

## GLOBAL ATTRACTOR AND INERTIAL SETS FOR A NONLOCAL KURAMOTO-SIVASHINSKY EQUATION

D. HILHORST

Analyse Numérique et EDP, CNRS  
Université de Paris Sud  
F-91405 Orsay, Cedex, France

L.A. PELETIER AND A.I. ROTARIU

Mathematical Institute  
PB 9512, 2300 RA, Leiden, The Netherlands

G. SIVASHINSKY

School of Mathematical Sciences  
Tel Aviv University, Israel

*This paper is dedicated to Mark Vishik on the occasion of his eightieth birthday*

**Abstract.** We consider a nonlinear fourth order parabolic equation with a nonlocal term which describes the time evolution of a flame front. After having established the existence of a global attractor for a corresponding boundary value problem, we prove the existence of inertial sets.

**1. Introduction.** The premixed gas flame (a self-sustained wave of an exothermic chemical reaction) is frequently situated in a nonuniform flow field and subjected to large-scale flame stretch [7]. The latter, apart from affecting the burning rate intensity, may also have a significant impact on the flame stability. It has long been observed that wrinkled structures, occurring spontaneously in freely propagating flames, may easily be suppressed in a stagnation point flow provided its intensity is high enough [6].

To study the dynamics of an intrinsically unstable flame held in a stagnation point flow, two model equations for the flame interface dynamics have recently been proposed. The first model [9] deals with diffusively unstable flames while the second model [8] deals with hydrodynamically unstable flames. These models may easily be combined to describe the situation where the flame is subject to diffusive and hydrodynamic instabilities occurring simultaneously. One thus ends up with a unified formulation which in appropriately chosen units results in the following initial-boundary value problem

$$(\mathcal{P}) \begin{cases} u_t = -u_{xxxx} - u_{xx} - \frac{1}{2}(u_x)^2 \\ \quad + \gamma I(u) - \alpha(xu)_x & \text{for } (x, t) \in Q, \\ u_x = 0, \quad u_{xxx} = 0 & \text{for } x = \pm\ell, t \in \mathbf{R}^+, \\ u(x, 0) = u_0(x) & \text{for } x \in (-\ell, \ell), \end{cases} \quad (1.1)$$

---

1991 *Mathematics Subject Classification.* 35K30, 35B40, 35B45.

*Key words and phrases.* Kuramoto-Sivashinsky Equation, Dynamical Systems, Global Attractors, Inertial Sets.

where  $Q = (-\ell, \ell) \times \mathbf{R}^+$  and

$$I(u) = \sum_{n=1}^{\infty} \frac{n}{\ell} \left\{ \int_{-\ell}^{\ell} u(y, t) \cos\left(\frac{n\pi}{\ell}y\right) dy \right\} \cos\left(\frac{n\pi}{\ell}x\right). \tag{1.2}$$

The differential equation in Problem (1.1) is a generalisation of the classical Kuramoto-Sivashinsky (KS) equation

$$u_t + u_{xxxx} + u_{xx} + \frac{1}{2}(u_x)^2 = 0. \tag{1.3}$$

In these equations,  $u(x, t)$  denotes the profile of the flame interface. The negative diffusion term  $-u_{xx}$  represents the diffusive flame instability occurring in premixed gas flames with sufficiently light deficient reactant (e.g. lean hydrogen-air mixtures). The term  $u_{xxxx}$  models the dissipation of small-scale disturbances, and the nonlinear term  $\frac{1}{2}(u_x)^2$  controls saturation of the growing large-scale disturbances.

Of the two terms in Problem (1.1) that have been added to the Kuramoto-Sivashinsky equation (1.3):

$$+\gamma I(u) - \alpha(xu)_x,$$

the nonlocal term  $\gamma I(u)$  is caused by the hydrodynamic flame instability induced by the thermal expansion of the burnt gas,  $\gamma$  being the thermal expansion intensity ( $0 < \gamma < 1$ ). The term involving  $\alpha$  describes the stabilizing effect of the stretch induced by the flow,  $\alpha$  being the stretch intensity ( $\alpha > 0$ ).

Rather than directly working with the unknown function  $u$ , it is more convenient to work with its spatial gradient,  $v = u_x$ . It satisfies

$$(P) \begin{cases} v_t = -v_{xxxx} - v_{xx} - vv_x \\ \quad + \gamma J(v) - \alpha xv_x - 2\alpha v & \text{for } (x, t) \in Q, \\ v = 0, \quad v_{xx} = 0 & \text{for } x = \pm\ell, t \in \mathbf{R}^+, \\ v(x, 0) = v_0(x) & \text{for } x \in (-\ell, \ell), \end{cases} \tag{1.4}$$

where

$$J(v) = \sum_{n=1}^{\infty} \frac{n}{\ell} \left\{ \int_{-\ell}^{\ell} v(y, t) \sin\left(\frac{n\pi}{\ell}y\right) dy \right\} \sin\left(\frac{n\pi}{\ell}x\right). \tag{1.5}$$

We assume that

$$v_0 \in L^2(-\ell, \ell). \tag{1.6}$$

Problem (P) is known to have a global solution in the distributional sense [5],[12]; for the precise definition we refer to Section 2. Specifically, we have the following proposition:

**Proposition 1.1.** *Let  $T$  be an arbitrary positive number. Then there exists one and only one weak solution  $v \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; \mathcal{H}^2(\Omega))$  of Problem (1.4). In addition,  $v \in C([0, T]; L^2(\Omega)) \cap L^2(\delta, T; \mathcal{H}^4(\Omega))$  for any  $\delta > 0$ .*

Here  $\Omega$  denotes the spatial domain:  $\Omega = (-\ell, \ell)$ ,  $\|\cdot\|$  denotes the norm in  $L^2(\Omega)$ ,  $\mathcal{H}^2(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$  and  $\mathcal{H}^4(\Omega) = H^4(\Omega) \cap H_0^2(\Omega)$ .

In this paper we are interested in the large time behaviour of solutions of Problem (1.4), and in the existence of a *Global Attractor* and of *Inertial Sets*. For the KS-equation such questions have been taken up in [10], [3], [13] and [14]. In these papers the size of the domain ( $2\ell$ ) proved to be an important parameter. In [2] analogous questions were discussed for the Cauchy Problem of the KS-equation with the additional damping term  $\alpha(xu)_x$ . In the present paper we study the effect

of this damping term, as well as the *destabilizing* nonlocal term  $\gamma I(u)$ . Thus, two new parameters have entered the problem:  $\alpha$  and  $\gamma$ . In fact, we find that rather than  $\gamma$ , the parameter

$$\kappa = \frac{\gamma \ell}{2\pi} \tag{1.7}$$

will play a central role in the analysis of Problem (1.4). If  $\alpha$  is sufficiently large, the asymptotic behaviour of solutions is particularly simple. We find the following critical value:

$$\alpha_0 \stackrel{\text{def}}{=} \frac{2}{3}(1 + \kappa^2). \tag{1.8}$$

**Theorem 1.2.** *If  $\alpha > \alpha_0$ , then there exist positive constants  $\nu$  and  $M$  such that the solution  $v(t)$  of Problem (1.4) satisfies*

$$\|v(t)\| \leq M e^{-\nu t} \quad \text{for } t \geq 0. \tag{1.9}$$

Thus, if  $\alpha > \alpha_0$  then the trivial solution  $v = 0$  is the unique global attractor.

For smaller values of  $\alpha$ , no such simple statements are available, and the large time behaviour may be more complicated. We shall study this case for *odd* solutions of Problem (1.4), i.e. for solutions in the space

$$H = \{w \in L^2(\Omega) : w(-x) = -w(x), x \in \Omega\}.$$

Plainly, if  $v_0 \in H$ , then so is  $v(t) \in H$  for every  $t \geq 0$  and therefore this subspace of  $L^2(\Omega)$  is invariant. Henceforth we shall mean by  $\{S(t) : t \geq 0\}$  the semigroup defined by Problem (1.4) on the space  $H$ .

Whereas the term involving  $\alpha$  is stabilizing, we find – as expected – that the term involving  $\gamma$  is destabilizing, and in the results we prove below we need to restrict it. Specifically we need to assume that

$$\kappa < \frac{1}{3^{7/4}}(1 + 3\alpha)^{3/4}. \tag{1.10}$$

In addition, we assume throughout that  $\ell > 2\pi$ . In the following theorem we give two global bounds, one in  $L^2(\Omega)$  and one in  $H^1(\Omega)$ . Below, and throughout the paper, we write

$$\mu = \frac{11}{5}.$$

**Theorem 1.3.** *Let  $\ell > 2\pi$ , and let  $\alpha$  and  $\kappa$  satisfy (1.10). Then*

(a) *The semigroup  $\{S(t) : t \geq 0\}$  possesses an absorbing set in  $H$ , and there exists a constant  $c_0$  and a nondecreasing function  $\tau_0(\cdot)$  such that*

$$\|S(t)v_0\| \leq c_0 \ell^\mu \quad \text{if } t \geq \tau_0(\|v_0\|).$$

(b) *The semigroup  $\{S(t) : t \geq 0\}$  possesses an absorbing set in  $H \cap H^1(\Omega)$  and there exists a constant  $c_1$  and a nondecreasing function  $\tau_1(\cdot)$  such that*

$$\|S(t)v_0\|_{H^1(\Omega)} \leq c_1 \ell^{5\mu/3} \quad \text{if } t \geq \tau_1(\|v_0\|).$$

It follows from Theorem 1.3 that the set

$$B = \{v \in H \cap H_0^1(\Omega) ; \|v\| \leq c_0 \ell^\mu \text{ and } \|v_x\| \leq c_1 \ell^{5\mu/3}\}$$

is absorbing for all bounded sets of  $H$ .

In view of [13], Theorem I.1, Theorem 1.3 immediately implies the existence of a global attractor.

**Theorem 1.4.** *The semigroup  $\{S(t) : t \geq 0\}$  possesses an attractor  $\mathcal{A}$  which is compact, connected, and maximal in  $H$ . In addition, it attracts the bounded sets of  $H$  and is also maximal among the functional invariant sets bounded in  $H$ .*

In order to characterize the global attractor we prove the existence of an inertial set, namely a compact set which contains the attractor, is positively invariant by the semigroup, has finite fractal dimension and attracts all solutions at an exponential rate. More precisely, we define the compact, connected set

$$Y = \overline{\cup_{t \geq \tau_1} S(t)B},$$

where  $\tau_1$  was introduced in Theorem 1.3, and we note that the semigroup  $\{S(t) : t \geq 0\}$  leaves  $Y$  invariant.

**Theorem 1.5.** *The semigroup  $\{S(t) : t \geq 0\}$  restricted to  $Y$  admits an exponential fractal attractor  $\mathcal{M}$  in  $Y$  whose fractal dimension is estimated by*

$$d_F(\mathcal{M}) \leq C\ell^{1+(2\mu/3)}, \tag{1.11}$$

where  $C$  is a positive constant which does not depend on  $\ell$ .

It was established in [5] and [12] that the global attractor is quite regular. For completeness we formulate this result in the following theorem.

**Theorem 1.6.** *The global attractor  $\mathcal{A}$  of the semigroup  $\{S(t) : t \geq 0\}$  is bounded in  $\mathcal{H}^{4n+2}(\Omega)$  for any  $n \geq 0$  and consequently,  $\mathcal{A}$  is contained in  $C^\infty(\overline{\Omega})$ .*

The plan of the paper is the following. In Section 2 we introduce the notion of a weak solution which we shall use in this paper. We derive a few calculus inequalities involving the linear elliptic operator in the parabolic equation in Problem (P) and prove the asymptotic estimate given in Theorem 1.2. In Section 3 we obtain uniform bounds in  $L^2(\Omega)$  and in  $H^1(\Omega)$ . The main ingredients here are a variation on a *Coercivity Lemma* established in [3] and the estimates obtained in Section 2. In Section 4 we use a result of [4] to prove the existence of inertial sets. Finally, in the Appendix we prove the Coercivity Lemma (Lemma 3.2) that is used to obtain the attractors.

**2. Preliminaries.** Before turning to the proofs of Theorems 1.2-1.5 we introduce some notation, give definitions, and present – for easy reference – a few technical lemmas which will be used throughout the text.

The analysis of Problem (1.4) will be carried out in the spaces  $L^2(\Omega)$ ,  $H_0^1(\Omega)$  and  $\mathcal{H}^2(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$ . Their norms will be defined by

$$\|v\|_{H_0^1} = \|v_x\| \quad \text{and} \quad \|v\|_{\mathcal{H}^2(\Omega)} = \|v_{xx}\|, \tag{2.1}$$

where  $\|\cdot\|$  denotes the norm in  $L^2(\Omega)$ , and  $\Omega = (-\ell, \ell)$ . The pairing between  $\mathcal{H}^2(\Omega)$  and its dual  $(\mathcal{H}^2(\Omega))'$  will be denoted by  $\langle \cdot, \cdot \rangle$ . We also introduce the linear second order operator

$$Lw \stackrel{\text{def}}{=} -w_{xx} + \gamma J(w) - \alpha x w_x - 2\alpha w, \tag{2.2}$$

so that the equation for  $v$  can be written compactly as

$$v_t + v_{xxxx} = Lv - vv_x. \tag{2.3}$$

**Definition 2.1.** Let  $T$  be an arbitrary positive number. A function  $v$  is called a weak solution of Problem (1.4) on  $(0, T)$  if

- (i)  $v \in L^2(0, T; \mathcal{H}^2(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ ;
- (ii)  $v_t \in L^2(0, T; (\mathcal{H}^2(\Omega))')$ ;
- (iii) Equation (2.3) is satisfied in the sense of distributions in  $Q_T$ , i.e.

$$\langle v_t, \varphi \rangle + \int_{\Omega} (v_{xx}\varphi_{xx} - Lv\varphi + vv_x\varphi) = 0$$

for all  $\varphi \in L^2(0, T; \mathcal{H}^2(\Omega))$  and for a.e.  $t \in (0, T)$ ;

- (iv)  $v(0) = v_0$ .

With the eigenvalues  $\lambda_n$  and the corresponding odd eigenfunctions  $\zeta_n$  of the Laplacian  $-d^2/d^2x$  on  $\Omega$ , with Dirichlet boundary conditions, the nonlocal term  $J(v)$  can be written more transparently as

$$J(w) = \sum_{n=1}^{\infty} n(w, \zeta_n)\zeta_n = \frac{\ell}{\pi} \sum_{n=1}^{\infty} \lambda_n^{1/2}(w, \zeta_n)\zeta_n, \tag{2.4}$$

where

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \text{ and } \zeta_n(x) = \frac{1}{\sqrt{\ell}} \sin\left(\frac{n\pi}{\ell}x\right) \quad n = 1, 2, \dots \tag{2.5}$$

**Lemma 2.1.** The functional  $J : H^1(\Omega) \rightarrow L^2(\Omega)$  is well defined and

$$\|J(w)\| \leq \frac{\ell}{\pi} \|w_x\| \quad \text{for } w \in H^1(\Omega). \tag{2.6}$$

If  $w$  is odd, then equality holds.

*Proof.* Because the sequence  $\{\zeta_n\}$  is orthonormal, we have

$$\|J(w)\|^2 = \frac{\ell^2}{\pi^2} \sum_{n=1}^{\infty} \lambda_n(w, \zeta_n)^2 \quad \text{for any } w \in H^1(\Omega).$$

If  $w$  is odd, we can write  $w = \sum_{n=1}^{\infty} (w, \zeta_n)\zeta_n$ . Therefore,

$$\|w_x\|^2 = \sum_{n=1}^{\infty} (w, \zeta_n)^2 \|(\zeta_n)_x\|^2 = \sum_{n=1}^{\infty} \lambda_n(w, \zeta_n)^2,$$

so that

$$\|J(w)\| = \frac{\ell}{\pi} \|w_x\|. \tag{2.7}$$

Let  $w \in H^1(\Omega)$ . Then we can write  $w = w_1 + w_2$ , where

$$w_1(x) = \frac{1}{2}\{w(x) - w(-x)\} \text{ is odd}$$

and

$$w_2(x) = \frac{1}{2}\{w(x) + w(-x)\} \text{ is even.}$$

Since  $w_2$  is even and  $\zeta_n$  is odd, it follows that  $(w_2, \zeta_n) = 0$  for every  $n \geq 1$ , and hence

$$J(w) = J(w_1) \quad \text{for } w \in H^1(\Omega). \tag{2.8}$$

Because  $w_1$  is odd, it follows from (2.7) and (2.8) that

$$\|J(w)\| = \|J(w_1)\| = \frac{\ell}{\pi} \|w_{1x}\|. \tag{2.9}$$

From the fact that  $(w_{1x}, w_{2x}) = 0$ , we deduce that

$$\|w_x\|^2 = \|w_{1x}\|^2 + \|w_{2x}\|^2 \geq \|w_{1x}\|^2.$$

Combining this with (2.9) we obtain

$$\|J(w)\| \leq \frac{\ell}{\pi} \|w_x\| \text{ for any } w \in H^1(\Omega).$$

□

The following two estimates involving  $L$  will prove very useful.

**Lemma 2.2.** *For any  $w \in \mathcal{H}^2(\Omega)$  and any  $s \in \mathbf{R}^+$  we have*

$$(w, Lw) \leq (1 + \kappa s) \|w_x\|^2 + \left(\frac{\kappa}{s} - \frac{3\alpha}{2}\right) \|w\|^2 \tag{2.10}$$

and

$$\|Lw\| \leq \|w_{xx}\| + (2\kappa + \alpha\ell) \|w_x\| + 2\alpha \|w\|. \tag{2.11}$$

*Proof.* We multiply  $Lw$  by  $w$  and integrate over  $\Omega$ . When we integrate by parts, and use the bound for  $J(w)$  of Lemma 2.1, we obtain

$$\begin{aligned} (w, Lw) &= -(w, w_{xx}) + \gamma(J(w), w) - \alpha(xw, w_x) - 2\alpha \|w\|^2 \\ &\leq \|w_x\|^2 + 2\kappa \|w_x\| \|w\| - \frac{3\alpha}{2} \|w\|^2 \\ &\leq \|w_x\|^2 (1 + \kappa s) + \left(\frac{\kappa}{s} - \frac{3\alpha}{2}\right) \|w\|^2, \end{aligned} \tag{2.12}$$

as asserted.

From the definition (2.2) of  $L$  we deduce that

$$\|Lw\| \leq \|w_{xx}\| + \gamma \|J(w)\| + \alpha \|xw_x\| + 2\alpha \|w\|.$$

Because  $\|xw_x\| \leq \ell \|w_x\|$ , the second assertion follows. □

**Lemma 2.3.** *Let  $v$  be the solution of Problem (1.4). Then*

$$\frac{d}{dt} \|v\|^2 + A \|v_{xx}\|^2 \leq B \|v\|^2, \tag{2.13}$$

where

$$A = 1 - \kappa s \quad \text{and} \quad B = 1 - 3\alpha + \kappa s + \frac{2\kappa}{s}, \tag{2.14}$$

and  $s$  is an arbitrary positive constant.

*Proof.* We take the duality product of (2.3) with  $v$ . This yields the equation

$$\frac{1}{2} \frac{d}{dt} \|v\|^2 + \|v_{xx}\|^2 = (v, Lv) - (v^2, v_x).$$

Applying Lemma 2.2 and using the fact that  $(v^2, v_x) = 0$ , it follows that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v\|^2 + \|v_{xx}\|^2 &\leq (1 + \kappa s) \|v_x\|^2 + \left(\frac{\kappa}{s} - \frac{3\alpha}{2}\right) \|v\|^2 \\ &\leq \frac{1}{2} (1 + \kappa s) \left\{ \|v_{xx}\|^2 + \|v\|^2 \right\} + \left(\frac{\kappa}{s} - \frac{3\alpha}{2}\right) \|v\|^2, \end{aligned} \tag{2.15}$$

where we have used the inequality

$$\|v_x\|^2 \leq \frac{1}{2} \left\{ \|v_{xx}\|^2 + \|v\|^2 \right\}. \tag{2.16}$$

With the constants  $A$  and  $B$  defined as in (2.14), the desired inequality follows from (2.15).  $\square$

**Corollary 2.4.** *If  $\alpha > \alpha_0$ , then there exist positive constants  $\nu$  and  $M$  such that*

$$\|v(t)\| \leq Me^{-\nu t} \quad \text{for } t > 0.$$

*Proof.* We use the estimate from Lemma 2.3 and put  $s = \frac{1}{\kappa}$ . Then  $A = 0$  and we obtain

$$\frac{d}{dt}\|v\|^2 \leq -2\nu\|v\|^2,$$

where

$$\nu = \frac{3}{2}(\alpha - \alpha_0).$$

The critical value  $\alpha_0$  is given by (1.8). One integration now yields (1.9) with  $M = \|v(0)\|$ .  $\square$

The *Uniform Gronwall Lemma* (see [13], p. 89) will be used in several places and for convenience we state it here. It applies to the differential inequality

$$\frac{dy}{dt} \leq gy + h \quad \text{for } t \geq t_0, \tag{2.17}$$

in which  $g$  and  $h$  are positive, locally integrable functions on  $(t_0, \infty)$ .

**Lemma 2.5.** *Let  $y(t)$  satisfy (2.17), and let*

$$\int_t^{t+r} g(s) ds \leq a_1, \quad \int_t^{t+r} h(s) ds \leq a_2, \quad \int_t^{t+r} y(s) ds \leq a_3,$$

where  $r, a_1, a_2,$  and  $a_3$  are positive constants. Then

$$y(t+r) \leq \left(\frac{a_3}{r} + a_2\right)\exp(a_1) \quad \text{for } t > t_0.$$

**3. Absorbing sets in  $L^2(\Omega)$  and in  $H^1(\Omega)$ .** In this section we prove the existence of absorbing sets in  $L^2(\Omega)$  and  $H^1(\Omega)$ . Using these results, we prove the existence of the global attractor for the semigroup  $\{S(t) : t \geq 0\}$ , defined by the solution  $v(t) = S(t)v_0$  of Problem (1.4).

We first prove a global bound in  $L^2(\Omega)$ ; we recall that  $\mu = 11/5$ .

**Theorem 3.1.** *Let  $\alpha, \gamma$  and  $\ell$  satisfy (1.10). Then the semigroup  $S(t)$  possesses an absorbing set in  $L^2(\Omega)$ . More precisely, there exists a constant  $c_0$  which does not depend on  $\ell$  and a nondecreasing function  $\tau_0(\cdot)$  such that*

$$\|S(t)v_0\|_{L^2(\Omega)} \leq c_0\ell^\mu \quad \text{if } t \geq \tau_0(\|v_0\|_{L^2(\Omega)}).$$

The main idea of the proof is the introduction of a suitably chosen translation of the dependent variable,

$$v(x, t) = \phi(x) + w(x, t).$$

The function  $\phi$  is then so chosen that we can derive a differential inequality for  $\|w(t)\|$  which enables us to obtain a uniform bound for  $\|w(t)\|$ .

Below we present a *Coercivity Lemma*, similar to one used in [3], that establishes the existence of a function  $\phi$  which has the required properties. For  $u, v \in X \stackrel{\text{def}}{=} H \cap \mathcal{H}^2(\Omega)$ , and  $\omega > 0$ , we define the weighted bilinear form

$$(u, v)_{\omega\phi} \stackrel{\text{def}}{=} (u_{xx}, v_{xx}) - (u_x, v_x) + \omega(uv, \phi_x), \tag{3.1}$$

where it is assumed that  $\phi_x \in L^2(\Omega)$ . Note that if we set  $\mathcal{L}_0 u = -u_{xxxx} - u_{xx}$ , then for  $\phi \in C_0^\infty(\Omega)$ ,

$$(u, v)_{\omega\phi} = - \int_{\Omega} u(\mathcal{L}_0 - \omega\phi_x)v.$$

**Lemma 3.2.** *For every  $\ell > 2\pi$  there exists a function  $\phi \in X$  such that for any  $\omega \in [\frac{1}{4}, 1]$  and any function  $w \in X$  the following inequality holds:*

$$(w, w)_{\omega\phi} \geq \frac{1}{4}\|w_{xx}\|^2 + \frac{1}{4}\|w\|^2. \tag{3.2}$$

The function  $\phi$  can be chosen so that

$$(\phi, \phi)_{\omega\phi} \leq C_1 \ell^{\mu+1} \tag{3.3}$$

and

$$\|\phi\| \leq C_2 \ell^\mu \quad \text{and} \quad \|\phi_x\| \leq C_3 \ell^{\mu-1}, \tag{3.4}$$

where  $C_1, C_2$  and  $C_3$  are positive constants.

**Corollary 3.3.** *For the function  $\phi$  and the constant  $\omega$  chosen in Lemma 3.2, the bilinear form  $(u, v)_{\omega\phi}$  is an inner product on  $X$ .*

For the proofs of Lemma 3.2 and Corollary 3.3 we refer to the Appendix.

*Proof.* (of Theorem 3.1) When we write the solution  $v$  of Problem (1.4) as

$$v(x, t) = \phi(x) + w(x, t),$$

where  $\phi$  is the function determined in Lemma 3.2, then  $w$  satisfies the equation

$$\begin{aligned} w_t &= -w_{xxxx} - \phi_{xxxx} - w_{xx} - \phi_{xx} - ww_x - w\phi_x - \phi w_x - \phi\phi_x \\ &\quad + \gamma J(w) + \gamma J(\phi) - \alpha x w_x - \alpha x \phi_x - 2\alpha w - 2\alpha\phi \end{aligned} \tag{3.5}$$

in the sense of distributions in  $Q_T$ . We first establish a differential inequality for  $\|w(t)\|$ :

**Lemma 3.4.** *Let  $\ell > 2\pi$ , and let*

$$\kappa \leq \frac{1}{3^{7/4}}(1 + 3\alpha)^{3/4}.$$

*Then, there exist positive constants  $\sigma$  and  $C$ , which do not depend on  $\ell$ , such that*

$$\frac{d}{dt}\|w\|^2 \leq -\sigma\|w\|^2 + C\ell^{2\mu}. \tag{3.6}$$

*Proof.* Let us take the duality product of (3.5) with  $w$ . Since  $\phi \in \mathcal{H}^2(\Omega)$ , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt}\|w\|^2 &= -(w, w)_{\frac{1}{2}\phi} - (w, \phi)_\phi + \gamma(J(w), w) + \gamma(J(\phi), w) \\ &\quad - \alpha(x, w w_x) - \alpha(x, w\phi_x) - 2\alpha\|w\|^2 - 2\alpha(w, \phi). \end{aligned} \tag{3.7}$$

For the first two terms on the right hand side of (3.7) we have

$$\begin{aligned} -(w, w)_{\frac{\phi}{2}} - (w, \phi)_{\phi} &\leq -(w, w)_{\frac{\phi}{2}} + \frac{\varepsilon}{2}(w, w)_{\phi} + \frac{1}{2\varepsilon}(\phi, \phi)_{\phi} \\ &= -\left(1 - \frac{\varepsilon}{2}\right)(w, w)_{\frac{1-\varepsilon}{2}\phi} + \frac{1}{2\varepsilon}(\phi, \phi)_{\phi}, \end{aligned} \tag{3.8}$$

where we have used Corollary 3.3 and  $\varepsilon$  is an arbitrary positive number such that  $\varepsilon < 1$ .

For the other terms on the right-hand-side of (3.7) we have that

$$(x, ww_x) = -\frac{1}{2}\|w\|^2, \tag{3.9}$$

$$|(x, w\phi_x)| \leq \frac{\ell}{2}\delta\|w\|^2 + \frac{\ell}{2\delta}\|\phi_x\|^2, \tag{3.10}$$

and

$$-2(w, \phi) \leq \frac{1}{\theta}\|\phi\|^2 + \theta\|w\|^2, \tag{3.11}$$

where  $\theta$  and  $\delta$  are arbitrary positive constants, and, in view of Lemma 2.1,

$$|\gamma|(J(w), w)| \leq 2\kappa\|w\|\|w_x\| \leq \kappa\eta\|w\|^2 + \frac{\kappa}{\eta}\|w_x\|^2 \tag{3.12}$$

and

$$|\gamma|(J(\phi), w)| \leq 2\kappa\|w\|\|\phi_x\| \leq \kappa\nu\|w\|^2 + \frac{\kappa}{\nu}\|\phi_x\|^2, \tag{3.13}$$

where  $\eta$  and  $\nu$  are arbitrary positive constants, and we recall that  $\kappa = \ell\gamma/2\pi$ . Substituting (3.8)-(3.13) into (3.7) and setting  $\varepsilon = \frac{2}{3}$  we arrive at the inequality

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w\|^2 &\leq -\frac{2}{3}(w, w)_{\frac{1}{4}\phi} + \left\{ \kappa(\eta + \nu) + \alpha \left( \theta + \frac{1}{2}\ell\delta - \frac{3}{2} \right) \right\} \|w\|^2 \\ &\quad + \frac{\kappa}{\eta} \|w_x\|^2 + R(\phi, \alpha, \kappa, \ell), \end{aligned} \tag{3.14}$$

where

$$R(\phi, \alpha, \kappa, \ell) = \frac{3}{4}(\phi, \phi)_{\phi} + \left( \frac{\alpha\ell}{2\delta} + \frac{\kappa}{\mu} \right) \|\phi_x\|^2 + \frac{\alpha}{\theta} \|\phi\|^2. \tag{3.15}$$

Using the coercivity of  $(v, v)_{\phi/4}$ , established in Lemma 3.2, in (3.14) we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w\|^2 &\leq -\frac{1}{6}\|w_{xx}\|^2 - \frac{1}{6}\|w\|^2 + \frac{\kappa}{\eta}\|w_x\|^2 \\ &\quad + \left\{ \kappa(\eta + \mu) + \alpha \left( \theta + \frac{1}{2}\ell\delta - \frac{3}{2} \right) \right\} \|w\|^2 + R(\phi, \alpha, \kappa, \ell). \end{aligned} \tag{3.16}$$

We now fix  $\eta$  and  $\nu$  so that

$$\kappa\eta = a \quad \text{and} \quad \kappa\nu = b,$$

where  $a$  and  $b$  will be fixed constants. When we use Young's inequality in the form

$$\frac{\kappa^2}{a} \|w_x\|^2 \leq \frac{1}{6} \|w_{xx}\|^2 + \frac{3\kappa^4}{2a^2} \|w\|^2.$$

we deduce from (3.16) that

$$\frac{1}{2} \frac{d}{dt} \|w\|^2 \leq -\left\{ \frac{1}{6} - (a + b) + \alpha \left( \frac{3}{2} - \theta - \frac{1}{2}\ell\delta \right) - \frac{3\kappa^4}{2a^2} \right\} \|w\|^2 + R(\phi, \alpha, \kappa, \ell). \tag{3.17}$$

Setting  $\delta\ell = 1$  and  $\theta = \frac{1}{2}$  in (3.17) results in

$$\frac{d}{dt} \|w\|^2 \leq -\sigma\|w\|^2 + 2R(\phi, \alpha, \kappa, \ell), \tag{3.18}$$

where

$$\sigma \stackrel{\text{def}}{=} \frac{1 + 3\alpha}{3} - 2(a + b) - \frac{3\kappa^4}{a^2}.$$

Since  $\phi$  is the function constructed in Lemma 3.2, it satisfies (3.2), (3.3), and (3.4), and thus, there exists a positive constant  $\mathcal{C}$  such that

$$R(\phi, \alpha, \kappa, \ell) \leq C_1 \ell^{\mu+1} + \left\{ \frac{\kappa^2}{b} + \frac{\alpha}{2} \ell^2 + 2\alpha \left( \frac{\ell}{\pi} \right)^2 \right\} \|\phi_x\|^2 \leq \mathcal{C} \ell^{2\mu}. \tag{3.19}$$

The coefficient  $\sigma$  will be positive if

$$\kappa^4 < \frac{1}{9} a^2 \{1 + 3\alpha - 6(a + b)\}.$$

Since we may choose  $b$  arbitrary small, it suffices that

$$\kappa^4 < \frac{1}{9} a^2 (1 + 3\alpha - 6a). \tag{3.20}$$

We now choose  $a$  such that the right hand side of (3.20) becomes largest, i.e. we set  $a = (1 + 3\alpha)/9$ . This then yields the condition

$$\kappa^4 < \frac{1}{3^7} (1 + 3\alpha)^3,$$

which we have assumed to be satisfied. □

From Lemma 3.4 it follows, using Gronwall’s Lemma applied to (3.6), that

$$\|w(t)\|^2 \leq \|w_0\|^2 e^{-\sigma t} + \frac{C}{\sigma} (1 - e^{-\sigma t}) \ell^{2\mu} \text{ for all } t \in \mathbf{R}^+, \tag{3.21}$$

in which  $C$  is a positive constant. We now introduce the time

$$\tau_0 = \frac{2}{\sigma} \log \|w_0\|.$$

Then

$$\|w(t)\|^2 < 1 + \frac{C}{\sigma} \ell^{2\mu} \leq C_0 \ell^{2\mu} \text{ for } t \geq \tau_0.$$

Since  $v = \phi + w$  and  $\|\phi\| = O(\ell^\mu)$ , this completes the proof of Theorem 3.1. □

In the next theorem we prove the existence of an absorbing set in  $H^1(\Omega)$ .

**Theorem 3.5.** *Let  $\ell > 2\pi$ , and let  $\alpha$  and  $\kappa$  satisfy (1.10). Then the semigroup  $S(t)$  possesses an absorbing set in  $H^1(\Omega)$ . There exists a constant  $c_1$  which does not depend on  $\ell$  and a nondecreasing function  $\tau_1(\cdot)$  such that*

$$\|S(t)v_0\|_{H^1(\Omega)} \leq c_1 \ell^{5\mu/3} \quad \text{if} \quad t \geq \tau_1(\|v_0\|_{L^2(\Omega)}).$$

As a first step we prove a bound for  $\|v_x\|$ .

**Lemma 3.6.** *We have*

$$\frac{d}{dt} \|v_x\|^2 + \|v_{xxx}\|^2 \leq K(\alpha, \kappa, \ell) \|v_x\|^2 + 12\alpha^2 \|v\|^2 + \frac{1}{4\vartheta} \|v\|^6, \tag{3.22}$$

in which

$$K(\alpha, \kappa, \ell) \stackrel{\text{def}}{=} 3(2\kappa + \alpha\ell)^2 + 8 + \frac{\vartheta}{4}, \tag{3.23}$$

and  $\vartheta$  an arbitrary positive constant.

*Proof.* Let  $t$  be such that  $v_{xx} \in \mathcal{H}^2(\Omega)$  and  $v_t \in (\mathcal{H}^2(\Omega))'$ . We take the duality product of equation (2.3) with  $v_{xx}$  and integrate by parts. This yields the identity

$$\frac{1}{2} \frac{d}{dt} \|v_x\|^2 + \|v_{xxx}\|^2 = -(Lv, v_{xx}) + (vv_x, v_{xx}). \tag{3.24}$$

To estimate the first term on the right hand side of (3.24), we use inequality (2.11) and obtain

$$\begin{aligned} |(Lv, v_{xx})| &\leq \{ \|v_{xx}\| + (2\kappa + \alpha\ell)\|v_x\| + 2\alpha\|v\| \} \|v_{xx}\| \\ &\leq \frac{1}{2} \{ \|v_{xx}\| + (2\kappa + \alpha\ell)\|v_x\| + 2\alpha\|v\| \}^2 + \frac{1}{2} \|v_{xx}\|^2 \\ &\leq \frac{3}{2} \{ \|v_{xx}\|^2 + (2\kappa + \alpha\ell)^2 \|v_x\|^2 + 4\alpha^2 \|v\|^2 \} + \frac{1}{2} \|v_{xx}\|^2 \\ &\leq \left\{ \frac{3}{2} (2\kappa + \alpha\ell)^2 + \frac{1}{\eta} \right\} \|v_x\|^2 + 6\alpha^2 \|v\|^2 + \eta \|v_{xxx}\|^2. \end{aligned} \tag{3.25}$$

Here we estimated  $\|v_{xx}\|$  by means of the inequality

$$2\|v_{xx}\|^2 = -2(v_x, v_{xxx}) \leq \frac{1}{\eta} \|v_x\|^2 + \eta \|v_{xxx}\|^2$$

in which  $\eta$  is an arbitrary positive constant.

To obtain a bound for the second term on the right hand side of (3.24), we use the following calculus inequality:

$$\|w\|_{L^4(\Omega)} \leq \|w_x\|^{1/4} \|w\|^{3/4}, \quad \text{for } w \in H_0^1(\Omega). \tag{3.26}$$

It is proved by first integrating the identity  $(w^2)_x = 2ww_x$  over  $(-\ell, x)$  to obtain

$$w^2(x) = 2 \int_{-\ell}^x ww_x \leq 2 \int_{-\ell}^x |w||w_x|,$$

and then over  $(x, \ell)$  to obtain a similar estimate involving an integral over  $(x, \ell)$ . Adding the two estimates then yields the uniform bound

$$w^2(x) \leq \|w\| \|w_x\| \text{ for any } x \in (-\ell, \ell). \tag{3.27}$$

Multiplying this inequality by  $w^2(x)$  and integrating it over  $(-\ell, \ell)$ , we obtain (3.26). To bound the second term on the right of (3.23) we integrate by parts, use Schwarz's inequality and (3.26):

$$\begin{aligned} |(vv_x, v_{xx})| &= \frac{1}{2} |(v^2, v_{xxx})| \leq \frac{\lambda}{4} \|v_{xxx}\|^2 + \frac{1}{4\lambda} \|v\|_{L^4(\Omega)}^4 \\ &\leq \frac{\lambda}{4} \|v_{xxx}\|^2 + \frac{1}{4\lambda} \|v_x\| \|v\|^3 \\ &\leq \frac{\lambda}{4} \|v_{xxx}\|^2 + \frac{\vartheta}{8\lambda} \|v_x\|^2 + \frac{1}{8\vartheta\lambda} \|v\|^6. \end{aligned} \tag{3.28}$$

Here  $\lambda$  and  $\vartheta$  are arbitrary positive constants. When we set  $\eta = \frac{1}{4}$  in (3.25) and  $\lambda = 1$  in (3.28), and substitute the resulting estimates into the right hand side of (3.24), we obtain the desired differential inequality.  $\square$

We complete the proof of Theorem 3.5 by applying the Uniform Gronwall Lemma to the differential inequality (3.22). We identify

$$y = \|v_x\|, \quad g = K(\alpha, \kappa, \ell) \quad \text{and} \quad h = 12\alpha^2 \|v\|^2 + \frac{1}{4\vartheta} \|v\|^6,$$

and set  $\vartheta = \ell^{4\mu/3}$ . Then there exists a constant  $C_1 = C_1(\alpha) > 0$  such that

$$K(\alpha, \kappa, \ell) \leq C_1 \ell^{4\mu/3},$$

and hence we have  $a_1 = C_1 r \ell^{4\mu/3}$ . Also, from Theorem 3.1 we conclude that there exists a constant  $C_2 > 0$  such that

$$12\alpha^2 \|v\|^2 + \frac{1}{4\vartheta} \|v\|^6 \leq C_2 \ell^{14\mu/3},$$

so that we can take  $a_2 = C_2 r \ell^{14\mu/3}$ .

It remains to determine  $a_3$ . We do this by interpolating between  $\|v\|$  and  $\|v_{xx}\|$ , as in inequality (2.16). For  $\|v\|$ , we use the bound from Theorem 3.1. To estimate  $\|v_{xx}\|$ , we use Lemma 2.3, where we set  $s = \frac{1}{2\kappa}$ , so that  $A = \frac{1}{2}$ . We then integrate the differential inequality (2.13) over  $(t, t+r)$  for some  $r > 0$  and  $t \geq \tau_0$ , and, once again, use the bound for  $\|v\|$ , this time at  $t$  and at  $t+r$ . We thus arrive at the estimate

$$\int_t^{t+r} \|v_{xx}\|^2 \leq C(1+r)\ell^{2\mu}$$

for some positive constant  $C$ . If we now integrate (2.16) over  $(t, t+r)$ , we obtain

$$\int_t^{t+r} \|v_x\|^2 \leq C_3(1+r)\ell^{2\mu} \stackrel{\text{def}}{=} a_3.$$

It follows that

$$\|v_x(t+r)\|^2 \leq C\left(\frac{1}{r}\ell^{2\mu} + r\ell^{14\mu/3}\right)e^{C_1 r \ell^{4\mu/3}},$$

where  $C$  is some positive constant. We choose  $r = \ell^{-4\mu/3}$ . Then

$$\|v_x(t+r)\|^2 \leq C\ell^{10\mu/3}e^{C_1}.$$

Since  $\ell > 2\pi$ , we conclude that

$$\|v_x(t+r)\| \leq C\ell^{5\mu/3} \text{ for } t \geq \tau_0,$$

and hence

$$\|v_x(t)\| \leq C\ell^{5\mu/3} \text{ for } t \geq \tau_1 = \tau_0 + \ell^{-4\mu/3}.$$

This completes the proof of Theorem 3.5.

From [13], Theorem 1.1, p. 23, we can now conclude the main result of this section.

**Theorem 3.7.** *The semigroup  $S(t)$  associated with Problem (1.4) maps  $H$  into itself. It possesses in  $H$  a global attractor that is compact and connected.*

**4. Inertial sets.** The object of this section is to prove the existence of inertial sets, namely compact sets which contain the attractor, are positively invariant by the semigroup, have a finite fractal dimension and attract all solutions at an exponential rate. We also obtain an upper bound for the fractal dimension of the attractor.

In Section 3 it was proved that the set

$$B = \{v \in H \cap H_0^1(\Omega) : \|v\| \leq \rho_0 \text{ and } \|v_x\| \leq \rho_1\}, \tag{4.1}$$

where

$$\rho_0 = c_0 \ell^\mu \quad \text{and} \quad \rho_1 = c_1 \ell^{5\mu/3}, \tag{4.2}$$

where  $\mu = 11/5$ , is absorbing for all bounded subsets of  $H$ , the set of odd functions in  $L^2(\Omega)$ . A direct consequence of these estimates and (3.27) is that for  $v \in B$ ,

$$\|v\|_{L^\infty(\Omega)} \leq \|v\|^{1/2} \|v_x\|^{1/2} \leq \sqrt{c_0 c_1} \ell^{4\mu/3}. \tag{4.3}$$

We define the positively invariant set

$$Y = \overline{\cup_{t \geq \tau_1} S(t)B}, \tag{4.4}$$

where  $\tau_1 = \tau_1(\|v_0\|)$  was introduced in Theorem 3.5.

To prove the existence of an inertial fractal set for  $(\{S(t)\}_{t \geq 0}, Y)$  we use a result of [4] (see Theorem 3.1 on p. 32 of [4]). To apply this theorem it is required to show that there exists a time  $t_* > 0$  such that the map

$$S_* \stackrel{\text{def}}{=} S(t_*) \tag{4.5}$$

has the following properties:

- (A)  $S_*$  is Lipschitz continuous on  $Y$ ;
- (B)  $S_*$  satisfies the *squeezing property* defined below:

**Definition 4.1.** *Let  $G$  be a Lipschitz continuous map from  $Y$  to itself. Then  $G$  is said to have the squeezing property in  $Y$  if for some  $\delta \in (0, \frac{1}{8})$  there exists an orthogonal projection  $P = P(\delta)$  in  $H$  of rank equal to  $N_0(\delta)$  such that for every  $v_1$  and  $v_2$  in  $H$  which satisfies*

$$\|(I - P)(Gv_1 - Gv_2)\|_H > \|P(Gv_1 - Gv_2)\|_H$$

the following inequality holds:

$$\|(Gv_1 - Gv_2)\|_H \leq \delta \|v_1 - v_2\|_H.$$

In Lemma 4.1 we prove the first property; in fact we show that  $S(t)$  is Lipschitz continuous for any  $t \in [0, T]$  where  $T$  is an arbitrary positive constant.

**Lemma 4.1.** *For any  $t \in [0, T]$ , the semigroup  $S(t) : Y \rightarrow Y$  associated with Problem (1.4) is Lipschitz continuous on  $H$ .*

*Proof.* Let  $v_1(x, t)$  and  $v_2(x, t)$  be two solutions of Problem (1.4) with the initial values  $v_{10}$  and  $v_{20}$  respectively in  $X$ . Their difference

$$w(x, t) = v_1(x, t) - v_2(x, t)$$

satisfies the equation

$$w_t + w_{xxxx} = Lw - (\bar{v}w)_x, \tag{4.6}$$

where  $\bar{v} = \frac{1}{2}(v_1 + v_2)$  is an element of  $B$  as well. When we multiply equation (4.6) by  $w$  and integrate over  $\Omega$ , we obtain

$$\frac{1}{2} \frac{d}{dt} \|w\|^2 + \|w_{xx}\|^2 = (Lw, w) + (\bar{v}, ww_x). \tag{4.7}$$

The first term on the right of (4.7) can be estimated by means of Lemma 2.2. To estimate the second term, we write

$$|(\bar{v}, ww_x)| \leq \|\bar{v}\|_{L^\infty(\Omega)} \|w\| \|w_x\|.$$

Thus, since  $\|w_x\|^2 \leq \|w\| \|w_{xx}\|$ , we obtain

$$\begin{aligned} |(\bar{v}, ww_x)| &\leq \|\bar{v}\|_{L^\infty(\Omega)} \|w\|^{3/2} \|w_{xx}\|^{1/2} \\ &\leq \frac{3}{4} \|\bar{v}\|_{L^\infty(\Omega)}^{4/3} \|w\|^2 + \frac{1}{4} \|w_{xx}\|^2, \end{aligned} \tag{4.8}$$

where we have applied Young's inequality

$$ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad a \geq 0, b \geq 0,$$

with  $p = \frac{4}{3}$ ,  $q = 4$ ,  $a = \|\bar{v}\|_{L^\infty(\Omega)}\|w\|^{3/2}$  and  $b = \|w_{xx}\|^{1/2}$ . From (4.2) we know that

$$\|\bar{v}\|_{L^\infty(\Omega)}^{4/3} \leq (c_0c_1)^{2/3}\ell^{16\mu/9},$$

and thus, (4.8) can be written as

$$|(\bar{v}, ww_x)| \leq C_2\ell^{16\mu/9}\|w\|^2 + \frac{1}{4}\|w_{xx}\|^2, \quad \text{where} \quad C_2 = \frac{3}{4}(c_0c_1)^{2/3}. \quad (4.9)$$

When we use (2.10) and (4.9) in (4.7) we obtain the estimate

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \frac{1}{4}(1 - 2\kappa s)\|w_{xx}\|^2 \leq \frac{1}{2}(1 + \kappa s + 2\frac{\kappa}{s} + 2C_2\ell^{16\mu/9})\|w\|^2.$$

Choosing  $s = 1/(4\kappa)$ , this inequality becomes

$$\frac{d}{dt}\|w\|^2 + \frac{1}{4}\|w_{xx}\|^2 \leq \left\{ \frac{5}{4} + 8\kappa^2 + 2C_2\ell^{16\mu/9} \right\} \|w\|^2.$$

Since  $\kappa$  is bounded we conclude that there exists a constant  $C_3 > 0$ , which does not depend on  $\ell$ , such that

$$\frac{d}{dt}\|w\|^2 + \frac{1}{4}\|w_{xx}\|^2 \leq 2C_3\ell^{16\mu/9}\|w\|^2. \quad (4.10)$$

This yields the estimate

$$\|w(t)\| \leq e^{C_3\ell^{16\mu/9}t}\|w(0)\|, \quad (4.11)$$

so that the Lipschitz constant  $\mathcal{L}$  is given by

$$\mathcal{L}(T) = e^{C_3\ell^{16\mu/9}T}. \quad (4.12)$$

Thus we have shown that  $S(t)$  is Lipschitz continuous for every  $t > 0$  and so satisfies condition (A) for every  $t > 0$ .  $\square$

Next we turn to condition (B) and show that for some  $t_*$  the operator  $S(t_*)$  possesses the squeezing property. We introduce some notation and write

$$Av = v_{xxxx}, \quad D(A) = \mathcal{H}^4(\Omega), \quad (4.13)$$

where we recall that  $\mathcal{H}^4(\Omega) = \{w \in \mathcal{H}^2(\Omega) : w_{xx} \in \mathcal{H}^2(\Omega)\}$ . It is readily verified that  $A$  is a positive self-adjoint linear operator. Since the injection  $D(A) \subset H$  is compact,  $A^{-1}$  can be considered as a self-adjoint compact operator on  $H$ , and we can use the spectral theory of self-adjoint compact operators in a Hilbert space (see [13], p. 56).

We are now allowed to define the powers  $A^s$  of  $A$  for  $s \in \mathbf{R}$ . Setting

$$V = D(A^{1/2}) = \mathcal{H}^2(\Omega), \quad (4.14)$$

it is clear that  $V$  is compactly imbedded in  $H$ . For the rest of this section we write

$$\|v\|_H = \|v\|_{L^2(\Omega)} = \|v\| \quad \text{and} \quad \|v\|_V = \|A^{1/2}v\|_H = \|v_{xx}\|_H = \|v\|_{\mathcal{H}^2(\Omega)}. \quad (4.15)$$

There exists a complete set of eigenvectors  $\{\zeta_n\}_{n=1}^\infty$  in  $H$  corresponding to the positive eigenvalues  $\{\sigma_n\}_{n=1}^\infty$  of  $A$ , that is

$$A\zeta_n = \sigma_n\zeta_n \quad \text{for all} \quad n \in \mathbf{N} \quad \text{and} \quad \sigma_n = \left(\frac{n\pi}{\ell}\right)^4. \quad (4.16)$$

We write  $H_n = \text{span} \{\zeta_1, \zeta_2, \dots, \zeta_n\}$ , denote by  $P_n$  the orthogonal projection of  $H$  onto  $H_n$ , and set  $Q_n = I - P_n$ . Plainly,  $Q_n$  is the orthogonal projection of  $H$  onto the orthogonal complement of  $H_n$ .

**Lemma 4.2.** *There exists a time  $t_*$  such that the operator  $S_* = S(t_*)$  satisfies the squeezing property.*

We first give an auxiliary result:

**Proposition 4.3.** *Suppose that for some  $n \geq 1$ ,*

$$\|Q_n(z)\| > \|P_n(z)\| \text{ for } z \in \mathcal{H}^4(\Omega). \tag{4.17}$$

Then

$$\frac{\|z''\|^2}{\|z\|^2} = \frac{\|z\|_V^2}{\|z\|_H^2} > \frac{1}{2}\sigma_{n+1}. \tag{4.18}$$

*Proof.* From the choice of the orthogonal projections  $P_n$  and  $Q_n$  it follows that  $P_n z$  is orthogonal to  $Q_n z$  with respect to the inner products both in  $H$  and  $V$ . Hence,

$$\frac{\|z\|_V^2}{\|z\|_H^2} = \frac{\|P_n(z) + Q_n(z)\|_V^2}{\|P_n(z) + Q_n(z)\|_H^2} = \frac{\|P_n(z)\|_V^2 + \|Q_n(z)\|_V^2}{\|P_n(z)\|_H^2 + \|Q_n(z)\|_H^2} > \frac{1}{2} \frac{\|Q_n(z)\|_V^2}{\|Q_n(z)\|_H^2},$$

where we have used the fact that  $\|Q_n(z)\| > \|P_n(z)\|$ . Now we can go back to the operator  $A$  to obtain

$$\|Q_n(z)\|_V^2 = \|A^{1/2}(Q_n(z))\|_H^2 = (A(Q_n(z)), Q_n(z))_H \geq \sigma_{n+1} \|Q_n(z)\|_H^2,$$

where the last inequality follows from the fact that  $\sigma_{n+1}$  is the smallest eigenvalue of  $A$  over  $Q_n H$ .

**Remark.** Suppose that (4.17) is satisfied for  $n = N_0$  at some  $t_*$ . Then, putting

$$z = w_* \stackrel{\text{def}}{=} w(t_*) = S(t_*)v_1 - S(t_*)v_2,$$

it follows that

$$\frac{\|w_*\|_V^2}{\|w_*\|_H^2} > \frac{1}{2}\sigma_{N_0+1}. \tag{4.19}$$

□

We have to show that (4.19) implies that

$$\|w_*\| \leq \delta \|v_1 - v_2\| \quad \text{for some } \delta \in (0, \frac{1}{8}).$$

To achieve this, we observe that  $w = v_1 - v_2$  satisfies

$$w_t + Aw - Rv_1 + Rv_2 = 0, \tag{4.20}$$

where  $A$  is the linear operator introduced in (4.13) and  $R$  denotes the right hand side of equation (2.3), namely

$$Rv = L(v) - vv_x. \tag{4.21}$$

We introduce the functions

$$\xi(t) = \frac{w(t)}{\|w(t)\|} \quad \text{and} \quad \eta(t) = \frac{\|w_{xx}(t)\|^2}{\|w(t)\|^2} = \frac{\|w(t)\|_{\mathcal{H}^2(\Omega)}^2}{\|w(t)\|^2} = \frac{\|w(t)\|_V^2}{\|w(t)\|_H^2}. \tag{4.22}$$

Note that  $\xi : \mathbf{R}^+ \rightarrow H$  and  $\eta : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ . Thus,

$$\|\xi_{xx}(t)\|^2 = \eta(t) \quad \text{and} \quad \eta(t_*) > \frac{1}{2}\sigma_{N_0+1}. \tag{4.23}$$

We can now write (4.10) as

$$\frac{d}{dt}\|w\|^2 + \left\{ \frac{1}{4}\eta(t) - K \right\} \|w\|^2 \leq 0, \tag{4.24}$$

where  $K = 2C_3\ell^{16\mu/9}$ . Hence, by Gronwall's Lemma

$$\|w(t)\|^2 \leq \exp\left(-\frac{1}{4} \int_0^t \eta(\tau) d\tau + Kt\right) \|w(0)\|^2. \tag{4.25}$$

Therefore, putting  $t = t_*$  and writing  $w(t_*) = w_*$  we obtain

$$\|S_*v_1 - S_*v_2\| = \|w_*\| \leq \delta(t_*) \|v_{10} - v_{20}\|, \tag{4.26}$$

in which

$$\delta_* = \delta(t_*) = \exp\left(-\frac{1}{8} \int_0^{t_*} \eta(\tau) d\tau + \frac{1}{2} Kt_*\right). \tag{4.27}$$

At this point we only know – from (4.23) – that  $\eta(t_*) > \frac{1}{2}\sigma_{N_0+1}$  and by (4.16) that  $\sigma_{N_0+1} \rightarrow \infty$  as  $N_0 \rightarrow \infty$ . In the following proposition we obtain a lower bound for  $\eta(t)$  which will be used to show that if  $t < t_*$ , then  $\eta(t) \rightarrow \infty$  as  $N_0 \rightarrow \infty$ .

**Proposition 4.4.** *For any given  $T > 0$ , and any positive constant  $\beta$ , the function  $\eta(t)$  has the following lower bound:*

$$\eta(t) \geq \eta(T)e^{-8(\beta+1)(T-t)} - \frac{K_0}{8(\beta+1)} \text{ for } 0 < t \leq T. \tag{4.28}$$

Here  $K_0 \leq C(\beta)\ell^{16\mu/3}$  and  $C(\beta) = \mathcal{O}(\beta^{-1})$  as  $\beta \rightarrow 0^+$ .

*Proof.* We start by differentiating equation (4.22):

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \eta(t) &= \frac{1}{\|w\|^2} \{(w_t, Aw) - (w_t, w)\eta(t)\} \\ &= \frac{1}{\|w\|} (w_t, (A - \eta)\xi) \\ &= \frac{1}{\|w\|} (-Aw + (Rv_1 - Rv_2), (A - \eta)\xi). \end{aligned} \tag{4.29}$$

We assert that

$$\frac{1}{\|w\|} (Aw, (A - \eta)\xi) = \|(A - \eta)\xi\|^2. \tag{4.30}$$

To prove this identity, note that

$$(\eta\xi, (A - \eta)\xi) = \eta(\xi, A\xi) - \eta^2 \|\xi\|^2 = \eta \|\xi\|_{\mathcal{H}^2(\Omega)}^2 - \eta^2 = \eta \frac{\|w\|_{\mathcal{H}^2(\Omega)}^2}{\|w\|^2} - \eta^2 = 0,$$

by the definition of  $\eta$ . Hence,

$$\|(A - \eta)\xi\|^2 = ((A - \eta)\xi, (A - \eta)\xi) = (A\xi, (A - \eta)\xi) = \frac{1}{\|w\|} (Aw, (A - \eta)\xi),$$

from which (4.30) follows. Combining (4.29) and (4.30) we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \eta(t) + \|(A - \eta)\xi\|^2 &= \frac{1}{\|w\|} (Rv_1 - Rv_2, (A - \eta)\xi) \\ &\leq \frac{\|Rv_1 - Rv_2\|}{\|w\|} \|(A - \eta)\xi\| \\ &\leq \frac{1}{2} \|(A - \eta)\xi\|^2 + \frac{1}{2} \frac{\|Rv_1 - Rv_2\|^2}{\|w\|^2}. \end{aligned}$$

Thus, we arrive at the inequality

$$\frac{d}{dt} \eta(t) + \|(A - \eta)\xi\|^2 \leq \frac{\|Rv_1 - Rv_2\|^2}{\|w\|^2}. \tag{4.31}$$

From the definition of  $R$ , we obtain

$$\frac{Rv_1 - Rv_2}{\|w\|} = \frac{-v_1v_{1x} + v_2v_{2x}}{\|w\|} + \frac{Lw}{\|w\|} = -(\bar{v}\xi)_x + L\xi. \tag{4.32}$$

Therefore, dropping the nonnegative term  $\|(A - \eta)\xi\|^2$  on the left-hand side of (4.31), and using (4.32), it follows that

$$\begin{aligned} \frac{d}{dt}\eta(t) &\leq \| -(\bar{v}\xi)_x + L\xi \|^2 \\ &= \| -\bar{v}_x\xi - \bar{v}\xi_x + L\xi \|^2 \leq 2\|\bar{v}_x\xi + \bar{v}\xi_x\|^2 + 2\|L\xi\|^2 \\ &\leq 4\|\bar{v}_x\xi\|^2 + 4\|\bar{v}\xi_x\|^2 + 2\|L\xi\|^2. \end{aligned} \tag{4.33}$$

We first evaluate

$$\begin{aligned} \|\bar{v}_x\xi\|^2 + \|\bar{v}\xi_x\|^2 &\leq \|\bar{v}_x\|^2\|\xi\|_{L^\infty(\Omega)}^2 + \|\bar{v}\|_{L^\infty(\Omega)}^2\|\xi_x\|^2 \\ &\leq \rho_1^2\|\xi\|\|\xi_x\| + \|\bar{v}\|\|\bar{v}_x\|\|\xi\|\|\xi_{xx}\| \\ &\leq \rho_1^2\|\xi\|^{\frac{3}{2}}\|\xi_{xx}\|^{\frac{1}{2}} + \rho_0\rho_1\eta(t)^{\frac{1}{2}} \\ &\leq \rho_1^2\eta(t)^{\frac{1}{4}} + \rho_0\rho_1\eta(t)^{\frac{1}{2}}, \end{aligned} \tag{4.34}$$

where we have used the Sobolev inequality (4.3) and the bounds for  $v$  and  $\xi$ . By (2.11), we also have

$$\begin{aligned} \|L\xi\|^2 &\leq 2\|\xi_{xx}\|^2 + 2\{(2\kappa + \alpha\ell)\|\xi_x\| + 2\alpha\|\xi\|\}^2 \\ &\leq 2\|\xi_{xx}\|^2 + 4(2\kappa + \alpha\ell)^2\|\xi_x\|^2 + 16\alpha^2\|\xi\|^2 \\ &\leq 2\{1 + \lambda(2\kappa + \alpha\ell)^2\}\eta(t) + 2\left\{8\alpha^2 + \frac{1}{\lambda}(2\kappa + \alpha\ell)^2\right\}\|\xi\|^2, \end{aligned}$$

where we have used the inequality

$$\|\xi_x\|^2 \leq \frac{\lambda}{2}\|\xi_{xx}\|^2 + \frac{1}{2\lambda}\|\xi\|^2,$$

in which  $\lambda$  is an arbitrary positive number. Setting  $\lambda = 1/(2\kappa + \alpha\ell)^2$ , and remembering that  $\|\xi\| = 1$ , we conclude that

$$\|L\xi\|^2 \leq 4\eta(t) + 2\{(2\kappa + \alpha\ell)^4 + 8\alpha^2\}. \tag{4.35}$$

Thus, it follows that

$$\|L\xi\|^2 \leq 4\eta(t) + C_4\ell^4, \tag{4.36}$$

where  $C_4$  does not depend on  $\ell$ . Substituting (4.34) and (4.36) into (4.33), we arrive at the differential inequality

$$\frac{d}{dt}\eta(t) \leq 8\eta(t) + 4\rho_1^2\eta(t)^{1/4} + 4\rho_0\rho_1\eta(t)^{1/2} + 2C_4\ell^4. \tag{4.37}$$

Using Young's inequality to estimate the two fractional powers of  $\eta$  in (4.37), we obtain after an elementary computation that

$$\frac{d}{dt}\eta(t) \leq 8(\beta + 1)\eta(t) + K_0, \tag{4.38}$$

where

$$K_0 = \frac{3}{(4\beta)^{1/3}}\rho_1^{8/3} + \frac{1}{\beta}\rho_0^2\rho_1^2 + 2C_4\ell^4,$$

and  $\beta$  is an arbitrary positive constant. Substituting the bounds for  $\rho_0$  and  $\rho_1$  from (4.2), we find that

$$K_0 \leq C(\beta)\ell^{16\mu/3}, \tag{4.39}$$

in which  $C(\beta) = \mathcal{O}(\beta^{-1})$  as  $\beta \rightarrow 0^+$ . By Gronwall’s Lemma, we obtain

$$\eta(T) \leq e^{8(\beta+1)(T-t)} \left\{ \eta(t) + \frac{K_0}{8(\beta+1)} \right\} \quad \text{for } 0 \leq t \leq T.$$

From the above inequality, the desired lower bound (4.28) for  $\eta(t)$  follows. This completes the proof of Proposition 4.4.  $\square$

We now put  $T = t_*$  in (4.28), integrate over  $(0, t_*)$  and use the lower bound (4.23) to obtain

$$\int_0^{t_*} \eta(t) dt \geq \frac{1}{16(\beta+1)} [(1 - e^{-8(\beta+1)t_*})\sigma_{N_0+1} - 2K_0t_*]. \tag{4.40}$$

Hence, by (4.27),

$$\delta_* = \delta(t_*) \leq \exp \left\{ \frac{1}{128(\beta+1)} \{ -(1 - e^{-8(\beta+1)t_*})\sigma_{N_0+1} + 2K_0t_* \} + \frac{1}{2}Kt_* \right\}.$$

When we now choose  $\beta$  so that

$$8(\beta+1)t_* = 1, \tag{4.41}$$

then  $1 - e^{-8(\beta+1)t_*} = 1 - e^{-1} > \frac{1}{2}$ , and we get

$$\delta_* \leq \exp \left\{ -\frac{1}{32}\sigma_{N_0+1}t_* + \frac{1}{8}K_0t_*^2 + \frac{1}{2}Kt_* \right\}. \tag{4.42}$$

Let us take

$$t_* = \ell^{-8\mu/3}.$$

Since we assume that  $\ell > 2\pi$  it follows that  $t_* < (2\pi)^{-8\mu/3}$ . According to (4.41) we have  $\beta^{-1} = \frac{8t_*}{1-8t_*}$ , so that we conclude that  $\beta^{-1} < 1$ . Thus, by (4.39), we obtain

$$K_0 \leq C_5 \ell^{16\mu/3},$$

where the constant  $C_5$  does not depend on  $\ell$ . Using also the fact that  $K = 2C_3 \ell^{16\mu/9}$  in which  $C_3$  does not depend on  $\ell$ , the inequality (4.42) becomes

$$\delta_* \leq \exp \left\{ -\frac{\pi^4(N_0+1)^4}{32\ell^{4+(8\mu/3)}} + C_6 \right\}, \tag{4.43}$$

where  $C_6$  does not depend on  $\ell$ . Hence, if  $N_0$  is the smallest natural number such that

$$N_0 + 1 \geq C_7 \ell^{1+(2\mu/3)} \tag{4.44}$$

where

$$C_7 = \frac{2}{\pi} \{2(\ln 8 + C_6)\}^{1/4}, \tag{4.45}$$

then  $\delta_* \leq \frac{1}{8}$ . This completes the proof of Lemma 4.2.

**Theorem 4.5.** *The flow  $\{S(t) : t \geq 0\}$  on  $H$  that is determined by Problem (1.3) admits an exponential fractal attractor  $\mathcal{M}$  in  $Y$  whose fractal dimension is estimated by*

$$d_F(\mathcal{M}) \leq C_0 \ell^{1+(2\mu/3)}. \tag{4.46}$$

where  $C_0$  is a constant that only depends on the absolute constants  $c_0$  and  $c_1$  introduced in Theorem 1.3. Moreover, there exist positive constants  $\mathcal{C}$  and  $\mathcal{K}$  such that for any  $t \geq 0$

$$\text{dist}_{L^2(\Omega)}(S(t)B, \mathcal{M}) \leq \mathcal{C}e^{-\mathcal{K}\ell^{8\mu/3}t}. \tag{4.47}$$

*Proof.* We have proved in Lemma 4.1 that the map  $F(t, x) = S(t)x$  is Lipschitz continuous from  $[0, T] \times Y$  into  $Y$  and in Lemma 4.2 we have shown that there exists a time  $t_* > 0$  such that  $S_* = S(t_*) : Y \rightarrow Y$  possesses the squeezing property.

By [4], Theorem 2.1, p. 14, this implies that the map  $S_*$  admits an exponential fractal attractor  $\mathcal{M}_* \subseteq B$  such that

$$d_F(\mathcal{M}_*) \leq C_8 N_0,$$

where the positive constant  $C_8$  does not depend on  $\ell$ . Thus, in view of (4.43),  $N_0 = C_9 \ell^{1+(2\mu/3)}$ , and it follows that

$$d_F(\mathcal{M}_*) \leq C_0 \ell^{1+(2\mu/3)}, \tag{4.48}$$

where  $C_0$  does not depend on  $\ell$ .

Now we set

$$\mathcal{M} = \cup_{0 \leq t \leq t_*} S(t)\mathcal{M}_*. \tag{4.49}$$

Then, from [4], Theorem 3.1, p.32, it follows that  $\mathcal{M}$  is an exponential fractal attractor for the semigroup  $\{S(t) : t \geq 0\}$ . Note that  $\mathcal{M}$  is the image of  $[0, t_*] \times \mathcal{M}_*$  under  $F$ , which is Lipschitz continuous. Since Lipschitz functions preserve fractal dimension,

$$d_F(\mathcal{M}) \leq d_F(\mathcal{M}_*) + 1 \leq 2C_0 \ell^2. \tag{4.50}$$

Moreover, by [4], equation (5.1.40), p. 44 we have

$$\text{dist}_{L^2(\Omega)}(S(t)B, \mathcal{M}) \leq C_{10} \mathcal{L}(t_*) [(\delta_*)^{\frac{1}{t_*}}]^t. \tag{4.51}$$

By (4.11) we have

$$\mathcal{L}(t_*) = \text{Lip}_B(S(t_*)) \leq e^{C_3 \ell^{16\mu/9} t_*} \leq e^{C_3}.$$

Also using the fact that  $\delta_* < \frac{1}{8}$ , we deduce from (4.49) that

$$\text{dist}_{L^2(\Omega)}(S(t)B, \mathcal{M}) \leq C_{10} \mathcal{L}(\delta_*)^{\ell^{8\mu/3} t} = \mathcal{C} e^{-\mathcal{K} \ell^{8\mu/3} t},$$

where  $\mathcal{C}$  is a constant which does not depend on  $\ell$ . □

**Appendix A.** In this Appendix we prove Lemma 3.2 which is similar to Proposition 2.1 of [3], but adapted to the type of boundary conditions we have in Problem (1.4). We recall that  $X = H \cup \mathcal{H}^2(\Omega)$  and

$$(u, v)_{\omega\phi} := (u_{xx}, v_{xx}) - (u_x, v_x) + \omega(uv, \phi_x), \quad u, v \in \mathcal{Q}, \quad \omega > 0 \tag{3.1}$$

where  $\phi_x \in L^2(\Omega)$ .

**Lemma 3.2.** *For every  $\ell > 2\pi$  there exists a function  $\phi \in X$  such that for any  $\omega \in [\frac{1}{4}, 1]$  and any function  $w \in X$  the following inequality holds:*

$$(w, w)_{\omega\phi} \geq \frac{1}{4} \|w_{xx}\|^2 + \frac{1}{4} \|w\|^2. \tag{3.2}$$

The function  $\phi$  can be chosen so that

$$(\phi, \phi)_{\omega\phi} \leq C_1 \ell^{\mu+1} \tag{3.3}$$

and

$$\|\phi\| \leq C_2 \ell^\mu \quad \text{and} \quad \|\phi_x\| \leq C_3 \ell^{\mu-1}, \tag{3.4}$$

where  $\mu = \frac{11}{5}$  and  $C_1, C_2$  and  $C_3$  are positive constants.

*Proof.* Since  $\phi \in L^2(\Omega)$  is odd, we can write it as

$$\phi(x) = -2 \sum_{n=1}^{\infty} \frac{\ell}{n\pi} \psi_n \sin\left(\frac{n\pi x}{\ell}\right). \tag{A.1}$$

Note that  $\phi(\pm\ell) = 0$ . We want to choose the coefficients  $\psi_n$  such that the inequalities (3.2) – (3.4) hold.

For a natural number  $M$ , to be chosen later, we define

$$\psi_n = \begin{cases} 4, & \text{when } 1 \leq |n| \leq 2M \\ 4f\left(\frac{|n|}{2M} - 1\right), & \text{when } 2M \leq |n|, \end{cases} \quad (\text{A.2})$$

where  $f$  is a non-increasing  $C^1$  function with the properties

$$\begin{aligned} f(0) &= 1, \quad f'(0) = 0, \quad f \geq 0 \text{ on } \mathbf{R}^+ \\ \sup |f'| &< 1, \quad \int_0^\infty (1 + k^{2\alpha})|f(k)|^2 dk < \infty, \\ n^\alpha f(n) &\rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned} \quad (\text{A.3})$$

where  $\alpha > 2$ . It is readily seen that

$$n^\alpha \psi_n \rightarrow 0 \text{ as } n \rightarrow \infty,$$

and that there exist positive constants  $K_1$  and  $K_2$  such that

$$\sum_{n=1}^\infty \psi_n^2 \leq K_1 M \quad \text{and} \quad \sum_{n=1}^\infty n^2 \psi_n^2 \leq K_2 M^3.$$

Since  $\alpha > 2$ , the series

$$\sum_{n=1}^\infty \frac{1}{n^\alpha} \cos\left(\frac{n\pi x}{\ell}\right)$$

is uniformly convergent in  $\Omega$ . Hence, because  $\psi_n \leq Cn^{-\alpha}$  for some constant  $C > 0$ , this implies that

$$2 \sum_{n=1}^\infty \psi_n \cos\left(\frac{n\pi x}{\ell}\right) \text{ converges uniformly on } \Omega. \quad (\text{A.4})$$

It is trivially verified that for  $x = 0$  the series of functions

$$2 \sum_{n=1}^\infty \frac{\ell}{n\pi} \psi_n \sin\left(\frac{n\pi x}{\ell}\right) \text{ is convergent.} \quad (\text{A.5})$$

We now apply the criterium of differentiability for series of functions:

If  $\{f_n\}$  is a sequence of differentiable functions on a bounded interval  $I$ , such that  $\sum_{n=1}^\infty f'_n$  converges uniformly on  $I$  to a function  $g$ , and there exists a point  $x_0 \in I$  such that  $\sum_{n=1}^\infty f_n(x_0)$  is convergent, then  $\sum_{n=1}^\infty f_n$  converges uniformly on  $I$  to a differentiable function  $f$  and  $f' = g$ , i.e.,

$$\left(\sum_{n=1}^\infty f_n\right)' = \sum_{n=1}^\infty f'_n.$$

In view of (A.4) and (A.5), we may apply this criterium to the series of functions  $f_n = \frac{\ell}{n\pi} \psi_n \sin\left(\frac{n\pi x}{\ell}\right)$ . We then find that

$$\phi_x(x) = -2 \sum_{n=1}^\infty \psi_n \cos\left(\frac{n\pi x}{\ell}\right). \quad (\text{A.6})$$

We will now apply this criterium again, this time to the series given by (A.6); we set  $f_n = \psi_n \cos\left(\frac{n\pi x}{\ell}\right)$ . To this end we need to show that

$$\sum_{n=1}^\infty \frac{n\pi}{\ell} \psi_n \sin\left(\frac{n\pi x}{\ell}\right) \text{ converges uniformly on } \Omega. \quad (\text{A.7})$$

Because  $\alpha - 1 > 1$  the series  $\sum_{n=1}^\infty \frac{1}{n^{\alpha-1}} \sin\left(\frac{n\pi x}{\ell}\right)$  converges uniformly on  $\Omega$ , and we can use the previous argument to prove the uniform convergence of the series in (A.7).

Because  $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$  is convergent, it follows that  $\sum_{n=1}^{\infty} \psi_n$  is convergent. Therefore, for  $x = 0$ , the series given by (A.6) is convergent. This, together with (A.7) shows that

$$\phi_{xx}(x) = 2 \sum_{n=1}^{\infty} \frac{n\pi}{\ell} \psi_n \sin\left(\frac{n\pi x}{\ell}\right). \tag{A.8}$$

We are now ready to prove the inequalities (3.2)–(3.4). By (A.3), (A.4) and (A.6), we have

$$\|\phi_x\|^2 = 4 \sum_{n=1}^{\infty} \psi_n^2 \int_{-\ell}^{\ell} \cos^2\left(\frac{n\pi x}{\ell}\right) dx = 4\ell \sum_{n=1}^{\infty} \psi_n^2 \leq 4K_1\ell M.$$

In order to prove (3.3), we observe that

$$(\phi, \phi)_{\omega\phi} \leq \|\phi_{xx}\|^2.$$

By (A.3), (A.7) and (A.8), we have

$$\|\phi_{xx}\|^2 = 4 \sum_{n=1}^{\infty} \psi_n^2 \left(\frac{n\pi}{\ell}\right)^2 \int_{-\ell}^{\ell} \sin^2\left(\frac{n\pi x}{\ell}\right) dx = 4\ell \sum_{n=1}^{\infty} \psi_n^2 \left(\frac{n\pi}{\ell}\right)^2 \leq \frac{4\pi^2}{\ell} K_2 M^3,$$

In Lemma A.2 we shall choose  $M \leq 4\left(\frac{\ell}{\pi}\right)^{7/5} + 1$ , so that (3.3) and (3.4) follow. It remains to prove the inequality (3.2). Since  $w$  is antisymmetric in the variable  $x$ , we can write it as

$$w(x) = 2 \sum_{n=1}^{\infty} w_n \sin \frac{n\pi}{\ell} x. \tag{A.9}$$

Using (A.6) and (A.9), we can simplify the expression for  $(w, w)_{\omega\phi}$ . We observe that

$$\begin{aligned} \frac{1}{2\ell} \int_{-\ell}^{\ell} w^2 \phi_x &= -\frac{8}{2\ell} \sum_{j>0, k>0, m>0} w_k w_m \psi_j \int_{-\ell}^{\ell} \sin \frac{k\pi}{\ell} x \sin \frac{m\pi}{\ell} x \cos \frac{j\pi}{\ell} x dx \\ &= \frac{1}{\ell} \sum_{j>0, k>0, m>0} w_k w_m \psi_j \int_{-\ell}^{\ell} \left\{ \cos \frac{(k+m-j)\pi}{\ell} x + \cos \frac{(k+m+j)\pi}{\ell} x \right. \\ &\quad \left. - \cos \frac{(k-m-j)\pi}{\ell} x - \cos \frac{(k-m+j)\pi}{\ell} x \right\} dx \\ &= \frac{2\ell}{\ell} \left\{ \sum_{k>0, m>0} w_k w_m \psi_{k+m} - \sum_{k>0, m>0} w_k w_m \psi_{k-m} - \sum_{m>0} w_k w_m \psi_{m-k} \right\} \\ &= 2 \left\{ 2 \sum_{k>0, m>0} w_k w_m \psi_{k+m} + \sum_{m>0} w_m^2 \psi_{2m} - 2 \sum_{k>0, m>0} w_k w_m \psi_{k-m} \right\} \\ &= 4 \sum_{k>0, m>0} w_k w_m (\psi_{k+m} - \psi_{k-m}) + 2 \sum_{m>0} w_m^2 \psi_{2m}. \end{aligned} \tag{A.10}$$

Setting  $E_n = \left(\frac{n\pi}{\ell}\right)^4 - \left(\frac{n\pi}{\ell}\right)^2$ , it follows from (A.10) that

$$\begin{aligned} (w, w)_{\omega\phi} &= \int_{-\ell}^{\ell} w_{xx}^2 - \int_{-\ell}^{\ell} w_x^2 + \omega \int_{-\ell}^{\ell} w^2 \phi_x \\ &= 4\ell \sum_{n>0} E_n w_n^2 + 8\ell\omega \sum_{k>0, m>0} w_k w_m (\psi_{k+m} - \psi_{k-m}) + 4\ell\omega \sum_{m>0} w_m^2 \psi_{2m} \\ &= 4\ell \left[ \sum_{n>0} (E_n + \omega\psi_{2n}) w_n^2 + 2\omega \sum_{k>0, m>0} w_k w_m (\psi_{k+m} - \psi_{k-m}) \right] =: 4\ell B. \end{aligned} \tag{A.11}$$

We introduce the notation

$$\tau_n = \sqrt{\frac{1}{2} \left\{ \left(\frac{n\pi}{\ell}\right)^4 + 1 \right\}} \text{ and } W_n = w_n \tau_n. \tag{A.12}$$

To prove the inequality (3.2), we claim that with the coefficients  $\psi_n$  defined by (A.2), the following inequality holds:

$$B \geq \frac{1}{2}(W, W) = \frac{1}{2} \sum_{n>0} W_n^2 \text{ for } \omega \in [\frac{1}{4}, 1], \tag{A.13}$$

where  $B$  as defined in (A.11).

Accepting this claim for the moment, we conclude from the definition of  $W$  that

$$\begin{aligned} (w, w)_{\omega\phi} &= 4\ell B \geq 2\ell(W, W) \\ &= 2\ell \sum_{n>0} W_n^2 = \ell \sum_{n>0} \left\{ \left( \frac{n\pi}{\ell} \right)^4 + 1 \right\} w_n^2 \\ &= \frac{1}{4} \|w_{xx}\|^2 + \frac{1}{4} \|w\|^2 \end{aligned}$$

which completes the proof of the inequality (3.2).

It remains to prove (A.13). We first observe that if  $\omega \geq \frac{1}{4}$ , and  $M > \frac{\ell}{\pi}$ , then by (A.2),

$$E_n + \omega\psi_{2n} \geq \frac{1}{2} \left\{ \left( \frac{n\pi}{\ell} \right)^4 + 1 \right\} = \tau_n^2.$$

Note that the condition we impose here on  $M$  is consistent with the condition on  $M$  we make in Lemma A.2. From the definition of  $B$  in (A.11) we conclude that

$$B \geq \sum_{n>0} W_n^2 + 2\omega \sum_{k>m>0} W_k \frac{\psi_{k+m} - \psi_{k-m}}{\tau_k \tau_m} W_m =: (W, (Id + 2\omega\Gamma)W) \tag{A.14}$$

in which  $\Gamma = \{\Gamma_{km}\}_{k>m>0}$  is an  $(N, N)$ -matrix with  $N = +\infty$  and  $\Gamma_{km} = \frac{\psi_{k+m} - \psi_{k-m}}{\tau_k \tau_m}$ .

We now give the definition of a Hilbert–Schmidt [HS for short] operator and define a norm on the space of HS operators—the Hilbert–Schmidt norm.

**Definition A.1.** A linear bounded operator  $T$  mapping a separable Hilbert space  $H_1$  into  $H_2$  is said to be Hilbert–Schmidt if, for some complete orthonormal sequence  $\{e_n\}_{n \geq 1}$  in  $H_1$ ,

$$\sum_{n \geq 1} (Te_n, Te_n) < \infty.$$

**Lemma A.1.** Let  $T$  be H.S. Then  $T$  is a compact operator and

$$\|T\|_{HS}^2 = \sum_{n \geq 1} (Te_n, Te_n),$$

which is independent of the particular orthonormal basis chosen, defines a norm on the (linear) space of HS operators—the Hilbert–Schmidt norm.

*Proof.* See [1].

To finish the proof of (3.2), we shall show that we can choose the constant  $M$  in the definition of the sequence  $\{\psi_n\}$  so that  $\Gamma$  becomes sufficiently small. Specifically, we prove

**Lemma A.2.** Let  $M$  be the smallest integer larger than  $4\left(\frac{\ell}{\pi}\right)^{7/5}$ . Then

$$\|\Gamma\|_{HS} = \left\{ \sum_{k>m>0} \left| \frac{\psi_{k+m} - \psi_{k-m}}{\tau_k \tau_m} \right|^2 \right\}^{1/2} < \frac{1}{4}, \tag{A.15}$$

where by  $\|\cdot\|_{HS}$  we have denoted the the Hilbert-Schmidt norm.

Accepting (A.15), we are ready to complete the proof of (A.13). By (A.14), we have

$$B \geq (W, (Id + 2\omega\Gamma)W) \geq (W, W) - 2\omega \|\Gamma\|_{HS}(W, W) \geq \frac{1}{2}(W, W), \tag{A.16}$$

as asserted.

*Proof.* (of Lemma A.2.) By (A.2), we have for any  $k > m > 0$

$$|\psi_{k+m} - \psi_{k-m}| = 0 \text{ if } k + m \leq 2M,$$

and

$$|\psi_{k+m} - \psi_{k-m}| \leq \frac{4m}{M} \text{ for all } k > m.$$

Therefore, the following estimate of the Hilbert-Schmidt norm of  $\Gamma$  holds

$$\begin{aligned} \|\Gamma\|_{HS}^2 &\leq \frac{16}{M^2} \sum_{m=1}^M m^2 \tau_m^{-2} \sum_{k=2M-m+1}^{\infty} \tau_k^{-2} + \frac{16}{M^2} \sum_{m=M+1}^{\infty} m^2 \tau_m^{-2} \sum_{k=m+1}^{\infty} \tau_k^{-2} \\ &\leq \frac{16}{M^2} \sum_{m=1}^M m^2 \tau_m^{-2} \int_{2M-m}^{\infty} \tau_k^{-2} dk + \frac{16}{M^2} \sum_{m=M+1}^{\infty} m^2 \tau_m^{-2} \int_m^{\infty} \tau_k^{-2} dk \\ &\leq \frac{32}{3} q^{-4} M^{-5} \sum_{m=1}^M m^2 \tau_m^{-2} + \frac{32}{3} q^{-4} M^{-2} \sum_{m=M+1}^{\infty} m^{-1} \tau_m^{-2}. \end{aligned}$$

Taking again integrals as upper bounds for the sums, we obtain

$$\begin{aligned} \|\Gamma\|_{HS}^2 &\leq \frac{64}{3} q^{-6} M^{-5} \int_0^{\infty} (1 + q^4 m^4)^{-\frac{1}{2}} dm + \frac{64}{3} q^{-4} M^{-2} \int_M^{\infty} m^{-1} (1 + q^4 m^4)^{-1} dm \\ &\leq \frac{128}{3} q^{-7} M^{-5} + \frac{16}{3} q^{-8} M^{-6}. \end{aligned}$$

We take  $M$  the smallest integer larger than  $4q^{-7/5} = 4(\frac{\pi}{\ell})^{-7/5}$ . Thus,

$$M - 1 \leq 4q^{-7/5} < M.$$

Since we assume that  $\ell > \pi$ , we only have to consider the case  $0 < q < 1$ , and we find that

$$\|\Gamma\|_{HS}^2 \leq \frac{11}{256} < \frac{1}{16}$$

and so, (A.15) follows. □

**Lemma A.3.** *Let  $\phi_x \in L^2(\Omega)$ . Then the bilinear form given by (3.1) defines a scalar product on  $X$ .*

*Proof.* We need to check the following properties: (i)  $(u, v)_{\omega\phi} = (v, u)_{\omega\phi}$  for all  $u, v \in X$ .

(ii)  $(\sigma u + \tau v, w)_{\omega\phi} = \sigma(u, w)_{\omega\phi} + \tau(v, w)_{\omega\phi}$  for all  $u, v, w \in X$  and for all  $\sigma, \tau \in \mathbf{R}$ .

(iii)  $(u, u)_{\omega\phi} \geq 0$  for all  $u \in X$  and  $(u, u) = 0$  if and only if  $u = 0$  in  $X$ . Properties (i) and

(ii) are trivially satisfied.

Lemma 3.1 implies that

$$\int_{-\ell}^{\ell} u_{xx}^2 - \int_{-\ell}^{\ell} u_x^2 + \omega \int_{-\ell}^{\ell} u^2 \phi_x \geq \frac{1}{4} \int_{-\ell}^{\ell} u_{xx}^2 + \frac{1}{4} \int_{-\ell}^{\ell} u^2 \geq 0. \tag{A.17}$$

If  $u = 0$  then  $\int_{-\ell}^{\ell} u_{xx}^2 - \int_{-\ell}^{\ell} u_x^2 + \omega \int_{-\ell}^{\ell} u^2 \phi_x = 0$ . On the other hand, if

$$\int_{-\ell}^{\ell} u_{xx}^2 - \int_{-\ell}^{\ell} u_x^2 + \omega \int_{-\ell}^{\ell} u^2 \phi_x = 0,$$

it follows from (A.17) that

$$\frac{1}{4} \int_{-\ell}^{\ell} u_{xx}^2 + \frac{1}{4} \int_{-\ell}^{\ell} u^2 = 0$$

so that  $u = 0$  in  $X$ , which completes the proof of Lemma A.3. □

**Acknowledgement.** It is a pleasure to thank Messoud Efendiev for several stimulating discussions and for pointing out an omission in the Appendix. The authors are grateful for support from the IHES in Bures-sur-Yvette (LAP), from TMR Grant ERBFM-RXCT980201 and from a grant of NWO, the Netherlands Foundation for Scientific Research (AIR). Support for GS from the Israel Science Foundation (Grant 67-01) and the US-Israel Science Foundation (Grant 98-00374) is also gratefully acknowledged.

#### REFERENCES

- [1] A.V. Balakrishnan, *Applied functional analysis*, Springer-Verlag, New York, 1976.
- [2] D. Brochet, *Large time behaviour and maximal attractor for a perturbed Kuramoto-Sivashinsky equation on  $\mathbf{R}$* , PhD Thesis, Université de Paris Sud, Centre d'Orsay, 1993.
- [3] P. Collet, J.-P. Eckmann, H. Epstein & J. Stubbe, *A global attracting set for the Kuramoto-Sivashinsky equation*, Comm. Math. Phys., **152** (1993) 203-214.
- [4] A. Eden, C. Foias, B. Nicolaenko & R. Temam, *Exponential attractors for dissipative evolution equations*, Masson, Paris and Wiley, New York, 1994.
- [5] D. Hilhorst, L.A. Peletier, A.I. Rotariu & G. Sivashinsky, *Global Attractor and Inertial Sets for a Nonlocal Kuramoto-Sivashinsky Equation*, Report Mathematical Institute MI 06-99, Leiden University 1999.
- [6] S. Ishizuka & C.K. Law, *An experimental study on extinction and stability of stretched premixed flames*. Nineteenth International Symposium on Combustion. The Combustion Institute, Pittsburgh, pp 327-335, 1982.
- [7] B. Karlovitz, D.W. Denniston Jr., D.H. Knappschaefer & F.F. Wells, *Studies in turbulent flames*, Fourth International Symposium on Combustion. The Combustion Institute.
- [8] Y. Kotsarts, I. Brailovsky & G. Sivashinsky, *On hydrodynamic instability of stretched flames*, Combustion Science and Technology **110** (1997) 524-529.
- [9] Y. Kotsarts, I. Brailovsky, S. Gutman & G. Sivashinsky, *On the stability of stretched flames*, Combustion Theory and Modelling **2** (1997) 153-156.
- [10] B. Nicolaenko, B. Scheurer & R. Temam, *Some global dynamical properties of the Kuramoto-Sivashinsky equations: Nonlinear stability and attractors*, Physica 16D (1985) 155-183.
- [11] A. Precupanu, *Analiza matematica, functii reale*, Ed. Didactica si pedagogica, Bucuresti, 1976.
- [12] A.I. Rotariu, *Fourth order model equations in physics and mechanics*, PhD Thesis Leiden University, 2000.
- [13] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Appl. Math. Sciences 68, Springer-Verlag, New York, 1997.
- [14] R. Temam & X. Wang, *Estimates on the lowest dimension of inertial manifolds for the Kuramoto-Sivashinsky equation in the general case*, Diff. & Int. Eqns **7** (1995) 1095-1108.

Received November 2001; revised September 2003.

*E-mail address:* peletier@math.leidenuniv.nl