightarrow \frac{t}{\varepsilon}$$

removes the parameter $\varepsilon$ from the equation (3.1a). Therefore, we fix $\varepsilon = 1$ and we show how for different values of $\tau$ and $u_B$ the solution $u(x, t)$ of Problem (3.1) converges as $t \rightarrow \infty$ to qualitatively different final profiles. Throughout we take $M = 2$. Computationally we found that (see also Figures 3 and 5)

$$\tau_* = 0.61.... \quad (3.2)$$

In the figures below we show graphs of solutions at various times $t$, appropriately scaled in space. Specifically, we show graphs of the function

$$w(\zeta, t) = u(x, t) \quad \text{where} \quad \zeta = \frac{x}{t}, \quad (3.3)$$

so that a front with speed $s$ will be located at $\zeta = s$.

We begin with a simulation when $(u_B, \tau) = (1, 0.2) \in A_1$. In Figure 9 we show the resulting solution $w(\zeta, t)$ at time $t = 1000$. It is evident that $w$ converges to the entropy solution constructed in Section 2 for $(u_B, \tau) \in A_1$. 

Figure 8: Case III: Solution graph (left) and flux function (right), with transition from $u_B = 0.55$ to 0.
Figure 9: Graph of \( w(\zeta, t) \) at \( t = 1000 \) when \( (u_B, \tau) = (1, 0.2) \in A_1 \). In this case \( \overline{\tau}(\tau) = 0.816... = \alpha \) and \( s = 1.112... \)

\[ \begin{array}{c}
\text{(a) } (u_B, \tau) = (1.5) \in A_2 \\
\text{(b) } (u_B, \tau) = (0.9, 5) \in B
\end{array} \]

Figure 10: Graphs of \( w(\zeta, t) \) at \( t = 1000 \) when \( (u_B, \tau) = (1.5) \in A_2 \) (left) and \( (u_B, \tau) = (0.9, 5) \in B \) (right). Here \( \overline{\tau}(\tau) \approx 0.98 \) and \( s = 1.02 \), while \( \zeta_\ell = 0.08 \) (left) and \( s_B = 0.28 \) (right)

In the next simulation we raise \( \tau \) to a value above \( \tau_* \): \( \tau = 5 \). In the first of these experiments, in which we keep \( u_B = 1 \), we see that for large time the graph consists of three pieces: one in which \( w \) gradually decreases from \( w = u_B = 1 \) to the "plateau" value \( w = \overline{\tau} \), one in which \( w \) is constant and equal to \( \overline{\tau} \), and one in which it drops down to \( u = 0 \), see Figure 10a. It is clear from the graph that \( \overline{\tau} > \alpha \). The plateau value \( \overline{\tau} \approx 0.98 \) computed here is in good agreement with the one obtained numerically when determining the graphs of \( \overline{\tau} \) and \( \underline{u} \) - see also Figures 3 and 5.

In the next experiment we keep \( \tau = 5 \), but we set \( u_B = 0.9 \). We are then in the region \( B \). For large times the solution \( w(\zeta, t) \) develops two shocks, one where it jumps up from \( u_B \) to the plateau at \( \overline{\tau} \), and one where it jumps down from \( \overline{\tau} \) to \( w = 0 \), see Figure 10(b).

In the next three experiments we decrease the value of \( u_B \) to values around the value \( \underline{u} \approx 0.68 \). The results are shown in Figure 11, where we have zoomed into the front. We
Figure 11: Graphs of $w(\zeta,t)$ with $\tau = 5$ at $t = 1000$ (dashed) and $t = 2000$ (solid); zoomed view: $0.9 \leq \zeta \leq 1.05$. Here $u(\tau) \approx 0.68$, and $u_B$ approaches $u(\tau)$ from above through 0.70 (left), 0.69 (middle), and 0.68 (right). Then $s_B$ increases from 0.95 (left) to 0.98 (middle) up to 1.02 (right). The other values are $\pi(\tau) \approx 0.98$ and $s = 1.02$.

see that, as $u_B$ decreases and approaches the boundary between the regions $\mathcal{B}$ and $C_2$ in Figure 5, the part of the graph where $w \approx u_B$ grows at the expense of the part where $w \approx \pi$.

Finally, in Figure 12 we show the graph of $w(\zeta,t)$ when $\tau = 5$ and $u_B$ is further reduced to 0.55, so that we are now in $C_2$. We find that the solution no longer jumps up to a higher plateau, but instead jumps right down after a small oscillation.

Note that the oscillations in Figures 11 and 12 contract around the shock as time progresses. This is due to the scaling, since we have plotted $w(\zeta,t)$ versus $\zeta = x/t$ for different values of time $t$.

We conclude from these simulations that the entropy solutions constructed in Section 2 emerge as limiting solutions of the Cauchy Problem (3.1). This suggests that these entropy solutions enjoy certain stability properties. It would be interesting to see whether these same entropy solutions would emerge when the initial value were chosen differently. We leave this question to a future study.
4 Proof of Theorem 1.1

In Theorem 1.1 we consider traveling wave solutions \( u(\eta) \) of equation (1.17) in which the limiting conditions have been chosen so that \( u(-\infty) = u_\ell \geq \alpha \) and \( u(\infty) = u_r = 0 \). Putting \( u_r = 0 \) in (1.18) we find that they are solutions of the problem

\[
(W_0) \begin{cases} 
    s\tau u'' - u' - su + f(u) = 0 & \text{for} \quad -\infty < \eta < \infty, \\
    u(-\infty) = u_\ell, \quad u(+\infty) = 0, 
\end{cases}
\]

in which the speed \( s \) is a-priori determined by \( u_\ell \) through

\[
s = s(u_\ell) \overset{\text{def}}{=} \frac{f(u_\ell)}{u_\ell}.
\]

The proof proceeds in a series of steps:

**Step 1:** We choose \( u_\ell \in (\alpha, \beta) \) and prove that there exists a unique \( \tau > 0 \) for which Problem \( (W_0) \) has a solution, which is also unique. This defines a function \( \tau = \tau(u_\ell) \) on \( (\alpha, \beta) \). We then show that \( \tau(u_\ell) \) is increasing, continuous and that

\[
\tau(u) \to \infty \quad \text{if} \quad u \to \beta.
\]

Finally, we write

\[
\tau_* \overset{\text{def}}{=} \lim_{u \to \alpha^+} \tau(u).
\]

**Step 2:** We show that for any \( \tau \in (0, \tau_*] \), Problem \( (W_0) \) has a solution with \( u_\ell = \alpha \).

The proof is concluded by defining the function \( \bar{\tau}_\ell(\tau) \) on \( (\tau_*, \infty) \) as the inverse of the function \( \tau(u_\ell) \) on the interval \( (\alpha, \beta) \). The resulting function \( \bar{\tau}(\tau) \), defined by (1.17) on \( \mathbb{R}^+ \), then has all the properties required in Theorem 1.1.

4.1 The function \( \tau(u) \)

As a first result we prove that \( \tau(u) \) is well defined on the interval \( (\alpha, \beta) \).

**Lemma 4.1** For each \( u_\ell \in (\alpha, \beta) \) there exists a unique value of \( \tau \) such that there exists a solution of Problem \( (W_0) \). This solution is unique and decreasing.

**Proof.** It is convenient to write equation (4.1a) in a more conventional form, and introduce the variables

\[
\xi = -\eta/\sqrt{s\tau} \quad \text{and} \quad \bar{u}(\xi) = u(\eta).
\]

In terms of these variables, Problem \( (W_0) \) becomes

\[
\begin{cases} 
    u'' + cu' - g(u) = 0 & \text{in} \quad -\infty < \xi < \infty, \\
    u(-\infty) = 0, \quad u(+\infty) = u_\ell, 
\end{cases}
\]

where

\[
c = \frac{1}{\sqrt{s\tau}} \quad \text{and} \quad g(u) = su - f(u),
\]
Figure 13: The function $g(u)$ for $M = 2$, and $s = 0.95$ (left) and $s = s(\alpha) = 1.113$ (right) and the tildes have been omitted. Graphs of $g(u)$ for $M = 2$ and different values of $s$ are shown in Figure 13.

We study Problem (4.3) in the phase plane and write equation (4.3a) as the first order system

$$
\mathcal{P}(c, s) \begin{cases} 
u' = v, \\ v' = -cv + g(u) \end{cases} \tag{4.5a}
$$

For $u_\ell \in (\alpha, \beta)$ the function $g(u)$ has three distinct zeros, which we denote by $u_i$, $i = 0, 1$ and 2, where

$$
u_0 = 0 \quad \text{and} \quad \alpha < u_1 < u_2 = u_\ell.
$$

Plainly the points $(u, v) = (u_i, 0)$, $i = 0, 1, 2$, are the equilibrium points of (4.5) with associated eigenvalues

$$
\lambda_\pm = -\frac{c}{2} \pm \frac{1}{2} \sqrt{c^2 + 4g'(u_i)}. \tag{4.6}
$$

Since $g'(u_0) > 0$, $g'(u_1) < 0$ and $g'(u_2) > 0$, the outer points, $(u_0, 0)$ and $(u_2, 0)$, are saddles and $(u_1, 0)$ is either a stable node or a stable spiral.

Since we are interested in a traveling wave with $u(-\infty) = 0$ and $u(+\infty) = u_\ell$, we need to investigate orbits which connect the points $(0, 0)$ and $(u_\ell, 0)$. The existence of a unique wave speed $c$ for which there exists such a solution of the system $\mathcal{P}(c, s)$, which is unique and decreasing, has been established in [19], see also [15]. This allows us to define the function $c = c(u_\ell)$ for $\alpha < u_\ell < \beta$.

By definition, $c(u_\ell)$ only takes on positive values. This is consistent with the identity, obtained by multiplying equation (4.3a) by $u'$ and integrating the result over $\mathbb{R}$:

$$
c \int_{\mathbb{R}} \{u'(\xi)\}^2 d\xi = \int_{\mathbb{R}} g(u(\xi)) u'(\xi) d\xi = \int_0^{u_\ell} g(t) dt \overset{\text{def}}{=} G(u_\ell), \tag{4.7}
$$

because $G(u_\ell) > 0$ when $0 < u_\ell < \beta$. 

17
Finally, by (4.2) and (4.4), we find that \( \tau \) is uniquely determined by \( u_\ell \) through the relation
\[
\tau(u_\ell) = \frac{1}{s(u_\ell)c^2(u_\ell)}. \tag{4.8}
\]
This completes the proof of Lemma 4.1. \( \square \)

Lemma 4.1 allows us to define a function \( \tau(u) \) on \( (\alpha, \beta) \) such that if \( u_\ell \in (\alpha, \beta) \) then Problem (TW_0) has a unique solution \( u(\eta) \) if and only if \( \tau = \tau(u_\ell) \). In the next lemma we show that the function \( \tau(u) \) is strictly increasing on \( (\alpha, \beta) \).

**Lemma 4.2** Let \( u_{\ell,i} = \gamma_i \) for \( i = 1, 2 \), where \( \gamma_1 \in (\alpha, \beta) \), and let \( \tau(\gamma_i) = \tau_i \). Then
\[
\gamma_1 < \gamma_2 \implies \tau_1 < \tau_2.
\]

**Proof.** For \( i = 1, 2 \) we write
\[
s_i = \frac{f(\gamma_i)}{\gamma_i} \quad \text{and} \quad g_i(u) = s_i u - f(u).
\]
Since
\[
\frac{d}{du} \left( \frac{f(u)}{u} \right) = \frac{1}{u} \left( f'(u) - \frac{f(u)}{u} \right) < 0 \quad \text{for} \quad \alpha \leq u < \beta,
\]
it follows that
\[
\gamma_1 < \gamma_2 \implies s_1 > s_2 \quad \text{and} \quad g_1(u) > g_2(u) \quad \text{for} \quad u > 0. \tag{4.9}
\]

To prove Lemma 4.2 we return to the formulation used in the proof of Lemma 4.1. Traveling waves correspond to heteroclinic orbits in the \( (u, v) \)-plane. Those associated with \( \gamma_1 \) and \( \gamma_2 \) we denote by \( \Gamma_1 \) and \( \Gamma_2 \). They connect the origin to respectively \( (\gamma_1, 0) \) and \( (\gamma_2, 0) \).

We shall show that
\[
\gamma_1 < \gamma_2 \implies c_1 = c(\gamma_1) > c(\gamma_2) = c_2. \tag{4.10}
\]
We can then conclude from (4.4) that
\[
\tau_2 s_2 > \tau_1 s_1 \implies \tau_2 > \frac{s_1}{s_2} \tau_1 > \tau_1,
\]
as asserted.

Thus, suppose to the contrary, that \( c_1 \leq c_2 \). We claim that this implies that near the origin the orbit \( \Gamma_1 \) lies below \( \Gamma_2 \). Orbits of the system \( P(c, s) \) leave the origin along the unstable manifold under the angle \( \theta \) given by
\[
\theta = \theta(c, s) = \frac{1}{2} \left\{ \sqrt{c^2 + 4s} - c \right\}. \tag{4.11}
\]
An elementary computation shows that
\[
\frac{\partial \theta}{\partial c} < 0 \quad \text{and} \quad \frac{\partial \theta}{\partial s} > 0. \tag{4.12}
\]
Hence, since \( s_1 > s_2 \) and we assume that \( c_1 \leq c_2 \), it follows that  
\[
\theta_1 = \theta(c_1, s_1) > \theta(c_2, s_2) = \theta_2,
\]
and hence that the orbit \( \Gamma_1 \) starts out above \( \Gamma_2 \).

Since \((\gamma_2, 0)\) lies to the right of the point \((\gamma_1, 0)\) we conclude that \( \Gamma_1 \) and \( \Gamma_2 \) must intersect. Let us denote the first point of intersection by \( P = (u_0, v_0) \). Then at \( P \) the slope of \( \Gamma_1 \) cannot exceed the slope of \( \Gamma_2 \). The slopes at \( P \) are given by  
\[
\frac{dv}{du} \bigg|_{\Gamma_i} = -c_i + g_i(u_0) \frac{1}{v_0}, \quad i = 1, 2.
\]
Because \( g_1(u) > g_2(u) \) for \( u > 0 \) by (4.9), it follows that  
\[
\frac{dv}{du} \bigg|_{\Gamma_1} > \frac{dv}{du} \bigg|_{\Gamma_2} \quad \text{at} \quad P,
\]
so that, at \( P \), the slope of \( \Gamma_1 \) exceeds the slope of \( \Gamma_2 \), a contradiction. Therefore we find that \( c_1 > c_2 \), as asserted. \( \square \)

In the next lemma we show that the function \( \tau(u) \) is continuous.

**Lemma 4.3** The function \( \tau : (\alpha, \beta) \to \mathbb{R}^+ \) is continuous.

**Proof.** Because the function \( s(\gamma) = \gamma^{-1}f(\gamma) \) is continuous, it suffices to show that the function \( c(\gamma) \) is continuous. Since we have shown in the proof of Lemma 4.1 that \( c(\gamma) \) is decreasing (cf. (4.10)), we only need to show that it cannot have any jumps.

Suppose to the contrary that it has a jump at \( \gamma_0 \), and let us write  
\[
\lim \inf_{\gamma \searrow \gamma_0^+} c(\gamma) = c^+ \quad \text{and} \quad \lim \sup_{\gamma \nearrow \gamma_0^-} c(\gamma) = c^-.
\]
Then, since \( c(\gamma) \) is decreasing, we may assume that \( c^- > c^+ \).

Thus, there exist sequences \( \{\gamma_n^-\} \) and \( \{\gamma_n^+\} \) with corresponding heteroclinic orbits \((u_n^\pm, v_n^\pm)\) and wave speeds \( c_n^\pm \), such that  
\[
c_n^+ \searrow c^+ \quad \text{and} \quad c_n^- \nearrow c^- \quad \text{as} \quad n \to \infty.
\]
Since the unstable manifold at \((0, 0)\) and the stable manifold at \((\gamma, 0)\) depend continuously on \( c \), it follows that the corresponding orbits also converge, i.e. that there exist orbits \((u^+, v^+)\) and \((u^-, v^-)\) such that  
\[
(u_n^\pm, v_n^\pm)(\xi) \to (u^\pm, v^\pm)(\xi) \quad \text{as} \quad n \to \infty,
\]
uniformly on \( \mathbb{R} \). This argument yields two heteroclinic orbits, one with speed \( c^+ \) and one with speed \( c^- \), which both connect the origin to the point \((\gamma_0, 0)\). Since by Lemma 4.1 there exists only one such orbit, we have a contradiction.

It follows that \( c^- = c^+ \), and continuity of the function \( c(\gamma) \), and hence of \( \tau(\gamma) \), has been established. \( \square \)

In the following lemma we prove the final assertion made in Step 1, which involves the behaviour of \( \tau(u) \) as \( u \to \beta \).
Lemma 4.4 We have
\[ \tau(\gamma) \to \infty \quad \text{as} \quad \gamma \to \beta - . \]

**Proof.** In view of the definition (4.7) of \( \tau \), it suffices to show that \( c(\gamma) \to 0 \) as \( \gamma \to \beta \). Proceeding as in the proof of Lemma 4.3 we find that \( c(\gamma) \) and the orbit \( \Gamma(c(\gamma)) \) converge to, say \( c_0 \) and \( \Gamma(c_0) = \{(u_0, v_0)(t) : t \in \mathbb{R}\} \), as \( \gamma \to \beta \). Note that
\[ c(\gamma) \int_\mathbb{R} v^2(\xi; \gamma) \, d\xi = \int_0^\gamma g(t; \gamma) \, dt, \]
where \( g(t; \gamma) = s(\gamma) t - f(t) \). If we let \( \gamma \to \beta \) in this identity we obtain
\[ c_0 \int_\mathbb{R} v_0^2(\xi) \, d\xi = \int_0^\beta g(t; \beta) \, dt = 0. \quad (4.13) \]
Because at the origin the unstable manifold points into the first quadrant when \( \gamma = \beta \) (cf. (4.9)), it follows that \( v_0 > 0 \) on \( \mathbb{R} \). Therefore, (4.13) implies that \( c_0 = 0 \), as asserted. \( \square \)

4.2 Traveling waves with \( u_\ell = \alpha \)

In Lemmas 4.1 and 4.2 we have shown that \( \tau(u) \) is an increasing function on \((\alpha, \beta)\). Since \( \tau(u) > 0 \) for all \( u \in (\alpha, \beta) \), the limit
\[ \tau_s \overset{\text{def}}{=} \lim_{u \to \alpha^+} \tau(u) \]
exists. In the following lemmas we show that \( \tau_s > 0 \) and that for all \( \tau \in (0, \tau_s] \), Problem (TW\(_0\)) has a unique solution with \( u_\ell = \alpha \).

Let \( S \subset \mathbb{R}^+ \) denote the set of values of \( \tau \) for which Problem (TW\(_0\)) has a unique solution with \( u_\ell = \alpha \).

**Lemma 4.5** There exists a constant \( \tau_0 > 0 \) such that \((0, \tau_0) \subset S\).

**Proof.** We shall show that there exists a wave speed \( c_0 > 0 \) such that if \( c > c_0 \), then Problem (4.5) has a heteroclinic orbit connecting the origin to the point \((\alpha, 0)\). This then yields Lemma 4.5 when we put
\[ \tau_0 = \frac{1}{c^2 s(\alpha)}. \]

In (4.6) we saw that the origin is a saddle and that the slope of the unstable manifold is given by
\[ \theta(c) = \frac{1}{2} \left\{ \sqrt{c^2 + 4s} - c \right\}. \]

Note that
\[ \theta(c) < \frac{1}{c} g'(0) = \frac{s}{c}. \]

Hence, near the origin the orbit lies below the isolcline \( \mathcal{I}_v = \{(u, v) : v = c^{-1} g(u), \: u \in \mathbb{R}\} \). Since \( u' > 0 \) and \( v' > 0 \) in the lens shaped region
\[ \mathcal{L} = \{(u, v) : 0 < u < \alpha, \: 0 < v < c^{-1} g(u), \: u \in \mathbb{R}\}, \]

20
the orbit will leave \( L \) again. To see what happens next, we consider the triangular region \( \Omega_m \) bounded by the positive \( u \)- and \( v \)-axis, and the line

\[
\ell_m \overset{\text{def}}{=} \{(u, v) : v = m(\alpha - u)\}, \quad m > 0.
\]

On the axes the vector field points into \( \Omega_m \), and on the line \( \ell_m \) it points inwards if

\[
\frac{dv}{du} \bigg|_{\ell_m} = -c + \frac{g(u)}{m(\alpha - u)} < -m.
\] (4.14)

Let

\[
m_0 = \inf \{m > 0 : g(u) < m(\alpha - u) \text{ on } (0, \alpha)\}.
\]

Then

\[
-c + \frac{g(u)}{m(\alpha - u)} < -c + \frac{m_0}{m},
\]

and (4.14) will hold for values of \( c \) and \( m \) which satisfy the inequality

\[
-c + \frac{m_0}{m} < -m,
\]

or

\[
c > m + \frac{m_0}{m}.
\]

To obtain the largest range of values of \( c \) for which the vector field points into \( \Omega_m \) we choose \( m \) so that the right hand side of this inequality becomes smallest, i.e. we put \( m = \sqrt{m_0} \). We thus find that for

\[
c > c_0 \overset{\text{def}}{=} 2\sqrt{m_0}
\]

the region \( \Omega_{\sqrt{m_0}} \) is invariant, and hence, that the orbit must tend to the point \((\alpha, 0)\). This completes the proof of Lemma 4.5 \( \square \)

The next lemma gives the structure of the set \( \mathcal{S} \).

**Lemma 4.6** If \( \tau_0 \in \mathcal{S} \), then \((0, \tau_0] \subset \mathcal{S}\).

**Proof.** As in earlier lemmas we prove a related result for Problem (4.5). Let \( \mathcal{S}^* \) be the set of values of \( c \) for which there exists a heteroclinic orbit of Problem (4.5) from \((0, 0)\) to \((\alpha, 0)\). We show that if \( c_0 \in \mathcal{S}^* \), then \([c_0, \infty) \subset \mathcal{S}^* \). Plainly this implies Lemma 4.6 with \( \tau_0 = 1/(c_0 s^2) \).

As before, we denote the orbit emanating from the origin by \( \Gamma(c) \). Suppose that \( c > c_0 \). Then, since \( \theta'(c) < 0 \) it follows that \( \theta(c_0) > \theta(c) \), so that near the origin \( \Gamma(c_0) \) lies above \( \Gamma(c) \). We claim that \( \Gamma(c_0) \) and \( \Gamma(c) \) will not intersect for \( u \in (0, \alpha) \). Accepting this claim for the moment, we conclude that since \( \Gamma(c_0) \) tends to \((\alpha, 0)\), the orbit \( \Gamma(c) \) must converge to \((\alpha, 0)\) as well.

It remains to prove the claim. Suppose that \( \Gamma(c_0) \) and \( \Gamma(c) \) do intersect at some \( u \in (0, \alpha) \), and let \((u_0, v_0)\) be the first point of intersection. Then

\[
\frac{dv}{du} \bigg|_{\Gamma(c)} \geq \frac{dv}{du} \bigg|_{\Gamma(c_0)} \quad \text{at} \quad (u_0, v_0). 
\] (4.16)
But, from the differential equations we deduce that
\[ \frac{dv}{du} \big|_{\Gamma(c)} = -c + \frac{g(u_0)}{v_0} < -c_0 + \frac{g(u_0)}{v_0} = \frac{dv}{du} \big|_{\Gamma(c_0)} \] at \((u_0, v_0)\),
which contradicts (4.16). This proves the claim and so completes the proof of Lemma 4.6. □

We conclude this section by showing that \(\tau_* \in \mathcal{S}\), and hence that \(\mathcal{S} = (0, \tau_*]\).

**Lemma 4.7** We have \(\mathcal{S} = (0, \tau_*]\).

**Proof.** It follows from Lemmas 4.1 and 4.2 that for every \(\varepsilon \in (0, \beta - \alpha)\), there exists a \(\tau_\varepsilon = \tau(\alpha + \varepsilon) > 0\) such that Problem (TW\(_0\)) has a unique traveling wave \(u_\varepsilon(\eta)\) with speed \(s_\varepsilon = s(\alpha + \varepsilon)\), such that
\[ u_\varepsilon(-\infty) = \alpha + \varepsilon \quad \text{and} \quad u_\varepsilon(\infty) = 0. \]
This wave corresponds to a heteroclinic orbit \(\Gamma_\varepsilon = \{(u_\varepsilon(\xi), v_\varepsilon(\xi)) : \xi \in \mathbb{R}\}\) of the system \(\mathcal{P}(c_\varepsilon, s_\varepsilon)\) where \(c_\varepsilon = 1/\sqrt{s_\varepsilon \tau_\varepsilon}\), which connects the points \((0, 0)\) and \((\alpha + \varepsilon, 0)\). It leaves the origin along the stable manifold under an angle \(\theta_\varepsilon = \theta(c_\varepsilon, s_\varepsilon)\) and enters the point \((\alpha + \varepsilon, 0)\) along the stable manifold under the angle
\[ \psi_\varepsilon = \psi(c_\varepsilon, s_\varepsilon) = \frac{1}{2} \left\{ -c_\varepsilon - \sqrt{c_\varepsilon^2 + 4g'(\alpha + \varepsilon)} \right\} \rightarrow -c_0 = -\frac{1}{\sqrt{s(\alpha) \tau_*}} \quad \text{as} \quad \varepsilon \to 0. \]
Reversing time, i.e., replacing \(\xi\) by \(-\xi\), we can view \(\Gamma_\varepsilon\) as the unique orbit emanating from the point \((\alpha + \varepsilon, 0)\) into the first quadrant, and entering the origin as \(\xi \to \infty\). In the limit, as \(\varepsilon \to 0\), we find that
\[ u_\varepsilon(\xi) \to u_0(\xi) \quad \text{and} \quad v_\varepsilon(\xi) \to v_0(\xi) \quad \text{as} \quad \varepsilon \to 0 \quad \text{for} \quad -\infty < \xi \leq \xi_0, \]
where \(\xi_0\) is any finite number. We claim that
\[ u_0(\xi) \to 0 \quad \text{and} \quad v_0(\xi) \to 0 \quad \text{as} \quad \xi \to \infty, \]
i.e., \(\Gamma_0 \overset{\text{def}}{=} \{(u_0(\xi), v_0(\xi)) : \xi \in \mathbb{R}\}\) is a heteroclinic orbit, which connects \((\alpha, 0)\) and the origin \((0, 0)\).

Suppose to the contrary that \(\Gamma_0\) does not enter the origin as \(\xi \to \infty\), and possibly not even exist for all \(\xi \in \mathbb{R}\). Then, since
\[ \frac{dv}{du} = -c_0 + \frac{g(u)}{v} > -c_0 \quad \text{if} \quad 0 < u < \alpha, \quad v > 0, \]
\(\Gamma_0\) must leave the first quadrant in finite time, either through the \(u\)-axis or through the \(v\)-axis. This means by continuity that for \(\varepsilon\) small enough \(\Gamma_\varepsilon\) must also leave the first quadrant in finite time. Since \(\Gamma_\varepsilon\) is known to enter the origin for every \(\varepsilon > 0\), and hence never to leave the first quadrant, we have a contradiction. This proves the claim that \(\Gamma_0\) is a heteroclinic orbit, which connects \((\alpha, 0)\) and \((0, 0)\). □
Remark 4.1 It is evident from Lemmas 4.6 and 4.7 that
\[ \tau_0 = \frac{1}{4m_0s(\alpha)}, \] (4.17)
where \( m_0 \) was defined in (4.15). For \( M = 2 \), we find that \( s(\alpha) = 1.112... \), \( m_0 = 0.704... \) and hence \( \tau_0 = 0.318... \). Numerically, we find that \( \tau_* = 0.61... \).

Remark 4.2 The function \( \overline{w}(\tau) \) has been computed numerically by means of a shooting technique. Specifically, we computed its inverse \( \tau(u) \) by fixing a left state \( u_0 \) and looking for a strictly decreasing solution \( u(\eta) \) of Problem (1.18). We can thus invert the function \( u(\eta) \) and so define the new dependent variable \( w(u) = -u'(\eta(u)) \) which satisfies the equation
\[ s\tau w' + w = g(u). \]
We then determine \( \tau > 0 \) such that \( w > 0 \) on \((0, u_\ell)\) and \( w(0) = w(u_\ell) = 0 \). With \( u_\ell \in (\alpha, \beta) \), there is a unique \( \tau > \tau_* \) yielding a solution \( w \) with the required properties.

5 Proof of Theorems 1.2 and 1.3

For the proofs of Theorems 1.2 and 1.3 we turn to the system \( \mathcal{P}(c, s) \) defined in Section 4. For convenience we restate it here
\[ \mathcal{P}(c, s) \left\{ \begin{array}{l}
u' = v, \\ v' = -cv + g_s(u),
\end{array} \right. \] (4.5a) (4.5b)
where
\[ c = \frac{1}{\sqrt{s\tau}} \quad \text{and} \quad g_s(u) = su - f(u). \]

Part (a) of Theorem 1.2 is readily seen to be a consequence of the following lemma:

Lemma 5.1 Let \( \tau > \tau_* \) be given. Then for every \( u_\ell \in (0, u_\ell) \), there exists a unique heteroclinic orbit of the system \( \mathcal{P}(c, s) \) in which
\[ s = s_\ell = \frac{f(u_\ell)}{u_\ell} \quad \text{and} \quad c = c_\ell = \frac{1}{\sqrt{s_\ell\tau}}, \]
which connects \((0,0)\) and \((u_\ell,0)\).

Proof. Let \( \Gamma_\ell \) and \( \overline{\Gamma} \) denote the orbits of \( \mathcal{P}(c_\ell, s_\ell) \) and \( \mathcal{P}(\overline{c}, \overline{s}) \), where \( \overline{c} = c(\overline{s}) \) and \( \overline{s} = s(\overline{s}) \), which enter the first quadrant from the origin. They do this under the angles \( \theta(c_\ell, s_\ell) \) and \( \theta(\overline{c}, \overline{s}) \), respectively. Since \( c_\ell > \overline{c} \) and \( s_\ell < \overline{s} \), it follows from (4.12) that
\[ \theta(c_\ell, s_\ell) < \theta(\overline{c}, \overline{s}). \]
Hence, near the origin, \( \Gamma_\ell \) lies below \( \overline{\Gamma} \). Thus, \( \Gamma_\ell \) enters the region \( \Omega \) enclosed between \( \overline{\Gamma} \) and the \( u \)-axis. Since
\[ \left. \frac{dv}{du} \right|_{\Gamma_\ell} = -c_\ell + \frac{s_\ell u - f(u)}{v} < -\overline{c} + \frac{\overline{s} u - f(u)}{v} = \left. \frac{dv}{du} \right|_{\overline{\Gamma}}, \]
it follows that \( \Gamma_\ell \) cannot leave \( \Omega \) though its “top” \( \overline{\Gamma} \). We define the following subsets of the bottom of \( \Omega \):

\[
S_1 = \{(u, v) : 0 < u < u_\ell, \ v = 0\},
\]

\[
S_2 = \{(u, v) : u = u_\ell, \ v = 0\},
\]

\[
S_3 = \{(u, v) : u_\ell < u < \overline{\tau}, \ v = 0\}.
\]

Inspection of the vector field show that orbits can only leave \( \Omega \) through \( S_3 \). Note that the set \( S_2 \) consists of an equilibrium point.

There are two possibilities: either \( \Gamma_\ell \) never leaves \( \Omega \), or \( \Gamma_\ell \) leaves \( \Omega \), necessarily through the set \( S_3 \). In the first case \( \Gamma_\ell \) is a heteroclinic orbit from \((0,0)\) to \((u_\ell,0)\), and the proof is complete.

Thus, let us assume that \( \Gamma_\ell \) leaves \( \Omega \) at some point \((u,v) = (u_0,0)\). Consider the energy function

\[
\mathcal{H}(u,v) = \frac{1}{2}v^2 - G_{s_\ell}(u),
\]

where \( G_s \) was defined in (4.7), and write \( H(\xi) = \mathcal{H}(u(\xi),v(\xi)) \), when \((u(\xi),v(\xi))\) is an orbit. Then differentiation shows that

\[
H'(\xi) = -c_\ell v^2(\xi) < 0.
\]

Since \( \mathcal{H}(0,0) = 0 \), it follows that

\[
\mathcal{H}(u_0,0) = -G_{s_\ell}(u_0) < 0.
\]

and that

\[
\mathcal{H}(u(\xi),v(\xi)) = \frac{1}{2}v^2 - G_{s_\ell}(u) < -G_{s_\ell}(u_0) \quad \text{for} \quad \xi > \xi_0.
\]

This means that

\[
G_{s_B}(u) > G_{s_\ell}(u_0) > 0 \quad \text{for} \quad \xi > \xi_0.
\]

Let

\[
u_1 = \inf \{ s \in \mathbb{R} : G_{s_\ell}(s) > G_{s_\ell}(u_0) \quad \text{on} \quad (s,u_0) \}.
\]

Since \( G_{s_\ell}(u_0) > 0 \) it follows that \( u_1 \in (0,u_\ell) \). Therefore

\[
0 < u_1 < u(x) < u_0 \quad \text{for} \quad x > x_0.
\]

From a simple energy argument we conclude that \((u(x),v(x)) \to (u_\ell,0)\) as \( x \to \infty \). This completes the proof of Lemma 5.1. \( \square \)

Part (b) follows from the following result.

**Lemma 5.2** Let \( \tau > \tau_* \) be given. For any \( u_\ell \in (\underline{u}(\tau),\overline{u}(\tau)) \) there exists no solution of the system \( \mathcal{P}(c_\ell,s_\ell) \), with

\[
s_\ell = \frac{f(u_\ell)}{u_\ell} \quad \text{and} \quad c_\ell = \frac{1}{\sqrt{s_\ell \tau}},
\]

which connects \((0,0)\) and \((u_\ell,0)\).
Proof. Let \( \Gamma \) denote the orbit corresponding to \( \sigma \) and \( \sigma' \), which connects \((0, 0)\) and the point \((\sigma, 0)\), and let \( \Gamma_\ell \) denote the orbit which corresponds to \( c_\ell \) and \( s_\ell \). Observe that
\[
s_\ell > \sigma \quad \text{and} \quad c_\ell < \sigma',
\]
and hence
\[
\theta(c_\ell, s_\ell) > \theta(\sigma, \sigma').
\]
Therefore, near the origin, \( \Gamma_\ell \) lies above \( \Gamma \). Hence, to reach the point \((u_\ell, 0)\), the orbit \( \Gamma_\ell \) has to cross \( \Gamma \) somewhere, and at the first point of crossing we must have
\[
\frac{dv}{du} \bigg|_{\Gamma} \geq \frac{dv}{du} \bigg|_{\Gamma_\ell}.
\]
However, by the equations, we have
\[
\frac{dv}{du} \bigg|_{\Gamma} = -\sigma + \frac{g_\sigma(u)}{u} < -c_\ell + \frac{g_{s_\ell}(u)}{u} = \frac{dv}{du} \bigg|_{\Gamma_\ell},
\]
so that we have a contradiction. \( \square \)

This completes the proof of Theorem 1.2.

The proof of Theorem 1.3 is entirely analogous to that of Theorem 1.2, and we omit it.

6 Entropy dissipation

In this section we study the Cauchy Problem
\[
\begin{align*}
\text{(CP)}: \quad u_t + (f(u))_x &= A_\varepsilon(u) \quad \text{in} \quad S = \mathbb{R} \times \mathbb{R}^+, \\
u(\cdot, 0) &= u_0(\cdot) \quad \text{on} \quad \mathbb{R},
\end{align*}
\]
where
\[
A_\varepsilon(u) = \varepsilon u_{xx} + \varepsilon^2 \tau u_{xxt}, \quad (\varepsilon > 0).
\]
With this choice (6.1a) becomes the regularized Buckley-Leverett equation (1.17) for which we obtained traveling wave solutions in the previous sections. In (6.1a) and (6.2) we introduce subscripts to denote partial derivatives. Without further justification we assume that Problem (CP) has a smooth, nonnegative and bounded solution \( u^\varepsilon \) for each \( \varepsilon > 0 \), and that there exist a limit function \( u : S \to [0, \infty) \) such that for each \((x, t) \in S\),
\[
u^\varepsilon(x, t) \to u(x, t) \quad \text{as} \quad \varepsilon \to 0.
\]
In addition we assume the following structural properties:

(i) \( \|u^\varepsilon\|_\infty < C \) for some constant \( C > 0 \), and for each fixed \( t > 0 \),
\[
u^\varepsilon(x, t) \to u_\ell \in \mathbb{R}^+ \quad \text{as} \quad x \to -\infty,
\]
\[
u^\varepsilon(x, t) \to u_r \in \mathbb{R}^+ \quad \text{as} \quad x \to +\infty.
\]
(ii) The partial derivatives of \(u^\varepsilon\) vanish as \(|x| \to \infty\).

(iii) Let \(U(s) = \frac{1}{2} s^2\) for \(s \geq 0\), \(U_\ell = U(u_\ell)\) and \(U_r = U(u_r)\). Then there exists a smooth function \(\lambda_\varepsilon : [0, \infty) \to \mathbb{R}\) which is uniformly bounded with respect to \(\varepsilon > 0\) in any bounded interval \((0, T)\), such that

\[
\int_{\mathbb{R}} \{U(u^\varepsilon(x, t)) - G_\varepsilon(x, t)\} \, dx = 0 \quad \text{for all} \quad t > 0,
\]

where \(G_\varepsilon\) is the step function

\[
G_\varepsilon(x, t) = U_\ell + (U_r - U_\ell)H(x - \lambda_\varepsilon(t)), \quad (x, t) \in S
\]

in which \(H\) denotes the Heaviside function.

Note that the traveling waves constructed in this paper all have these properties. The main purpose of this section is to show that \(U(u^\varepsilon)\) is an entropy for equation (6.1a) (see also [20]).

For completeness we recall some definitions. We say that the term \(A_\varepsilon(u)\) is conservative if

\[
\lim_{\varepsilon \to 0^+} \int_S A_\varepsilon(u^\varepsilon) \varphi = 0 \quad \text{for all} \quad \varphi \in C_0^\infty(S) \tag{6.3}
\]

and we say that \(A_\varepsilon(u)\) is entropy dissipative (for an entropy \(U\)) if

\[
\limsup_{\varepsilon \to 0^+} \int_S A_\varepsilon(u^\varepsilon) U'(u^\varepsilon) \varphi \leq 0 \quad \text{for all} \quad \varphi \in C_0^\infty(S), \ \varphi \geq 0. \tag{6.4}
\]

We establish the following theorem:

**Theorem 6.1** Let \(u^\varepsilon\) be the solution of Problem (CP), and let \(u^\varepsilon\) satisfy (i), (ii) and (iii). Then, the regularization \(A_\varepsilon(u)\) defined in (6.2) has the following properties:

(a) \(A_\varepsilon(u)\) is conservative.

(b) \(A_\varepsilon(u)\) is entropy dissipative for the entropy \(U(u) = \frac{1}{2} u^2\).

**Proof.** Part (a). For any \(\varphi \in C_0^\infty(S)\) we obtain after partial integration with respect to \(x\) and \(t\),

\[
\int_S A_\varepsilon(u^\varepsilon) \varphi = \varepsilon \int_S u^\varepsilon \varphi_{xx} - \varepsilon^2 \tau \int_S u^\varepsilon \varphi_{xxt} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

Part (b). To simplify notation, we drop the superscript \(\varepsilon\) from \(u^\varepsilon\). When we multiply (6.1a) by \(u\) we obtain

\[
\partial_t U(u) + \partial_x F(u) = u A_\varepsilon(u) = \varepsilon uu_{xx} + \varepsilon^2 \tau uu_{xxt}, \tag{6.5}
\]

where

\[
F(u) = \int_0^u U'(s)f'(s) \, ds = \int_0^u sf'(s) \, ds = uf(u) - \int_0^u f(s) \, ds. \tag{6.6}
\]

26
An elementary computation shows that
\[ \varepsilon u_{xx} = \varepsilon U_{xx} - \varepsilon u_x^2, \]
\[ \varepsilon^2 \tau u_{xxt} = \varepsilon^2 \tau \left( U_{xxt} - \frac{1}{2} (u_x^2)_{t} - (u_t u_x)_x \right). \]

Hence
\[ \int_S A_{\varepsilon}(u) u \varphi = \varepsilon \int_S U \varphi_{xx} - \varepsilon \int_S u_x^2 \varphi - \varepsilon^2 \tau \int_S U \varphi_{xxt} + \frac{1}{2} \varepsilon^2 \tau \int_S u_x^2 \varphi_t + \varepsilon^2 \tau \int_S u_t u_x \varphi_x. \]  

(6.7)

Plainly
\[ \varepsilon \int_S U \varphi_{xx} \to 0 \quad \text{and} \quad \varepsilon^2 \tau \int_S U \varphi_{xxt} \to 0 \quad \text{as} \quad \varepsilon \to 0. \]

Since \( \varphi \geq 0 \), it remains to estimate the last two terms in the right hand side of (6.7).

For this purpose we establish the following two estimates:

**Lemma 6.1** Let \( T > 0 \), and let \( S_T = \mathbb{R} \times (0, T] \). Then there exists a constant \( C > 0 \) such that for all \( \varepsilon > 0 \),
\[ \varepsilon \int_{S_T} u_x^2 \leq C \]  
and
\[ \varepsilon \int_{S_T} u_x^2 \leq C. \]

(6.8) (6.9)

**Proof of (6.8).** We write (6.5) as
\[ \partial_t U(u) + \partial_x F(u) = \varepsilon U_{xx} - \varepsilon u_x^2 + \varepsilon^2 \tau \left( U_{xxt} - \frac{1}{2} (u_x^2)_{t} - (u_t u_x)_x \right). \]

(6.10)

Using properties (i)–(iii), and writing \( F_\ell = F(u_\ell) \), \( F_r = F(u_r) \), we find that
\[ \frac{d}{dt} \int_\mathbb{R} \{ U(x, t) - G_\varepsilon(x, t) \} \, dx - \frac{d\lambda_\varepsilon}{dt} (U_r - U_\ell) \]
\[ + (F_r - F_\ell) + \varepsilon \int_\mathbb{R} u_x^2 + \frac{1}{2} \varepsilon^2 \tau \frac{d}{dt} \int_\mathbb{R} u_x^2 \leq 0, \]

or, when we integrate over \( (0, T) \)
\[ - \{ \lambda_\varepsilon(t) - \lambda_\varepsilon(0) \} (U_r - U_\ell) + (F_r - F_\ell) t + \varepsilon \int_{S_T} u_x^2 + \frac{1}{2} \varepsilon^2 \tau \int_\mathbb{R} u_x^2(t) \leq \frac{1}{2} \varepsilon^2 \tau \int_\mathbb{R} (u_0^\varepsilon)^2, \]

from which (6.8) immediately follows.

**Proof of (6.9).** We multiply (6.1) by \( u_t \). This yields
\[ u_t^2 + (f(u))_x u_t = u_t A_\varepsilon(u) = \varepsilon u_t u_{xx} + \varepsilon^2 \tau u_t u_{xxt} \]  

(6.11)
Using the identities
\[ u_t u_{xx} = (u_x u_t)_x - \frac{1}{2} (u_x^2)_t \quad \text{and} \quad u_t u_{xxt} = (u_{xt} u_t)_x - (u_{xt})^2, \]
we find that
\[ u_t^2 + \frac{\varepsilon}{2} (u_x^2)_t \leq -f'(u) u_t u_x + \varepsilon (u_x u_t)_x + \varepsilon^2 (u_{xt} u_t)_x. \]
When we integrate over \( \mathbb{R} \) and use Schwarz’s inequality and properties (i) and (ii), we obtain
\[ \int_{\mathbb{R}} u_t^2 + \frac{\varepsilon}{2} d \int_{\mathbb{R}} u_x^2 \leq \frac{1}{2} \int_{\mathbb{R}} u_t^2 + \frac{K^2}{2} \int_{\mathbb{R}} u_x^2, \]
where \( K = \max\{|f'(s)| : s > 0\} \). Hence, when we integrate over \((0, t)\),
\[ \int_{S_t} u_t^2 \leq \varepsilon \int_{\mathbb{R}} (u_0')^2 + \frac{K^2}{2} \int_{S_t} u_x^2. \]
In view of the first estimate this establishes (6.9), and completes the proof of Lemma 6.1.

We now return to the proof of Part (b) of Theorem 6.1. For each \( \varphi \in C_0^\infty (S) \) we choose \( T > 0 \) so that \( \text{supp} \varphi \subset S_T \). Then (6.8) implies that
\[ \varepsilon^2 \int_{S_T} u_x^2 \varphi_t \leq \varepsilon^2 K_1 \int_{S_T} u_x^2 \leq \varepsilon K_1 C \quad \text{with} \quad K_1 = \sup |\varphi_t|, \] (6.12)
and (6.8) and (6.9) together imply that
\[ \varepsilon^2 \int_{S_T} u_t u_x \varphi_x \leq \varepsilon^2 K_2 \int_{S_T} |u_t||u_x| \leq \varepsilon K_2 C \quad \text{with} \quad K_2 = \sup |\varphi_x|. \] (6.13)
Using (6.12) and (6.13) in (6.7) we conclude that, writing \( u = u^\varepsilon \) again,
\[ \limsup_{\varepsilon \searrow 0} \int_S A_\varepsilon (u^\varepsilon) u^\varepsilon \varphi \leq 0, \]
which is what was claimed in Theorem 6.1. \( \square \)

It now follows from (6.5) that in the limit as \( \varepsilon \to 0 \),
\[ \partial_t U(u) + \partial_x F(u) \leq 0 \] (6.14)
holds in a weak or distributional sense. This shows that \((U, F)\) is an entropy pair for equation (1.1).

The inequality in (6.14) indicates entropy dissipation. Across shocks \( \{u_\ell, u_r\} \) it can be computed explicitly. Let
\[ u(x, t) = \begin{cases} u_\ell & \text{for } x < st, \\ u_r & \text{for } x > st. \end{cases} \]

28
Then (6.14) implies that
\[-s(U_r - U_\ell) + (F_r - F_\ell) \leq 0.\]

Hence the entropy dissipation is given by
\[E(u_\ell, u_r) \overset{\text{def}}{=} -s(U_r - U_\ell) + (F_r - F_\ell). \tag{6.15}\]

We conclude by observing that if \(u = u(\eta)\) is a travelling wave satisfying Problem (TW), then (6.15) can be written as
\[E(u_\ell, u_r) = \int_{\mathbb{R}} \left\{ -s(U(u))' + (F(u))' \right\} d\eta. \tag{6.16}\]

Applying (6.6) and the definition of \(U\) gives
\[E(u_\ell, u_r) = \int_{\mathbb{R}} u \left( -s + \frac{df}{du} \right) u' d\eta = \int_{u_\ell}^{u_r} u \left( -s + \frac{df}{du} \right) du. \tag{6.17}\]

Rewriting further
\[-s + \frac{df}{du} = \frac{d}{du}(-s(u - u_\ell) + f(u) - f(u_\ell)),\]

integrating (6.17) by parts, and using the Rankine-Hugoniot condition yields
\[E(u_\ell, u_r) = \int_{u_\ell}^{u_r} \{ f(u) - f(u_\ell) - s(u - u_\ell) \} du.\]

In the special case when \(u_r = 0\) we have \(s = f(u_\ell)/u_\ell\) and thus
\[E(u_\ell, 0) = \int_{0}^{u_\ell} \{ f(u) - su \} du.\]

Returning to the proof of Proposition 1.1 we observe that the integral is negative provided \(u_\ell < \beta\). Thus this condition acts as an entropy condition in the sense that \(E(u_\ell, 0) < 0\) only if \(u_\ell < \beta\).

**Acknowledgment** The authors are grateful for the support from the Centrum voor Wiskunde en Informatica in Amsterdam (CWI).

**References**


