Three approaches to extend the Heston model

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1 Introduction

The stock price in the Heston model [8] is given by the following stochastic differential equation

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sqrt{v_t}S_t\mathrm{d}W_{1,t}, \quad S_0 > 0,$$

where $r > 0$ denotes the risk-free interest rate, which is assumed to be constant in time. Since S_t follows a geometric Brownian motion, it is advantageous to consider $X_t = \ln S_t$ instead. By the Itô–Doeblin formula one then has

$$\mathrm{d}X_t = \mathrm{d}\ln S_t = (r - \frac{1}{2}v_t)\mathrm{d}t + \sqrt{v_t}\mathrm{d}W_{1,t}.$$

The volatility of the instantaneous stock returns $\mathrm{d}S_t/S_t$ follows the process

$$\mathrm{d}v_t = \kappa(\eta - v_t)\mathrm{d}t + \lambda\sqrt{v_t}\mathrm{d}W_{2,t}, \quad v_0 > 0,$$

in which $\kappa > 0$ determines the speed of adjustment of the volatility towards its theoretical mean $\eta > 0$, and $\lambda > 0$ is the second-order volatility, i.e., the variance of the volatility. Note that this has exactly the form as the Cox-Ingersoll-Ross (CIR) [6] interest rate process.

The money-market account evolves according to the ordinary differential equation $\mathrm{d}M_t = rM_t\mathrm{d}t$ with solution $M_t = M_0e^{rt}$. The importance of the Heston model comes from the fact that it allows for a semi-analytical solution in terms of characteristic functions (see Section 3).

2 Extension of the Heston model

Although the Heston model incorporates stochastic volatility, the fixed interest rate is an unrealistic assumption. Let us therefore consider (following [14]) a generalized Hull–White process [9] for the interest rate,

$$\mathrm{d}r_t = (\theta_t - ar_t)\mathrm{d}t + \sigma\mathrm{d}W_{3,t},$$

where $\theta_t > 0$, $t \in \mathbb{R}$, is a nonconstant drift term. Usually, stock rate, volatility, and interest rate are correlated; a phenomenon known as the leverage effect [2, 3]. Assume that

$$\mathrm{d}W_{i,t}\mathrm{d}W_{j,t} = \rho_{ij}\mathrm{d}t,$$

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where

$$C = (\rho_{ij})_{1\le i,j\le 3} = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{21} & 1 & \rho_{23} \\ \rho_{31} & \rho_{32} & 1 \end{pmatrix}$$

is a constant[1] covariance matrix, and therefore positive semi-definite. In fact, for the application in finance, we can assume that C is nonsingular[2].

From the spectral theorem of linear algebra we see that C, being positive definite and symmetric, has a unique matrix square root $A = (a_{ij})_{1\le i,j\le 3}$, such that

$$C = U\Sigma U^t = (U\Sigma^{1/2})(U\Sigma^{1/2})^t = AA^t, \tag{2.1}$$

where $U\Sigma U^t$ is the singular-value decomposition of C. Explicitly, we have

$$\sum_{k=1}^{3} a_{ik}a_{jk} = \rho_{ij}, \quad \text{for all } i, j = 1, 2, 3.$$

There now exist adapted, independent Brownian motions $B_{i,t}$, $i = 1, 2, 3$, such that

$$dW_{i,t} = \sum_{j=1}^{3} a_{ij}\, \mathrm{d}B_{j,t},$$

and the general model we consider here is the following:

$$\mathrm{d}S_t = r_t S_t \mathrm{d}t + \sqrt{v_t} S_t\, a_{1i} \mathrm{d}B_{i,t} \quad \text{or} \quad \mathrm{d}X_t = (r_t - \frac{1}{2} v_t)\mathrm{d}t + \sqrt{v_t} a_{1i} \mathrm{d}B_{i,t} \tag{2.2}$$

$$\mathrm{d}v_t = \kappa(\eta - v_t)\mathrm{d}t + \lambda \sqrt{v_t}\, a_{2j} \mathrm{d}B_{j,t} \tag{2.3}$$

$$\mathrm{d}r_t = (\theta_t - a r_t)\mathrm{d}t + \sigma\, a_{3k} \mathrm{d}B_{k,t}, \tag{2.4}$$

where the Einstein convention for summation of repeated indices is used. The money market account develops according to

$$M_t = M_0 \exp\left(\int_0^t r_s \mathrm{d}s\right).$$

In this generality, the model is probably not solvable (semi-) analytically. Therefore three different constraints, arising from different strategies are discussed that lead to partial solutions.

3 Independent interest process

The first simplification is to assume that the interest rate process r_t evolves independently from the stock price and volatility processes S_t and v_t, keeping the correlation between the latter two,

$$\mathrm{d}W_{1,t}\mathrm{d}W_{2,t} = \rho \mathrm{d}t$$
$$\mathrm{d}W_{1,t}\mathrm{d}W_{3,t} = \mathrm{d}W_{2,t}\mathrm{d}W_{3,t} = 0.$$

[1] The decomposition of correlated Brownian motions into independent ones we are about to describe is also possible if $C = C(t)$ is an adapted process in time.

[2] This is possible since we will never have a perfectly linear relation between the driving Brownian motions of stock price, volatility, and interest rate — this would be rather contradictory to the assumption of stochasticity, and in such a case we could do with a simpler model than the one considered.

The first relation can be rewritten[3] as

$$\mathrm{d}W_{1,t} = \rho \mathrm{d}W_{2,t} + \sqrt{1-\rho^2}\mathrm{d}W'_{2,t},$$

where $W'_{2,t}$ is another Brownian motion, independent of $W_{2,t}$.

Define the integrated interest $R_t = \int_0^t r_t \mathrm{d}t$. We want to find the European call option price at maturity time T, given an initial stock price S_0, volatility v_0 and interest rate r_0 (and initial time $t = 0$),

$$\begin{aligned} C_T(S_0, v_0, r_0) &= \mathbb{E}[e^{-R_T}(S_T - K)^+ \mid S_0, v_0, r_0] \\ &= \mathbb{E}[e^{-R_T} S_T \cdot 1_{(\ln S_T > \ln K)}] - K\mathbb{E}[e^{-R_T} \cdot 1_{(\ln S_T > \ln K)}] \\ &= \mathbb{E}[e^{-R_T} S_T] \frac{\mathbb{E}[e^{-R_T} S_T \cdot 1_{(\ln S_T > \ln K)}]}{\mathbb{E}[e^{-R_T} S_T]} - K\mathbb{E}[e^{-R_T}] \frac{\mathbb{E}[e^{-R_T} \cdot 1_{(\ln S_T > \ln K)}]}{\mathbb{E}[e^{-R_T}]}, \end{aligned}$$

where $x^+ = \max(0, x)$ denotes the positive part of x, and 1_A is the indicator function of the event A. Note that under the risk-neutral measure the process $(e^{-R_t} S_t)_{t \geq 0}$ is a martingale, such that $\mathbb{E}[e^{-R_t} S_t] = S_0$.

Define an (analytic) function

$$\Psi(z) = \mathbb{E}[e^{-R_T + z \ln S_T}], \quad z \in \mathbb{C},$$

such that

$$\Psi(0) = \mathbb{E}[e^{-R_T}] = P(r_0, T)$$

is the discount price function, i.e., the price of a zero-coupon bond at time T.

Consider now the two (scaled) characteristic functions

$$\begin{aligned} \Phi_1(z) &= \frac{\Psi(1 + iz)}{\Psi(1)} = \frac{\mathbb{E}[e^{-R_T} S_T e^{iz \ln S_T}]}{\mathbb{E}[e^{-R_T} S_T]} \\ \Phi_2(z) &= \frac{\Psi(iz)}{\Psi(0)} = \frac{\mathbb{E}[e^{-R_T} e^{iz \ln S_T}]}{\mathbb{E}[e^{-R_T}]} \end{aligned}$$

for two distribution functions F_1, F_2.

The particular form of these functions is a consequence of the generalized Bayes theorem [12, pg. 231] for conditional expectations, when we require

$$C_T(S_0, v_0, r_0) = S_0 \int_{\ln K}^{\infty} \mathrm{d}F_1(x) - KP(r_0, T) \int_{\ln K}^{\infty} \mathrm{d}F_2(x). \tag{3.1}$$

Fourier inversion[4] then allows to numerically evaluate the probability distributions [4], such that

[3] This is nothing else than the two-dimensional analogue of the matrix square root decomposition, Eq. (2.1).

[4] The inversion formula goes back to Gurland [7], who showed that

$$F(x) + F(x - 0) = 1 - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-iux}\Phi(u)}{iu} \mathrm{d}u,$$

where the integral has to be interpreted as a Cauchy principal value. For (left-) continuous $F(x)$ this reduces to

$$P(X \leq x) = F(x) = \frac{1}{2} + \frac{1}{2\pi} \int_0^{\infty} \frac{\Phi(-u)e^{iux} - \Phi(u)e^{-iux}}{iu} \mathrm{d}u,$$

such that

$$P(X \geq \ln K) = 1 - F(\ln K) = \frac{1}{2} - \frac{1}{2\pi} \int_0^{\infty} \frac{\Phi(-u)e^{iu \ln K} - \Phi(u)e^{-iu \ln K}}{iu} \mathrm{d}u.$$

the option pricing function at time t is

$$C_{T-t}(S_t, v_t, r_t) = S_t\left(\frac{1}{2} - \frac{1}{2\pi}\int_0^\infty \frac{\Phi_1(-u)e^{iu\ln K} - \Phi_1(u)e^{-iu\ln K}}{iu}\mathrm{d}u\right) - KP(r_t, T-t)\left(\frac{1}{2} - \frac{1}{2\pi}\int_0^\infty \frac{\Phi_2(-u)e^{iu\ln K} - \Phi_2(u)e^{-iu\ln K}}{iu}\mathrm{d}u\right).$$

The remaining work is to find an expression for $\Psi(z)$. This method is due to Scott [11], and we just follow his calculations (see Appendix for details), to arrive at

$$\Psi(z) = e^{-z(v_0+\kappa\eta T)} \cdot \mathbb{E}[e^{(z-1)R_T}] \cdot \mathbb{E}[e^{wV_T + z\frac{\rho}{\lambda}v_T}], \tag{3.2}$$

where we used the integrated volatility $V_t = \int_0^t v_t \mathrm{d}t$, and

$$w = (z-1)z\frac{1}{2}(1-\rho^2) + z\left(\frac{\rho}{\lambda}\kappa - \frac{1}{2}\rho^2\right).$$

From the theory of Bessel bridges [10, 5] we have the following closed form for the second expectation:

$$\mathbb{E}[e^{-s_1 V_T - s_2 v_T} \mid v_0] = e^{a_T - b_T v_0}, \qquad \mathrm{Re}\ s_i \geq 0, \quad i = 1, 2,$$

where

$$a_T = 2\kappa\eta \cdot \ln \frac{2\gamma e^{\frac{1}{2}(\kappa-\gamma)T}}{2\gamma e^{-\gamma T} + (\kappa + \gamma + s_2)(1 - e^{-\gamma T})}$$

$$b_T = \frac{(1 - e^{-\gamma T})(2s_1 - \kappa s_2) + \gamma s_2(1 + e^{-\gamma T})}{2\gamma e^{-\gamma T} + (\kappa + \gamma + s_2)(1 - e^{-\gamma T})},$$

and $\gamma = \sqrt{\kappa^2 + 2s_1}$. The parameters κ and η are taken from the volatility process:

$$\mathrm{d}v_t = \kappa(\eta - v_t)\mathrm{d}t + \lambda\sqrt{v_t}\mathrm{d}W_{2,t}, \quad v_0 > 0. \tag{3.3}$$

This almost solves the problem, since we still need to find an expression for the first expectation in Eq. (3.2).

If we now replace[5] the generalized Hull–White interest rate process with a CIR type interest process,

$$\mathrm{d}r_t = (\theta - ar_t)\mathrm{d}t + \sigma\sqrt{r_t}\mathrm{d}W_{3,t},$$

then this is also of the above form (3.3) (replacing κ by a, and η by θ/a), giving us a semi-analytical solution.

4 Constrained correlations

We now present an alternative method. Consider the model (2.2-2.4) again.

The change of variable $S_t = \exp(X_t)$ leads to $G_t(X_t, \cdot, \cdot, \cdot) = C_t(S_t, \cdot, \cdot, \cdot)$, such that $(e^{-R_t}G_t)$ is a martingale (under the appropriate, equivalent risk-neutral measure). Following the strategy of the

[5] In fact, it should be possible to arrive at a similar expression for the (standard) Hull–White interest rate process, too, by following the lines of the proof of above formula in [10, 5]. This is one possible direction for future research.

multi-dimensional Feynman-Kac theorem for independent Brownian motions [13], we expand the differential $\mathrm{d}(e^{-R_t}G_t)$ in $\mathrm{d}t$ and $\mathrm{d}B_{i,t}$ terms ($i = 1, 2, 3$), and set the $\mathrm{d}t$ term equal to zero, leading[6] to the following PDE:

$$
\begin{aligned}
r_t G_t = \frac{\partial G_t}{\partial t} + (r_t - \frac{1}{2}v_t)\frac{\partial G_t}{\partial X_t} + \kappa(\eta - v_t)\frac{\partial G_t}{\partial v_t} + (\theta_t - a r_t)\frac{\partial G_t}{\partial r_t} \\
+ \frac{1}{2}v_t \frac{\partial^2 G_t}{\partial X_t^2} + \frac{1}{2}\lambda^2 v_t \frac{\partial^2 G_t}{\partial v_t^2} + \frac{1}{2}\sigma^2 \frac{\partial^2 G_t}{\partial r_t^2} \\
+ \lambda\rho_{12} v_t \frac{\partial^2 G_t}{\partial X_t \partial v_t} + \sigma\rho_{13}\sqrt{v_t}\frac{\partial^2 G_t}{\partial X_t \partial r_t} + \lambda\sigma\rho_{23}\sqrt{v_t}\frac{\partial^2 G_t}{\partial v_t \partial r_t}.
\end{aligned}
$$

The ansatz[7]

$$G_t = e^{A(T-t)+v_t B(T-t)+r_t C(T-t)+\sqrt{v_t}D(T-t)+iuX_t}$$

now gives[8] the following system of equations:

$$
\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}t} &= \theta_t C(t) + \frac{1}{2}\sigma^2 C^2(t) + \kappa\eta B(t) + \frac{1}{2}\lambda\sigma\rho_{23} D(t)C(t) + \frac{1}{8}\lambda^2 D^2(t) \\
\frac{\mathrm{d}B}{\mathrm{d}t} &= -\frac{iu}{2} - \frac{u^2}{2} - \kappa B(t) + \lambda\rho_{12} iu B(t) + \frac{1}{2}\lambda^2 B^2(t) \\
\frac{\mathrm{d}C}{\mathrm{d}t} &= iu - aC(t) \\
\frac{\mathrm{d}D}{\mathrm{d}t} &= iu\sigma\rho_{13} C(t) - \frac{1}{2}\kappa D(t) + iu\frac{1}{2}\lambda\rho_{12} D(t) + \lambda\sigma\rho_{23} B(t)C(t) + \frac{1}{2}\lambda^2 B(t)D(t) \\
0 &= 8D(t)\left(4\kappa\eta - \lambda^2\right)
\end{aligned}
$$

which is a system of ODEs, either (i) if we set

$$\lambda = 2\sqrt{\kappa\eta} \qquad \text{(Forced volatility variance)},$$

or (ii) if we set D(t) = 0. The latter is possible, if we let $B(t) = -iu\frac{\rho_{13}}{\rho_{23}}\frac{1}{\lambda}$, which gives us two constraints on the parameters (from $\frac{\mathrm{d}B}{\mathrm{d}t} = 0$):

$$\rho_{23} = \frac{2\kappa}{\lambda}\rho_{13}, \qquad \rho_{12} = \frac{4\kappa^2 + \lambda^2}{4\kappa\lambda} \qquad \text{(Forced volatility correlation)}.$$

In this case, the equation in $A(t)$ can be integrated easily, since $C(t)$ is readily available,

$$C(t) = \frac{iu}{a}\left(e^{at} - 1\right), \quad \text{when } C(0) = 0.$$

Furthermore, if θ_t is assumed constant, the solution is given analytically by the characteristic function of G_t, as in the solution of the Heston model.

[6]Note that $a_{ik}\mathrm{d}B_{k,t} \cdot a_{jl}\mathrm{d}B_{l,t} = a_{ik}a_{jl}\delta_{kl}\mathrm{d}t = a_{ik}a_{jk}\mathrm{d}t = \rho_{ij}\mathrm{d}t$, where δ_{kl} is the Kronecker delta.

[7]Which fulfills the necessary boundary condition $G_T = e^{iuX_T}$, given the initial conditions $A(0) = B(0) = C(0) = D(0) = 0$.

[8]Use that

$$
\begin{aligned}
\frac{\partial G_t}{\partial v_t} &= G_t\left[B(t) + \frac{1}{2\sqrt{v_t}}D(t)\right] \\
\frac{\partial^2 G_t}{\partial v_t^2} &= G_t\left[B^2(t) + \frac{B(t)D(t)}{\sqrt{v_t}} + \frac{1}{4v_t}D^2(t) - \frac{1}{4(v_t)^{3/2}}D(t)\right].
\end{aligned}
$$

5 Volatility-interest coupling

The third method discussed considers an interest rate process that is coupled[9] to the volatility, via

$$\mathrm{d}r_t = (\theta_t - ar_t)\mathrm{d}t + \sigma\sqrt{v_t}a_{3k}\mathrm{d}B_{k,t}.$$

The Feynman–Kac partial differential equation for the martingale $(e^{-R_t}G_t)$ then reads

$$\begin{aligned} r_tG_t = \frac{\partial G_t}{\partial t} + (r_t - \frac{1}{2}v_t)\frac{\partial G_t}{\partial X_t} + \kappa(\eta - v_t)\frac{\partial G_t}{\partial v_t} + (\theta_t - ar_t)\frac{\partial G_t}{\partial r_t} \\ + \frac{1}{2}v_t\frac{\partial^2 G_t}{\partial X_t^2} + \frac{1}{2}\lambda^2 v_t\frac{\partial^2 G_t}{\partial v_t^2} + \frac{1}{2}\sigma^2 v_t\frac{\partial^2 G_t}{\partial r_t^2} \\ + \lambda v_t\frac{\partial^2 G_t}{\partial X_t\partial v_t}\rho_{12} + \sigma v_t\frac{\partial^2 G_t}{\partial X_t\partial r_t}\rho_{13} + \lambda\sigma v_t\frac{\partial^2 G_t}{\partial v_t\partial r_t}\rho_{23}. \end{aligned}$$

Following Heston, we make a similar ansatz for the characteristic function:

$$G_t = e^{A(T-t)+B(T-t)v_t+C(T-t)r_t+iuX_t}.$$

Grouping together terms with v_t, respectively r_t, we get the following system of ordinary differential equations,

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= \kappa\eta B(t) + \theta_t C(t) \\ \frac{\mathrm{d}B}{\mathrm{d}t} &= b_0 + b_1 B(t) + \frac{1}{2}\lambda^2 B(t)^2 + \frac{1}{2}\sigma^2 C(t)^2 \\ &\qquad + \lambda\sigma\rho_{23}B(t)C(t) + iu\lambda\rho_{12}C(t) \\ \frac{\mathrm{d}C}{\mathrm{d}t} &= (iu - 1) + aC(t) \end{aligned}$$

where $b_0 = -\frac{1}{2}iu(1 - iu)$, and $b_1 = iu\sigma\rho_{13} - \kappa$.

The initial conditions are $A(0) = B(0) = C(0) = 0$, and the last equation has solution:

$$C(t) = \frac{1 - iu}{a}(e^{-at} - 1).$$

The second equation is a Riccati equation of form

$$\frac{\mathrm{d}B(t)}{\mathrm{d}t} = \frac{1}{2}\lambda^2 B(t)^2 + g(t)B(t) + h(t)$$

with coefficient functions

$$\begin{aligned} g(t) &= g_0 + g_1 e^{-at} \\ h(t) &= h_0 + h_1 e^{-at} + h_2 e^{-2at} \end{aligned}$$

[9]The form of this coupling is only motivated by the mathematical structure. In fact, whether this coupling is of any value in the modelling of real-world finance, is quite unclear, though one might expect it not to be.

where, setting $q = (1 - iu)$,

$$g_0 = iu\sigma\rho_{13} - \kappa - \lambda\sigma\frac{q}{a}\rho_{23}, \qquad h_0 = -\frac{1}{2}iuq + \frac{q^2\sigma^2}{2a^2} - iu\lambda\frac{q}{a}\rho_{12}$$

$$g_1 = \lambda\sigma\frac{q}{a}\rho_{23}, \qquad h_1 = iu\lambda\frac{q}{a}\rho_{12} - \frac{q^2\sigma^2}{a^2}$$

$$h_2 = \frac{q^2\sigma^2}{2a^2}.$$

Although the quadratic term $B(t)^2$ makes it impossible to split this equation into real and imaginary parts, there exists[10] an analytical solution of this equation in terms of Whittaker functions [1], such that it can be evaluated efficiently with tabulated values. Yet the equation for $A(t)$ makes it necessary to solve the whole system numerically. Still, this is more efficient than integration of the partial differential equation or direct (Monte-Carlo) simulation, and makes this approach also interesting.

6 Discussion

In this short note we have discussed three different ways of obtaining efficient solutions to extensions of the Heston model. Unfortunately, the page limitation in this contribution does not allow for numerical experiments with these methods.

A The method of Scott

Write

$$\begin{aligned}
\ln S_t &= \int_0^t r_s \mathrm{d}s + \int_0^t \sqrt{v_s}\left(\rho \mathrm{d}W_{2,s} + \sqrt{1-\rho^2}\mathrm{d}W'_{2,s}\right) - \frac{1}{2}\int_0^t v_s \mathrm{d}s \\
&= R_t + \left(\rho\int_0^t \sqrt{v_s}\mathrm{d}W_{2,s} - \frac{1}{2}\rho^2\int_0^t v_s \mathrm{d}s\right) \\
&\quad + \left(\sqrt{1-\rho^2}\int_0^t \sqrt{v_s}\mathrm{d}W'_{2,s} - \frac{1}{2}(1-\rho^2)\int_0^t v_s \mathrm{d}s\right) \\
&= R_t + \eta_t + \xi_t.
\end{aligned}$$

Since v_s develops independently from $\mathrm{d}W'_{2,s}$, we can calculate

$$\begin{aligned}
\mathbb{E}[\xi_t \mid W_{2,t}] &= -\frac{1}{2}(1-\rho^2)V_t, \\
\mathrm{Var}[\xi_t \mid W_{2,t}] &= (1-\rho^2)V_t,
\end{aligned}$$

where $V_t = \int_0^t v_s \mathrm{d}s$.

Furthermore, we now can use $\sqrt{v_t}\mathrm{d}W_{2,t} = \frac{1}{\lambda}(\mathrm{d}v_t - \kappa(\eta - v_t)\mathrm{d}t)$ to write

$$\eta_t = \frac{\rho}{\lambda}(v_t - v_0 - \kappa\eta t + \kappa V_t) - \frac{1}{2}\rho^2 V_t.$$

[10]The commercial software package MAPLE can be used to derive the analytical solution of this ODE.

Considering $\Psi(z) = \mathbb{E}\left[e^{-R_T + z \ln S_T}\right]$, we see that $\Psi(z) = \mathbb{E}\left[e^{(z-1)R_T}\right] \cdot \mathbb{E}\left[e^{z\xi_T + z\eta_T}\right]$. Now ξ_T, being an Itô integral, is normally distributed. Therefore $e^{z\xi_T}$ has a log-normal distribution, such that

$$\mathbb{E}[e^{z\xi_T} \mid W_{2,t}] = e^{(z-1)z\frac{1}{2}(1-\rho^2)V_t} \qquad \text{(conditional on } W_{2,t}\text{)}$$

and we arrive at the formula given in the text, Eq. (3.2).

References

[1] G. E. Andrews, R. A. Askey, and R. Roy. *Special Functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2001.

[2] F. Black. Studies of stock price volatility changes. In *Proceedings of the 1976 Meetings of the Business and Economic Statistics Section*, pages 177–181. American Statistical Association, 1976.

[3] J.-P. Bouchaud, A. Matacz, and M. Potters. Leverage effect in financial markets: The retarded volatility model. *Phys. Rev. Let.*, 87(22):228701–228704, 2001.

[4] Peter Carr and Dilip B. Madan. Option valuation using the Fast Fourier Transform. *Journal of Computational Finance*, 2(4):61–73, 1998.

[5] Marc Chesney, Robert J. Elliott, and Rajna Gibson. Analytical solutions for the pricing of American bond and yield options. *Mathematical Finance*, 3(3):277–294, 1993.

[6] J. C. Cox, J. E. Ingersoll, and S. A. Ross. An intertemporal general equilibrium model of asset prices. *Econometrica*, 53(2):363–384, 1985.

[7] J. Gurland. Inversion formulae for the distribution of ratios. *The Annals of Mathematical Statistics*, 19(2):228–237, 1948.

[8] S. L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies*, 6(2):327–343, 1993.

[9] J. C. Hull and A. White. Pricing interest-rate derivative securities. *Rev. Finan. Stud.*, 3(4):573–592, 1990.

[10] J. Pitman and M. Yor. A decomposition of Bessel bridges. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 59:425–457, 1982.

[11] L. O. Scott. Pricing stock options in a jump-diffusion model with stochastic volatility and interest rates: Applications of Fourier inversion methods. *Mathematical Finance*, 7(4):413–424, 1997.

[12] A. N. Shiryaev. *Probability*, volume 95 of *GTM*. Springer, second edition, 1996.

[13] S. E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Springer, 2004.

[14] A.P. C. van der Ploeg and J. Tistaert. Improving an equity option pricing model for ING: Deriving a call option pricing formula in the hybrid Heston–Hull–White model. Technical report, ING Corporate Market Risk Management, 2007.