Curves on $\mathbf{P}^1 \times \mathbf{P}^1$

Peter Bruin
16 November 2005

1. Introduction

One of the exercises in last semester’s Algebraic Geometry course went as follows:

Exercise. Let $k$ be a field and $Z = \mathbf{P}^1_k \times_k \mathbf{P}^1_k$. Show that the Picard group $\text{Pic} Z$ is the free Abelian group generated by the classes of a horizontal and a vertical line.

Here $\text{Pic} Z$ is to be interpreted as the divisor class group $\text{Cl} Z$, to which it is naturally isomorphic for Noetherian integral separated locally factorial schemes [Hartshorne, Corollary 6.16]. We view the first $\mathbf{P}^1$ as the result of glueing $\text{Spec}(k[x])$ and $\text{Spec}(k[1/x])$ via $\text{Spec}(k[x, 1/x])$, and similarly for the second $\mathbf{P}^1$ with $y$ instead of $x$. Then $Z = \mathbf{P}^1_k \times_k \mathbf{P}^1_k$ is the result of glueing the spectra of $k[x, y]$, $k[x, 1/y]$, $k[1/x, y]$ and $k[1/x, 1/y]$ in the obvious way.

To prove the claim (see [Hartshorne, Example II.6.6.1] for a different approach), let $L_x$ and $L_y$ be the vertical and horizontal lines $x = \infty$ and $y = \infty$. More precisely, $L_x$ is determined by the coherent sheaf of ideals $\mathcal{I}_{L_x}$ with

$$\mathcal{I}_{L_x}|_{\text{Spec } A} = \begin{cases} \hat{A} & \text{for } A = k[x, y] \text{ and } A = k[1/x, y] \\ 1/x \cdot \hat{A} & \text{for } A = k[1/x, y] \text{ and } A = k[1/x, 1/y], \end{cases}$$

and similarly for $L_y$. If $Y$ is a curve on $Z$ different from $L_x$ and $L_y$ (curves are assumed to be integral), the intersection of $Y$ with $\text{Spec}(k[x, y])$ is a plane curve defined by an irreducible polynomial $f \in k[x, y]$. Let $a$ be the degree of $f$ as a polynomial in $x$ and $b$ is its degree as a polynomial in $y$; then the divisor of $f$ as a rational function on $Z$ equals

$$(f) = Y - a \cdot L_x - b \cdot L_y,$$

so we see that the divisor class of $Y$ is equal to

$$[Y] = a[L_x] + b[L_y].$$

This shows that $\text{Cl} Z$ is generated by $[L_x]$ and $[L_y]$; because there are no rational functions $f \in k(x, y)$ with the property that $(f) = a \cdot L_x + b \cdot L_y$ as a divisor on $Z$ unless $a = b = 0$, the classes $[L_x]$ and $[L_y]$ are linearly independent. If $Y$ is a divisor on $Z$ and $a$, $b$ are the unique integers with $[Y] = a[L_x] + b[L_y]$, we say that $Y$ is of type $(a, b)$.

The isomorphism $\text{Cl} Z \to \text{Pic} Z$ sends the class of a divisor $Y$ of type $(a, b)$ to the invertible sheaf $\mathcal{O}_Z(Y) \cong \mathcal{O}_Z(a \cdot L_x + b \cdot L_y)$. Note that $\mathcal{O}_Z(a \cdot L_x)$ is isomorphic to the pullback $p_1^*(\mathcal{O}_{\mathbf{P}^1_k}(a \cdot \infty))$, where the invertible sheaf $\mathcal{O}_{\mathbf{P}^1_k}(a \cdot \infty)$ on $\mathbf{P}^1_k$ is defined by

$$\mathcal{O}_{\mathbf{P}^1_k}(a \cdot \infty)_{|\text{Spec } k[x]} = (k[x])^a,$$

$$\mathcal{O}_{\mathbf{P}^1_k}(a \cdot \infty)_{|\text{Spec } k[1/x]} = x^a \cdot (k[1/x])^a.$$

On the other hand, there is the invertible sheaf $\mathcal{O}_{\mathbf{P}^1_k}(a)$ with

$$\mathcal{O}_{\mathbf{P}^1_k}(a)_{|\text{Spec } k[x/y]} = y^a \cdot (k[x/y])^a,$$

$$\mathcal{O}_{\mathbf{P}^1_k}(a)_{|\text{Spec } k[y/x]} = x^a \cdot (k[y/x])^a,$$

which is clearly isomorphic to $\mathcal{O}_{\mathbf{P}^1_k}(a \cdot \infty)$, so

$$\mathcal{O}_Z(a \cdot L_x) \cong p_1^*(\mathcal{O}_{\mathbf{P}^1_k}(a)).$$

Something similar is true for the second projection. Using

$$\mathcal{O}_Z(a \cdot L_x + b \cdot L_y) \cong \mathcal{O}_Z(a \cdot L_x) \otimes \mathcal{O}_Z(b \cdot L_y)$$

1
we conclude that $\mathcal{O}_Z(Y)$ is isomorphic to the invertible sheaf $\mathcal{O}(a, b)$ on $Z$ defined by

$$\mathcal{O}(a, b) = p_1^*(\mathcal{O}_{\mathbb{P}_k^1}(a)) \otimes_{\mathcal{O}_Z} p_2^*(\mathcal{O}_{\mathbb{P}_k^1}(b)).$$

The aim of this talk is to study the cohomology of the sheaves $\mathcal{O}(a, b)$ and to derive some consequences for the kind of curves that exist on $Z$. We will do the following:

1. Prove the Künneth formula: if $X$ and $Y$ are Noetherian separated schemes over a field $k$, there is a natural isomorphism

$$H(X \times_k Y, p_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_k Y}} p_2^*\mathcal{G}) \cong H(X, \mathcal{F}) \otimes_k H(Y, \mathcal{G})$$

for all quasi-coherent sheaves $\mathcal{F}$ on $X$ and $\mathcal{G}$ on $Y$.

2. Deduce a connectedness result for closed subschemes and a genus formula for curves on $Z$.

3. Prove Bertini’s theorem: if $X$ is a non-singular subvariety of $\mathbb{P}_k^n$ with $k$ an algebraically closed field, there exists a hyperplane $H \subset \mathbb{P}_k^n$ not containing $X$ such that $H \cap X$ is a regular scheme.

4. Deduce that if $k$ is algebraically closed field, there exist non-singular curves of type $(a, b)$ on $Z$ for all $a, b > 0$.

2. Tensor products of complexes

Let $A$ be a ring, $(C, d)$ a complex of right $A$-modules and $(C', d')$ a complex of left $A$-modules, i.e. $C$ and $C'$ are graded $A$-modules

$$C = \bigoplus_{n \in \mathbb{Z}} C^n \quad \text{and} \quad C' = \bigoplus_{n \in \mathbb{Z}} C'^n$$

and $d, d'$ are $A$-module endomorphisms such that $dd = 0$ and $d(C^n) \subseteq C^{n+1}$ (similarly for $d'$). Let $C \otimes_A C'$ be the usual tensor product, graded in such a way that

$$(C \otimes_A C')^n = \bigoplus_{p+q=n} C^p \otimes_A C'^q.$$

There is a group endomorphism $D$ of $C \otimes_A C'$ defined by

$$D(x \otimes y) = dx \otimes y + (-1)^p x \otimes d' y \quad \text{for} \quad x \in C^p;$$

it fulfills $D((C \otimes_A C')^n) \subseteq (C \otimes_A C')^{n+1}$ and $DD = 0$, so $((C \otimes_A C'), D)$ is a complex of Abelian groups.

For any complex $(C, d)$ of Abelian groups, we write $Z(C)$ for the subgroup of cocycles, $B(C)$ for the subgroup of coboundaries and $H(C)$ for the cohomology of $C$:

$$Z(C) = \ker d, \quad B(C) = \text{im} d, \quad H(C) = Z(C)/B(C).$$

If $x$ and $y$ are cocycles in $C$ and $C'$, respectively, then $x \otimes y$ is a cocycle in $C \otimes_A C'$, because

$$D(x \otimes y) = dx \otimes y + (-1)^p x \otimes d' y = 0 \quad \text{for} \quad x \in C^p.$$

This means that there is a natural $A$-bilinear map

$$Z(C) \times Z(C') \rightarrow Z(C \otimes_A C')$$

$$(x, y) \mapsto x \otimes y.$$ 

If either $x \in B(C)$ or $y \in B(C')$, then the image of $(x, y)$ under this map is in $B(C \otimes_A C')$, because for example

$$dx \otimes y = D(x \otimes y) \quad \text{for all} \quad x \in C, y \in Z(C').$$

This means that we can divide out by the coboundaries in each of the groups and get a natural $A$-bilinear map

$$H(C) \times H(C') \rightarrow H(C \otimes_A C')$$

and therefore (by the universal property of the tensor product) a natural group homomorphism

$$\gamma_{C, C'} : H(C) \otimes_A H(C') \rightarrow H(C \otimes_A C').$$

In the next section we will need the following result:
Lemma. Let $A$ be a ring, $(C, d)$ a complex of right $A$-modules and $(C', d')$ a complex of left $A$-modules. Assume $d = 0$. Then $H(C) \cong C$ and $\gamma_{C, C'}$ induces a natural group homomorphism

$$C \otimes_A H(C') \rightarrow H(C \otimes_A C')$$

$$x \otimes y \mapsto x \otimes y.$$ 

(1)

If $C$ is flat over $A$, then this map is an isomorphism.

Proof. We only need to prove the last claim. Because $C$ is flat, $\ker(D) = \ker(1 \otimes d') = C \otimes_A \ker(d')$, so the natural map $C \otimes_A Z(C') \rightarrow Z(C \otimes_A C')$ is an isomorphism. Furthermore, the image of $C \otimes_A B(C')$ in $C \otimes_A Z(C')$ corresponds to the subgroup $B(C \otimes_A C')$ under this isomorphism, since both are generated by elements of the form $x \otimes d'y$ with $x \in C$ and $y \in C'$. This implies the map defined above is an isomorphism.

3. The Künneth formula

From now on we restrict our attention to the case where $A$ is a field $k$. Then all complexes have the structure of $k$-vector spaces, and all modules are flat. For a treatment without this restriction, see [Bourbaki]. We will prove the following theorem (note that the previous lemma is a special case of this):

Theorem (Künneth formula). Let $(C, d)$ and $(C', d')$ be complexes over $k$. Then the natural $k$-linear map

$$\gamma_{C, C'}: H(C) \otimes_k H(C') \rightarrow H(C \otimes_k C')$$

is an isomorphism.

Proof. Write $Z = Z(C)$, $B = B(C)$, $H = H(C)$ and $H' = H(C')$. Consider the short exact sequence of complexes defining $Z(C)$ and $B(C)$:

$$0 \rightarrow Z \rightarrow C \xrightarrow{d} B(1) \rightarrow 0.$$ 

Here $B(1)$ denotes the complex $B$ shifted one place to the left, i.e. $B(1)^n = B^{n+1}$. Taking the tensor product with $C'$ gives a short exact sequence of complexes

$$0 \rightarrow Z \otimes_k C' \xrightarrow{j \otimes 1} C \otimes_k C' \xrightarrow{d \otimes 1} (B \otimes_k C')(1) \rightarrow 0.$$ 

We take the cohomology sequence of this short exact sequence. The coboundary map will go from $H(B \otimes_k C')$ to $H(Z \otimes_k C')$. To find out what it does, we write down the following diagram with exact rows:

$$0 \rightarrow (Z \otimes_k C')^{n-1} \xrightarrow{j \otimes 1} (C \otimes_k C')^{n-1} \xrightarrow{d \otimes 1} (B \otimes_k C')^n \rightarrow 0$$

$$0 \rightarrow (Z \otimes_k C')^n \xrightarrow{j \otimes 1} (C \otimes_k C')^n \xrightarrow{d \otimes 1} (B \otimes_k C')^{n+1} \rightarrow 0.$$ 

Because $d = 0$ on $B$ and because $B$ is flat over $k$, the kernel of $D: (B \otimes_k C')^n \rightarrow (B \otimes_k C')^{n+1}$ equals

$$\ker(D) = \ker(1 \otimes d') \cong B \otimes_k \ker(d'),$$

so $\ker D$ is generated by elements of the form $dx \otimes y$ with $x \otimes y \in (C \otimes_k C')^n$ such that $y \in Z(C')$. The image of $x \otimes y \in (C \otimes_k C')^n$ in $(Z \otimes_k C')^n$ is now $D(x \otimes y) = dx \otimes y$, which is in $(Z \otimes_k C')^n$. We see therefore that the coboundary map sends the class of $dx \otimes y$ to that of $(i \otimes 1)(dx \otimes y)$, where $i: B \rightarrow Z$ is the inclusion. In other words, the coboundary map equals $H(i \otimes 1)$. The long exact sequence is now

$$H^n(B \otimes_k C') \xrightarrow{H(i \otimes 1)} H^n(Z \otimes_k C') \xrightarrow{H(j \otimes 1)} H^n(C \otimes_k C') \xrightarrow{H(d \otimes 1)} H^{n+1}(B \otimes_k C') \xrightarrow{H(i \otimes 1)} H^{n+1}(Z \otimes_k C').$$
We can also take the tensor product with $H'$ of the short exact sequence defining $H$ to obtain an exact sequence

$$0 \xrightarrow{} B \otimes_k H' \xrightarrow{i \otimes 1} Z \otimes_k H' \xrightarrow{p \otimes 1} H \otimes_k H' \xrightarrow{} 0.$$  

We connect this sequence with the long exact sequence above via the natural maps

$$
\begin{aligned}
\gamma_{B,C'} & : B \otimes_k H' \to H(C \otimes_k C') \\
\gamma_{Z,C'} & : Z \otimes_k H' \to H(C \otimes_k C') \\
\gamma_{C,C'} & : H \otimes_k H' \to H(C \otimes_k C'),
\end{aligned}
$$

the first two of which are the isomorphisms occurring in the lemma from Section 2. This gives a commutative diagram with exact rows

$$
\begin{array}{c}
(B \otimes_k H')^n \xrightarrow{i \otimes 1} (Z \otimes_k H')^n \xrightarrow{p \otimes 1} (H \otimes_k H')^n \xrightarrow{} 0 \\
\downarrow \gamma_{B,C'} \downarrow \gamma_{Z,C'} \downarrow \gamma_{C,C'} \\
H^n(B \otimes_k C') \xrightarrow{H(i \otimes 1)} H^n(Z \otimes_k C') \xrightarrow{H(p \otimes 1)} H^n(H \otimes_k C') \xrightarrow{H(d \otimes 1)} H^{n+1}(Z \otimes_k C') \\
\downarrow \gamma_{B,C'} \downarrow \gamma_{Z,C'} \downarrow \gamma_{C,C'} \\
0 \xrightarrow{} (B \otimes_k H')^{n+1} \xrightarrow{i \otimes 1} (Z \otimes_k H')^{n+1}
\end{array}
$$

The lower right part shows that $H(i \otimes 1)$ is injective, so $H(d \otimes 1) = 0$ by exactness. From the rest of the diagram we now see that $\gamma_{C,C'}$ is an isomorphism.

4. The cohomology of sheaves of the form $\mathcal{F} \otimes_k \mathcal{G}$

Let $X$ and $Y$ be two compact separated schemes over a field $k$. Consider the scheme $Z = X \times_k Y$ together with its projection morphisms $p_1: Z \to X$ and $p_2: Z \to Y$. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent sheaves on $X$ and $Y$, respectively. Recall that the pullbacks $p_1^* \mathcal{F}$ and $p_2^* \mathcal{G}$ of $\mathcal{F}$ and $\mathcal{G}$ to $Z$ are defined by

$$
\begin{aligned}
p_1^* \mathcal{F} & = O_Z \otimes_{p_1^{-1}O_X} p_1^{-1} \mathcal{F} \\
p_2^* \mathcal{G} & = O_Z \otimes_{p_2^{-1}O_Y} p_2^{-1} \mathcal{G}.
\end{aligned}
$$

It is a general fact that the pullback of a quasi-coherent sheaf is quasi-coherent. We use this for $p_1^* \mathcal{F}$ and $p_2^* \mathcal{G}$. Suppose $U = \text{Spec} A$ and $V = \text{Spec} B$ are affine opens of $X$ and $Y$, respectively, $M$ is an $A$-module such that $\mathcal{F}|_U \cong M^\sim$ and $N$ is a $B$-module such that $\mathcal{F}|_V \cong N^\sim$. Then the restrictions of $p_1^* \mathcal{F}$ and $p_2^* \mathcal{G}$ to the affine open subscheme $W = U \times_k V = \text{Spec}(A \otimes_k B)$ of $Z$ are

$$
\begin{aligned}
p_1^* \mathcal{F}|_W & \cong (A \otimes_B \mathcal{F}(U))^\sim \\
p_2^* \mathcal{G}|_W & \cong (A \otimes_B \mathcal{G}(V))^\sim.
\end{aligned}
$$

From this we get the following expression for the sheaf $p_1^* \mathcal{F} \otimes_{O_Z} p_2^* \mathcal{G}$:

$$
\begin{aligned}
p_1^* \mathcal{F} \otimes_{O_Z} p_2^* \mathcal{G}|_W & \cong ((B \otimes_k M) \otimes_B (A \otimes_k N))^\sim \\
& \cong (M \otimes_k N)^\sim.
\end{aligned}
$$

In particular, we see that

$$
p_1^* \mathcal{F} \otimes_{O_Z} p_2^* \mathcal{G}(U \times_k V) \cong \mathcal{F}(U) \otimes_k \mathcal{G}(V)
$$

for all open affine subschemes $U$ of $X$ and $V$ of $Y$. It seems therefore useful to introduce the abbreviated notation

$$\begin{align*}
\mathcal{F} \otimes_k \mathcal{G} = p_1^* \mathcal{F} \otimes_{O_Z} p_2^* \mathcal{G}
\end{align*}$$

for quasi-coherent sheaves $\mathcal{F}$ on $X$ and $\mathcal{G}$ on $Y$. (To prevent confusion, this notation should only be used if the sheaves are quasi-coherent.)

We are now going to compare the cohomology of the sheaf $\mathcal{F} \otimes_k \mathcal{G}$ on $Z$ to the cohomology of $\mathcal{F}$ on $X$ and $\mathcal{G}$ on $Y$. This we will do using a variant of Čech cohomology with respect to finite affine coverings of $X$, $Y$ and $Z$. 

4
Definition. The unordered Čech complex of a sheaf \( \mathcal{F} \) of Abelian groups on a topological space \( X \) with respect to an open covering \( \mathcal{U} = \{ U_i \}_{i \in I} \) is the complex defined by

\[
C^n(\mathcal{U}, \mathcal{F}) = \prod_{i_0, \ldots, i_n \in I} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_n})
\]

where, as usual,

\[
U_{i_0, \ldots, i_n} = U_{i_0} \cap \cdots \cap U_{i_n}.
\]

The maps \( d : C^n \to C^{n+1} \) are defined using the same formula as for the usual (alternating) Čech complex:

\[
d\left( \{ s_{i_0, \ldots, i_n} \}_{i_0, \ldots, i_n \in I} \right) = \left\{ \sum_{j=0}^{n+1} (-1)^j s_{i_0, \ldots, \hat{i}_j, \ldots, i_n} | U_{i_0, \ldots, \hat{i}_j, \ldots, i_n} \cap U_{i_0, \ldots, i_n+1} \right\}
\]

Notice that, in contrast to the alternating Čech cohomology, all the \( C^n(\mathcal{U}, \mathcal{F}) \) are non-zero (unless \( X = \emptyset \)), but that the product occurring in the definition of \( C^n(\mathcal{U}, \mathcal{F}) \) is finite if \( I \) is finite.

Let \( \mathcal{U} = \{ U_i \}_{i \in I} \) and \( \mathcal{V} = \{ V_j \}_{j \in J} \) be finite coverings by open affine subschemes of \( X \) and \( Y \), respectively. Because \( X \) and \( Y \) are separated over \( k \), the intersection of any positive number of such affines is again affine [Hartshorne, Exercise II.4.3]. We look at the unordered Čech complex of the sheaf \( \mathcal{F} \otimes_k \mathcal{G} \) on \( Z \) with respect to the affine open covering \( \mathcal{U} \times \mathcal{V} \). By the property (1) of \( \mathcal{F} \otimes_k \mathcal{G} \) and because \( I \) and \( J \) are finite,

\[
C^n(\mathcal{U} \times_k \mathcal{V}, \mathcal{F} \otimes_k \mathcal{G}) = \prod_{(i_0, j_0), \ldots, (i_n, j_n) \in I \times J} \mathcal{F}(U_{i_0} \times_k V_{j_0} \cap \cdots \cap U_{i_n} \times_k V_{j_n})
\]

\[
\cong \bigoplus_{i_0, \ldots, i_n \in I, j_0, \ldots, j_n \in J} \mathcal{F}(U_{i_0, \ldots, i_n}) \otimes_k \mathcal{G}(V_{j_0, \ldots, j_n}).
\]

Since the tensor product is distributive over direct sums, we see that

\[
C^n(\mathcal{U} \times_k \mathcal{V}, \mathcal{F} \otimes_k \mathcal{G}) \cong \left( \bigoplus_{i_0, \ldots, i_n \in I} \mathcal{F}(U_{i_0, \ldots, i_n}) \right) \otimes_k \left( \bigoplus_{j_0, \ldots, j_n \in J} \mathcal{G}(V_{j_0, \ldots, j_n}) \right)
\]

\[
\cong C^n(\mathcal{U}, \mathcal{F}) \otimes_k C^n(\mathcal{V}, \mathcal{G}).
\]

We take the direct sum over all \( n \) and conclude that

\[
C(\mathcal{U} \times_k \mathcal{V}, \mathcal{F} \otimes_k \mathcal{G}) \cong \bigoplus_{n=0}^{\infty} C^n(\mathcal{U}, \mathcal{F}) \otimes_k C^n(\mathcal{V}, \mathcal{G}). \tag{2}
\]

Fact. There exists a natural homotopy equivalence of complexes

\[
\bigoplus_{n=0}^{\infty} C^n(\mathcal{U}, \mathcal{F}) \otimes_k C^n(\mathcal{V}, \mathcal{G}) \sim C(\mathcal{U}, \mathcal{F}) \otimes_k C(\mathcal{V}, \mathcal{G}).
\]

After applying this fact, which follows from the Eilenberg–Zilber theorem [Godement, Théorème 3.9.1], to the right-hand side of (2) and taking cohomology, we obtain a natural isomorphism

\[
H(C(\mathcal{U} \times_k \mathcal{V}, \mathcal{F} \otimes_k \mathcal{G})) \cong H(C(\mathcal{U}, \mathcal{F}) \otimes_k C(\mathcal{V}, \mathcal{G})).
\]

Now the Künneth formula implies that

\[
\bar{H}(\mathcal{U} \times_k \mathcal{V}, \mathcal{F} \otimes_k \mathcal{G}) \cong \bar{H}(\mathcal{U}, \mathcal{F}) \otimes_k \bar{H}(\mathcal{V}, \mathcal{G}).
\]

If \( X \) and \( Y \) are Noetherian, then from the fact that the Čech cohomology is isomorphic to the derived functor cohomology for open affine coverings (the proof of [Hartshorne, Theorem III.4.5] also works for the unordered Čech cohomology) we get the following theorem:

Theorem. Let \( X \) and \( Y \) be Noetherian separated schemes over a field \( k \). For all quasi-coherent sheaves \( \mathcal{F} \) on \( X \) and \( \mathcal{G} \) on \( Y \), there is a natural isomorphism of \( k \)-vector spaces

\[
H(\mathcal{X}, \mathcal{F}) \otimes_k H(\mathcal{Y}, \mathcal{G}) \cong H(\mathcal{X} \times_k \mathcal{Y}, \mathcal{F} \otimes_k \mathcal{G}).
\]
5. Application to the sheaves $\mathcal{O}(a, b)$ and curves on $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$

We have seen in Dirard’s talk (see also [Hartshorne, Section III.5]) that for any ring $A$ the cohomology of the sheaves $\mathcal{O}(n)$ on $X = \mathbb{P}^n_A$ is given by

\[
\begin{align*}
H^0(X, \mathcal{O}(n)) &\cong S_n \\
H^i(X, \mathcal{O}(n)) &\cong 0 \quad \text{for } 0 < i < r \\
H^r(X, \mathcal{O}(n)) &\cong \text{Hom}_A(S^r, A)
\end{align*}
\]

for all $n \in \mathbb{Z}$, where $S_n$ is the component of degree $n$ in $S = A[x_0, \ldots, x_r]$. In particular, for $A$ equal to the field $k$ and for $r = 1$,\[H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(n)) \cong k[x_0, x_1]^n \]
\[H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(n)) \cong k[x_0, x_1]_{n-2}\]

The dimensions are therefore equal to

\[
\begin{align*}
\dim_k H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(n)) &= \max\{n + 1, 0\} \\
\dim_k H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(n)) &= \max\{-n - 1, 0\}.
\end{align*}
\]

It is now a matter of simple calculations and applying the Künneth formula to find the following table for the cohomology of the sheaves $\mathcal{O}(a, b)$ on $Z = \mathbb{P}^1_k \times_k \mathbb{P}^1_k$:

<table>
<thead>
<tr>
<th>$a \geq -1, \ b \geq -1$</th>
<th>$a \geq -1, \ b \leq -1$</th>
<th>$a \leq -1, \ b \geq -1$</th>
<th>$a \leq -1, \ b \leq -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(a + 1)(b + 1)$</td>
<td>$0$</td>
<td>$(a + 1)(-b - 1)$</td>
<td>$0$</td>
</tr>
<tr>
<td>$0$</td>
<td>$(a + 1)(-b - 1)$</td>
<td>$0$</td>
<td>$(a + 1)(b + 1)$</td>
</tr>
</tbody>
</table>

We can now look at a few applications of this. Let $Y$ be a locally principal closed subscheme of $Z$, and let $i: Y \to Z$ be the inclusion map, which is a closed immersion. Viewing $Y$ as a divisor on $Z$, we have an exact sequence of coherent sheaves:

\[0 \to \mathcal{O}_Z(-Y) \to \mathcal{O}_Z \to i_* \mathcal{O}_Y \to 0.\]

The corresponding long exact cohomology sequence is

\[0 \to H^0(Z, \mathcal{O}_Z(-Y)) \to H^0(Z, \mathcal{O}_Z) \to H^0(Z, i_* \mathcal{O}_Y) \]
\[\to H^1(Z, \mathcal{O}_Z(-Y)) \to H^1(Z, \mathcal{O}_Z) \to H^1(Z, i_* \mathcal{O}_Y) \]
\[\to H^2(Z, \mathcal{O}_Z(-Y)) \to H^2(Z, \mathcal{O}_Z) \to H^2(Z, i_* \mathcal{O}_Y) \to 0.\]

Because $i$ is a closed immersion, we know that

\[H(Z, i_* \mathcal{O}_Y) \cong H(Y, \mathcal{O}_Y).\]

Furthermore, the case $a = b = 0$ gives us that $H^0(Z, \mathcal{O}_Z) \cong k$, $H^1(Z, \mathcal{O}_Z) = 0$ and $H^2(Z, \mathcal{O}_Z) = 0$, so the long exact sequence breaks down into two exact sequences

\[0 \to H^0(Z, \mathcal{O}_Z(-Y)) \to k \to H^0(Y, \mathcal{O}_Y) \to H^1(Z, \mathcal{O}_Z(-Y)) \to 0\]

and

\[0 \to H^1(Y, \mathcal{O}_Y) \to H^2(Z, \mathcal{O}_Z(-Y)) \to 0.\]
If \( Y \) is of type \((a, b)\) with \(a, b > 0\), then \( O_Z(-Y) \cong O(-a, -b)\); for these sheaves we have by the bottom row of the table above

\[
H^0(Z, O_Z(-Y)) = 0, \quad H^1(Z, O_Z(-Y)) = 0, \quad \dim_k H^2(Z, O_Z(-Y)) = (-a + 1)(-b + 1) = (a - 1)(b - 1).
\]

Therefore,

\[
H^0(Y, O_Y) \cong k \quad \text{and} \quad \dim_k H^1(Y, O_Y) = (a - 1)(b - 1) \quad \text{if} \ a, b > 0.
\]

The interpretation of this is that \( Y \) is connected, and if \( Y \) is a non-singular curve it has genus \((a - 1)(b - 1)\).

6. Bertini’s theorem

In this section we study intersections of projective varieties with hyperplanes. A hyperplane \( H \subset P^n \) is by definition the zero set of a single homogeneous polynomial \( f \in k[x_0, \ldots, x_n] \) of degree 1. Let \( V \) be the subspace of homogeneous elements of degree 1 in \( k[x_0, \ldots, x_n] \). Form the projective space

\[
\mathfrak{H} = (V \setminus \{0\})/k^\times
\]

and view it as a projective variety over \( k \); it is isomorphic to \( P^n_k \). Because two non-zero sections of \( O_{P^n} \) determine the same hyperplane if and only if one is a multiple of the other by an element of \( k^\times \), there is a canonical bijection between \( \mathfrak{H} \) and the set of hyperplanes in \( P^n_k \).

**Theorem (Bertini).** Let \( X \) be a non-singular closed subvariety of \( P^n_k \), where \( k \) is an algebraically closed field. Then there exists a hyperplane \( H \subset P^n_k \), not containing \( X \), such that the scheme \( H \cap X \) is regular. Moreover, the set of all hyperplanes with this property is an open dense subset of \( \mathfrak{H} \).

**Proof.** Consider a closed point \( x \) of \( X \). There is an \( i \in \{0, 2, \ldots, n\} \) such that \( x \) is not in the hyperplane defined by \( x_i \); after renaming the coordinates we may assume \( i = 0 \). Then \( f/x_0 \) is a regular function in a neighbourhood of \( x \) for all \( f \in V \), so there is a \( k \)-linear map

\[
\phi_x : V \to O_{X,x}
\]

\[
f \mapsto f/x_0,
\]

where \( O_{X,x} \) is the local ring of \( X \) at \( x \). If \( X \) is contained in the hyperplane \( H \) defined by \( f \), then \( \phi_x(f) = 0 \); conversely, \( \phi_x(f) = 0 \) means that \( f \) vanishes on some open neighbourhood of \( x \) in \( X \), hence on all of \( X \) since \( X \) is irreducible. We conclude that \( \phi_x(f) = 0 \iff X \subseteq H \). Furthermore, \( \phi_x(f) \in m_x \iff x \in H \).

Assume \( X \not\subseteq H \) but \( x \in X \cap H \), so that \( \phi_x(f) \in m_x \setminus \{0\} \). Then \( f = \phi_x(f)O_{X,x} \) is a non-zero ideal of \( O_{X,x} \) contained in \( m_x \). Now the local ring of \( H \cap X \) at \( x \) is \( O_{X,x}/f \), and its maximal ideal is \( n = m_x/f \). The fact that \( O_{X,x} \) is an integral domain and \( f \) is a non-zero principal ideal implies that

\[
\dim (O_{X,x}/f) = \dim (O_{X,x}) - 1.
\]

Furthermore, \( n^2 = (m_x^2 + f)/f \) and \( n/n^2 \cong m_x/(m_x^2 + f) \). In particular,

\[
\dim_k n/n^2 \leq \dim_k m_x/m_x^2
\]

with equality if and only if \( f \subseteq m^2 \). Recall that \( \dim_k m_x/m_x^2 \geq \dim O_{X,x} \) with equality if and only if \( O_{X,x} \) is a regular local ring. Applying this also to \( O_{X,x}/f \) we see that \( O_{X,x}/f \) is regular if \( f \not\subseteq m^2 \) (in which case \( \dim_k n/n^2 = \dim O_{X,x}/f \)), and not regular if \( f \subseteq m \). Hence \( O_{X,x}/f \) is a regular local ring if and only if \( \phi_x(f) \in m_x \setminus m_x^2 \).

Let \( B_x \subset \mathfrak{H} \) be the set of hyperplanes that are defined by an element \( f \in V \) for which \( \phi_x(f) \in m_x^2 \). In other words, if we put

\[
\tilde{\phi}_x : V \to O_{X,x}/m_x^2
\]

\[
f \mapsto f/x_0 \mod m_x^2
\]
then

\[ B_x = (\ker \bar{\phi}_x \setminus \{0\})/k^x \subseteq \mathcal{H}. \]

This is a subvariety of \( \mathcal{H} \), the interpretation of which is as follows: a hyperplane \( H \) is in \( B_x \) if and only if either \( H \supseteq X \) or \( x \in H \cap X \) and \( x \) is a singular point of \( H \cap X \). Let us take a closer look at \( B_x \). We put \( y_i = x_i/x_0 \) for \( 1 \leq i \leq n \), so that \( \text{Spec } k[y_1, \ldots, y_n] \) is an affine open neighbourhood of \( x \). Let \( g_1, \ldots, g_m \in k[y_1, \ldots, y_n] \) be local equations for \( X \), and let \( (a_1, \ldots, a_n) \) be the coordinates of the point \( x \). Then \( \mathcal{O}_{X,x} \) is isomorphic to \( \mathcal{A}_p \), where

\[
A = (k[y_1, \ldots, y_n]/(g_1, \ldots, g_m)), \quad p = (y_1 - a_1, \ldots, y_n - a_n),
\]

and \( m_x \) corresponds to \( pA_p \) under this isomorphism. Furthermore, the \( k \)-vector space \( \mathcal{O}_{X,x}/m_x^2 \) has dimension

\[
\dim_k (\mathcal{O}_{X,x}/m_x^2) = \dim_k (\mathcal{O}_{X,x}/m_x) + \dim_k (m_x/m_x^2) = 1 + \dim X
\]

and is spanned over \( k \) by the elements \( 1, y_1 - a_1, \ldots, y_n - a_n \) (easy check). This shows that \( \bar{\phi}_x \) is surjective, and

\[
\dim \ker \bar{\phi}_x = \dim_k V - \dim_k (\mathcal{O}_{X,x}/m_x^2)
\]

\[
= (n + 1) - (1 + \dim X)
\]

\[
= n - \dim X,
\]

from which we conclude that \( \dim B_x = n - \dim X - 1 \).

The polynomials \( g_1, \ldots, g_m \) which locally define \( X \) are modulo \( m_x^2 \) congruent to the polynomials

\[
g_i = \sum_{j=1}^{n} (y_j - a_j) \frac{\partial g_i}{\partial y_j} (a_1, \ldots, a_n) \quad (1 \leq i \leq m).
\]

Because \( \phi_x(f) \) is of the form \( b_0 + \sum_{j=1}^{n} b_j y_j \), we see that

\[
\phi_x(f) \in m_x^2 \iff f/x_0 \in \sum_{i=1}^{m} k \bar{g}_i,
\]

or, equivalently,

\[
\ker \bar{\phi}_x = \sum_{i=1}^{m} k x_0 \bar{g}_i, \quad B_x = \left( \sum_{i=1}^{m} k x_0 \bar{g}_i \setminus \{0\} \right)/k^x.
\]

Consider the fibre product \( X \times_k \mathcal{H} \). Because of the above characterisation of \( \ker \bar{\phi}_x \), there is a closed subscheme \( B \) of \( X \times_k \mathcal{H} \) such that the closed points of \( B \) are precisely the points of \( X \times_k \mathcal{H} \) corresponding to the pairs \((x, H)\) with \( x \) a closed point of \( X \) and \( H \in B_x \).

We have seen that the fibre of \( B \) above each point of \( X \) has dimension \( n - \dim X - 1 \), so \( B \) itself has dimension \((n - \dim X - 1) + \dim X = n - 1 \). Because \( X \) is proper over \( k \) and proper morphisms are preserved under base extension, the projection \( p_2 : X \times_k \mathcal{H} \to \mathcal{H} \) is proper too. This implies that \( p_2(B) \) is a closed subset of \( \mathcal{H} \) of dimension at most \( n - 1 \), and from this we conclude that \( \mathcal{H} \setminus p_2(B) \) is an open dense subset of \( \mathcal{H} \). For each \( H \in \mathcal{H} \setminus p_2(B) \), the scheme \( H \cap X \) is regular at every point by the construction of \( B \).
7. Application to the existence of non-singular curves of type \((a, b)\)

Let \(k\) be an algebraically closed field, and let \(a, b\) be positive integers. We want to show that there are non-singular curves of type \((a, b)\) on \(\mathbb{P}^1_k \times_k \mathbb{P}^1_k\). First we embed \(\mathbb{P}^1_k \times_k \mathbb{P}^1_k\) into \(\mathbb{P}^n_k\), where \(n = ab + a + b\), using the \(a\)-uple, \(b\)-uple and Segre embeddings:

\[
P^1_k \times_k P^1_k \longrightarrow P^a_k \times_k P^b_k \longrightarrow P^n_k.
\]

Recall that the \(a\)-uple embedding is defined by

\[
(x_0 : x_1) \mapsto (x_0^a : x_0^{a-1}x_1 : \ldots : x_1^a)
\]

and similarly for the \(b\)-uple embedding; the Segre embedding is defined by

\[
((s_0 : \ldots : s_a), (t_0 : \ldots : t_b)) \mapsto (\ldots : s_it_j : \ldots)
\]

in lexicographic order. Let \(j\) denote the composed embedding \(\mathbb{P}^1_k \times_k \mathbb{P}^1_k \rightarrow \mathbb{P}^n_k\). The image of \(j\) is a non-singular surface \(X\) in \(\mathbb{P}^n_k\) that is isomorphic to \(\mathbb{P}^1_k \times_k \mathbb{P}^1_k\). We apply Bertini’s theorem to find a hyperplane \(H\) in \(\mathbb{P}^n_k\) such that \(H \cap X\) is a one-dimensional regular closed subscheme of \(X\). This hyperplane is given by a homogeneous linear polynomial in the coordinates \(\{z_{i,j} : 0 \leq i \leq a, 0 \leq j \leq b\}\) of \(\mathbb{P}^n_k\). Now

\[
z_{i,j} = j(x_0^{a-i}x_1^j y_0^{b-j} y_1^j),
\]

so \(Y = j^{-1}(H \cap X)\), viewed as a divisor on \(\mathbb{P}^1_k \times_k \mathbb{P}^1_k\), is of type \((a, b)\). We have seen earlier that this implies that \(Y\) is connected. The local rings of \(Y\) are regular local rings, so in particular they are integral domains [Hartshorne, Remark II.6.11.1A]. This means that there cannot be two irreducible components of \(Y\) intersecting each other; therefore \(Y\) is irreducible, and hence a non-singular curve.

References