Chapter 1 (week of Monday February 6)

Together with Chapter 2, Chapter 1 forms the prelude to the rest of the book. Chapter 1 basically introduces the three main types of Banach spaces which will occur in the book: \( l_p \), \( L_p \) and \( C(K) \). Here, as in the future, \( L_p \) denotes the \( L_p \) space corresponding to the Lebesgue measure on the Lebesgue \( \sigma \)-algebra on \([0, 1]\). Aside, one could also take the Borel \( \sigma \)-algebra, since the corresponding \( L_p \) spaces are canonically isomorphic (which is not entirely trivial).

Although they are mentioned in this chapter, the spaces \( l_p \) or \( L_p \) for \( 0 < p < 1 \) play no role in the book. For those who are interested anyway, it’s nice to know that the spaces \( L_p \) for such \( p \) have zero dual. See, e.g., Rudin’s FA Section 1.47.

Likewise, we won’t be paying attention to the spaces mentioned in connection with (1.3) and (1.5) which are metrized but not normed. After all, the book is on Banach space and not on the more general class of \( F \)-spaces. There is a slight inaccuracy here: if \( X \) is not empty, then the space \( C(X) \) on page 5 is of course always metrizable, as is any non-empty set.

If you are not too familiar with the general topological notions mentioned in this chapter there is time to catch up, since these spaces \( C(K) \) start playing a major role from Chapter 12 onward only. There is some topology in the appendix. If you know what a locally compact Hausdorff space is, what a completely regular space is and if you know what the Urysohn lemma and the Tietze extension lemma are, you are already well on your way.

It’s not so obvious that \( l_\infty \) is actually of type \( C(K) \), as is mentioned on page 6. This will become clear only in Corollary 15.2.(i).

Chapter 2 (week of Monday February 13)

Most of the material in this chapter is standard and it is perhaps for this reason that the author has payed somewhat less attention to the accuracy of the statements and proofs. Fortunately, this Chapter is not representative for the rest of the book.

Note the nice way of showing that a normed space has a Banach space completion on page 15. Of course one can show that the metric completion of a normed space is canonically a normed vector space and construct it this way, but by the standard embedding on page 15 you get it for free.

There is a mistake in the last paragraph on page 16: it has to be assumed that \( X \) is Banach here. Otherwise one cannot apply the closed graph theorem. The upshot is that there is a canonical bijection between decompositions \( X = L \oplus M \), where the subspaces are \( closed \), and \( continuous \) projections \( P \). The correspondence is obtained by assigning to such a decomposition the projection of \( X \) onto \( L \) along \( M \). The “miracle” is that this projection is necessarily continuous.

If you are not familiar with quotients as normed spaces, and want to see more details, you could try Bollobás’ book (reference [18]).

The word “smallest” in line -14 on page 18 should be replaced by “largest”.

The first paragraph on page 19 is not quite accurate: the map $S$ uniquely exists by general principles, and it is then an additional result that $\|S\| = \|T\|$.

The second and third paragraph on page 19 are concerned with the question when there is a copy of $X/M$ contained in $X$ which maps isomorphically onto the quotient under the canonical map. This italicized condition is not mentioned in the text. The answer is that (if $X$ is a Banach space) this holds precisely when $M$ is complemented.

The isomorphism between $(X/M)^*$ and $M^\perp$ and between $X^*/M^\perp$ and $M^*$ are standard results. Although the first one is proved again in the text (with an error, since $M$ need not be of codimension one), it is implied immediately by the preceding discussion which shows the following: if $X$ and $Y$ are normed, and $M$ is closed in $X$, then $B(X/M, Y)$ is isometric to the subspace of $B(X, Y)$ of operators annihilating $M$. Now take $Y = \mathbb{R}$. As to the second one, the map obtained by restriction from $X$ to $M$ establishes a linear bijection between $X^*/M^\perp$ and $M^*$ (use the Hahn-Banach theorem for surjectivity) and it requires just slightly more argumentation that this is actually an isometry (the Hahn-Banach theorem is then used again; see, e.g., Rudin’s FA section 4.9).

I am puzzled by the argumentation in the final paragraph on page 19. The result holds for direct sums of normed spaces (not just Banach spaces) and it is essentially obvious, without using any isomorphism theorems.

Theorem 2.2 is not needed for the rest of the book. In the statement of the result, the kernel of the projection is actually equal to $i(X)^\perp$, not just isometric to it.

Chapter 3 (week of Monday February 20)

This chapter contains the basics about bases. Note that “basis” is short for “Schauder basis”, so unless the Banach space is finite dimensional a “basis” as we will call it is never a basis in the algebraic sense (since a “true” basis, also called a Hamel basis, is then necessarily uncountable).

The fact (due to Mazur) that every Banach space contains a basic sequence, as mentioned on page 25, will appear as Corollary 4.2. The isomorphic embedding of separable spaces into $C[0, 1]$, which is also mentioned on page 25, is stated as Corollary 12.14 (it is not extremely deep though; but Chapter 12 is the logical place for this result given the structure of the book).

On page 28, in the last line one should replace $2^{m+1}$ by $2^m + 1$ (twice).

The fact, as mentioned on page 29, that $C[0, 1]$ has a bases consisting of polynomials, will reappear on page 41 in the Notes.

The result which is proved on page 29/30 is that the Haar system is a basis in $L_p[0, 1]$ for $1 \leq p < \infty$. I am not particularly fond of the proof in the book, which I find not too precise, but there are scans of three pages of one of the famous books by Lindenstrauss and Tzafriri ([94] and [95]) on the webpage which contain a more insightful approach on the third page (basically the answer to Exercise 3 on page 3 of our book).

The terminology monotone basis for a basis with basis constant 1 is motivated by the third part in the following result:

**Theorem 1.** Let $X$ be a Banach space with a Schauder basis $(x_n)$ and associated natural projections $(P_n)$. Then the following are equivalent:
1. The basis constant $K = \sup_n \| P_n \|$ is equal to 1.

2. $\| P_n \| = 1$ for all $n$.

3. For all sequences of scalars $(a_n)$, the sequence $(\| \sum_{i=1}^n a_i x_i \|)$ is non-decreasing in $n$.

Try to convince yourself of this result. (Hint: use the first two lines of the proof of Theorem 3.2.) As remarked in the text, one can then always find an equivalent norm in which the given basis is monotone.

Chapter 4 (week of Monday February 27)

Although not even every separable Banach space has a basis, every infinite dimensional Banach space has an infinite dimensional closed subspace that has a basis, as is stated in Corollary 4.2 (the infinite dimensionality of the subspace should be added in Corollary 4.2, otherwise it is trivial). Of course, this is an abstract general result and it is nice to have concrete examples of such basic sequences in concrete spaces: this is part of the content of Lemma 4.3 and 4.4. Theorem 4.5 and 4.7 show how to obtain new basic sequences from given ones, possibly preserving the property of being complemented.

After this general layout, here are some detailed remarks:

In the proof of Corollary 4.2, Mazur’s lemma is Proposition 4.1. Mazur’s lemma can be thought of as the existing of “almost orthoplements of finite dimensional subspaces”. This becomes clear when one assumes that $\varepsilon = 0$ and that $X$ is a Hilbert space: the inequality is then equivalent to stating that $x$ is orthogonal to $F$.

In Lemma 4.3 the isometric isomorphism is in fact explicit: it sends $e_n$ to $f_n$. In the proof, one has to check that the series defining $P f$ is actually convergent. When doing this by verifying that the partial sums are a Cauchy sequence, it also becomes clear that $P$ is continuous and has norm one.

Although it is not mentioned explicitly in the discussion of $c_0$ on page 36, it is not only the case that there is an isometric copy in $c_0$ via an isomorphism sending $e_n$ to $y_n$, but in addition this copy is complemented by a norm one projection, just as for $L_p(\mu)$.

In Lemma 4.4, add the condition that $1 \leq p < \infty$. In the proof, the sequence $(a_n)$ is tacitly assumed to be in $\ell_p$. The proof also shows that the “fast enough” convergence occurs certainly if $\sum_{n=1}^{\infty} \varepsilon_n < \frac{1}{2}$ and that the isomorphism sends $e_n$ to $f_n$ again. The fact that $[f_n]$ is complemented follows from Theorem 4.5(ii) under the condition that $\sum_{n=1}^{\infty} \varepsilon_n < \frac{1}{8}$.

In the proof of Theorem 4.5, it should be checked that the series $\sum a_n y_n$ is convergent if $\sum a_n x_n$ is (easily done by showing that the partial sums are a Cauchy sequence). The second inequality in the second displayed equation is true but superfluous (the series need not even converge). In the statement of part (i), add that $(y_n)$ is a basis of $X$ if $(x_n)$ is: it is easy to check that $\|1 - T\| < 1$, so that $T$, which is then defined on all of $X$, is invertible and in particular surjective.

Likewise, in the statement of part (i) of Theorem 4.7, it is the case that $(y_n)$ is a basis of $X$ if $(x_n)$ is. This is a consequence of the proof of this theorem in the scan of page 41 of Wojtaszczyk’s “Banach spaces for analysts” which you find on the seminar’s page.
Chapter 5 (week of Monday March 6)

In this Chapter we obtain the first answers to classification questions about $\ell_p$-spaces, $c_0$ and their subspaces: Theorem 5.8, Corollary 5.9 (implying that different members of this family contain no isomorphic subspaces at all!) and Theorem 5.10. Although it is no longer visible in the statements of these results, they are actually proved using basis theory and the first section on Block Basic Sequences contains the necessary preparations for this, building on previous results. This first section also contains an alternative approach to Corollary 4.2, but for this the general embedding result of separable spaces into $C[0,1]$ has to be used, so the direct approach in Chapter 4 using Mazur’s lemma is to be preferred.

Here are the detailed remarks:

In Lemma 5.1 it should be added as a hypothesis that $(z_n)$ contains a subsequence which is bounded away from zero. The remark following the proof that one should insist that $(z_{n_k})$ is bounded away from zero does not repair the proof, since this is evidently impossible to arrange if $(z_n)$ converges to zero.

Corollary 5.4 follows from Lemma 5.2, together with the yet to be proved isometric embedding of the separable Banach space $[z_n]$ into $C[0,1]$.

That Proposition 5.5 holds for $c_0$ as well, and not just for $\ell_p$ (1 ≤ p < ∞), follows from the discussion following Lemma 4.3 rather than from the Lemma itself.

Corollary 5.7 is vital in the proof of Corollary 5.9 and the discussion preceding it, as well as in the proof of Pelczyński’s theorem 5.10. Unfortunately its proof is only sketched, so the proof in the first book by Lindenstrauss and Tzafriri can be found in the scans on the seminar’s page.

In the definition of a strictly singular operator on pp. 48-49, it should be required that it is not an isomorphism on any infinite dimensional subspace. Clearly, any non-zero operator has a finite dimensional subspace (namely: a one dimensional one) on which it is an isomorphism, so without this addition only the zero operator would be strictly singular.

On page 50, it is claimed that e.g. $(\ell_p \oplus \ell_p \oplus \cdots)_p = \ell_p$. This is in fact easy to see: consider one quadrant of the lattice in the plane. Given an element of $\ell_p$, distribute its elements along the finite diagonals, starting at the origin. If you think about it for a moment, it will become obvious that this establishes an isomorphism as required. Likewise for $c_0$, with a slight addition in the argument.

Chapter 6 (week of Monday March 13)

Chapter 6 concludes the group of Chapters 3-6, where the Banach space theory has really started and in which chapters the basic theory of bases is treated with an emphasis on the spaces $\ell_p$ and $c_0$. Three particularly important spaces $\ell_1$, $\ell_\infty$ and $c_0$ are now investigated in more detail. After this chapter, Chapter 7 will be an intermezzo on duality and then the story continues with the Chapters 8-10 on $L_p$-spaces.

Here are the detailed remarks:

In Theorem 6.1 a more precise statement is that every separable Banach space is isometric to a quotient of $\ell_1$. For this to be the case it is equivalent that there is a linear map $Q$ as in the proof, mapping the open unit ball in $\ell_1$ onto the open unit ball in $X$. If you doubt this, realize that the canonical map from a normed linear space to a quotient has this property and that a linear bijection between
two normed spaces is an isometry precisely when it is a bijection between the two corresponding open unit balls. In the proof on page 56, the condition that \( j \neq n_1 \) appears to be a typo.

Corollary 6.3, as well as the corresponding result for \( \ell_p \) \((1 < p < \infty)\) instead of \( c_0, \) also follows from Corollary 5.9. Using Schur’s result is of course much easier. On page 58 it is claimed that we now understand why there are uncountably many isomorphically distinct uncomplemented subspaces in \( \ell_1, \) but I fail to see the reason for this. Anyone who does?

Theorem 6.4 states that the only situation in which \( X \) can map onto \( \ell_1 \) is the situation in which this is trivially possible since \( X = Y \oplus \ell_1 \) isomorphically.

The fact that one can take \( X = \ell_1 \) in the proof of Corollary 6.5 follows of course from Theorem 6.1.

In the proof of Theorem 6.6, the inequalities in the first line should read \( \alpha |||x||| \leq \alpha^{-1} \) \( ||x|| \) and then the inequalities in the sixth line of the proof change into \( \alpha \leq \lambda \leq \alpha^{-1}. \) In line 7-9 of the proof, the construction of the sequence \( (y_k) \) goes as follows. First, for \( n \geq n_0, \) take \( z_n \in \text{Ker} \ P_n \subset \text{Ker} \ P_{n_0} \) with \( |||z_n||| = 1 \) and \( ||z_n||_1 \) very close to \( \lambda_n. \) Next, realize that the \( e_n \) for \( n \geq n_0 \) are still a Schauder basis for the Banach space \( \text{Ker} \ P_{n_0} \) in the norm \( ||| \cdot |||, \) since this is an equivalent norm. Therefore, Lemma 5.1 provides a block basic sequence \( (y_k) \) in \( \text{Ker} \ P_{n_0} \) with \( y_k \) very close to \( z_n \) in \( ||| \cdot ||| \). In particular, \( |||y_k||| \) is almost one. Rescaling changes the norm \( \|y_k\| \) into 1 and since \( ||z_n||_1 \) is almost \( \lambda_n, \) the gap between \( \lambda_n \) and \( \lambda/(1 + \varepsilon) \) can be used to choose all approximations so close that one still has \( ||y_k||_1 > \lambda/(1 + \varepsilon). \)

In Lemma 6.7, add that one can assume that the \( f_n \) are norm one.

On page 62 I don’t see why the proof of Theorem 6.9 computes the norm in the quotient. In the proof, \( x \) is modified by elements of \( c_{00}, \) whereas for the quotient norm one allows modification by elements of \( c_0, \) which is much larger.

In the proof of Corollary 6.10, \( T^{-1}PT^{**} \) should be replaced with \( JT^{-1}PT^{**} \) where \( J : c_0 \to \ell_\infty \) is canonical (otherwise it is not an operator on \( \ell_\infty). \) Incidentally, if one wants to prove only that \( c_0 \) is not isometric to a dual space, then another (standard) approach is possible. Indeed, by the Banach-Alaoglu theorem the unit ball in a dual space is compact in the weak* topology and by the Krein-Milman theorem it is then the closed convex hull of its extremal points. In particular, the closed unit ball in a dual space has extremal points. For \( c_0, \) however, the unit ball has no extremal points.

The notion “injective Banach space” on page 62 stems from the general theory of abelian categories, such as the category of Banach spaces. (For “projective Banach space”: reverse the arrows.) We are very close to homological algebra here. Note that the (unproven) statement in this paragraph implies that Theorem 5.10 also holds for \( \ell_\infty. \)

In the proof of Theorem 6.12 the arguments are not well presented. Of course one can perturb by elements of \( K \) without affecting the value on \( Y: \) this is trivially true by definition. The correct argument goes as follows. What one needs in the conclusion of the proof is a sequence \( (z_n^*) \) in \( K \) such that the distance between \( z_n^* \) and \( x_n^* \) tends to zero. To obtain these \( z_n^* \), it is clearly sufficient (in fact equivalent) to show that \( d(x_n^*, K) \) tends to zero. And to understand why this is true: assume that it is not and that there is a subsequence staying away from \( K. \) By compactness, however, this subsequence has another subsequence which converges. The limit must be outside \( K \) by construction, but on the other hand (and this is proved correctly) any weak* limit point of the \( x_n^* \) must be in \( K; \) contradiction.
In Theorem 6.13, if \( Y^\ast \) were separable and containing \( c_0 \), then by the first part \( \ell_\infty \) would be a subspace of \( Y^\ast \), hence separable, which is absurd. Therefore no separable dual contains \( c_0 \). The proof as given is an imprecise copy of the one on page 103-104 in Lindenstrauss and Tzafriri I. It uses Goldstine’s theorem (the weak*-density of the unit ball of \( X \) in that of \( X^{**} \)) without mentioning it and also an additional result (I.a.12 on page 7 in L&T I) about Schauder bases which is not covered in Carothers’ book. If you want to convince yourself of the details you can look it up in L&T, but personally I am satisfied with Bessaga and Pelczynski stating the result and knowing it to be in L&T.

**Chapter 7 (week of Monday March 20)**

This chapter investigates the relation between a basis \( (x_n) \) in \( X \) and the canonical candidate \( (x_n^\ast) \) for a basis in \( X^\ast \). What is mentioned here is just the tip of the iceberg.

Some detailed remarks:

On page 67, it is claimed that \( (x_n^\ast) \) is a basic sequence with the same basis constant \( K \) as the basis \( (x_n) \). One can, however, only conclude that the basis constant is at most \( K \). On the same page, it is meant that \( P_n^\ast \) is the canonical projection in the Banach space \( [x_n^\ast] \). It must be formulated this way since \( (x_n^\ast) \) need not be a basis of \( X^\ast \) itself.

On page 68, the fact that the span of the \( x_n^\ast \) is weak*-dense in \( X^\ast \) has nothing to do with the fact that the \( P_n^\ast \) converge strongly (i.e., pointwise) to the identity. It is always true that the span of a set \( S \subseteq X^\ast \) which separates the points in \( X \) is weak*-dense and this is a special case thereof.

In the first equation in the proof of Theorem 7.1, replace equality with \( \leq \).

In the proof of Theorem 7.2, note that the norm of the \( P_n^\ast \) and of the \( P_n^{**} \) is equal to one, since this holds for the \( P_n \) and taking adjoints preserves the norm. Also you may want to use the following result (easily proved): if \( (y_n^\ast) \subset Y^\ast \) converges weak* to \( y^\ast \) in some dual space \( Y^\ast \), then \( \|y^\ast\| \leq \lim \sup \|y_n^\ast\| \). This is used with \( Y = X^\ast \). In the part of the proof which shows that the map is surjective, it is used that the bounded sequence \( (\sum_{i=1}^{n} a_i x_i^\ast)_{n=1}^{\infty} \) (viewed canonically embedded in \( X^{**} \)) has a weak* convergent subsequence. This is always true in the dual of a separable space, hence it applies here because \( X^\ast \) is separable as it has a basis. The last sentence of the proof contains the argument why the map is injective.

The first part of the proof in Theorem 7.3 where it is shown that the basis \( (x_n^\ast) \) is boundedly complete is incorrect. If you look closely you will see that it is used that a bounded weak*-convergent sequence is uniformly convergent, which is nonsense. Here is a correct argument. If \( (\sum_{i=1}^{n} a_i x_i^\ast) \) is bounded in \( X^\ast \), then as a consequence of the Banach-Alaoglu theorem and the separability of \( X \), there is a weak*-convergent subsequence:

\[
x^\ast = \lim_{k \to \infty} \sum_{i=1}^{n_k} a_i x_i^\ast
\]

where \( n_k \to \infty \) as \( k \to \infty \). Since we already know that the \( x_n^\ast \) are a basis of \( X^\ast \), there is a convergent series

\[
x^\ast = \sum_{i=1}^{\infty} b_i x_i^\ast.
\]
Looking at the weak*-convergent subsequence one sees that \( a_i = b_i \), so the series with the \( a_i \) is convergent as required.

The second part of the proof of Theorem 7.3, starting with a boundedly complete basis, is also incorrect: on page 71 it is stated that \( P_n x^* \in Y \), but there is no reason why this should be so. This is more or less beyond repair within the scope of a reading guide and if you want you can look up a correct proof in Megginson’s “An introduction to Banach space theory”, Theorem 4.4.13. In fact, combining Lemma 4.4.3 and 4.4.12 in that book, we have the following result.

**Theorem 2.** Let \( X \) be a Banach space that has a Schauder basis \((x_n)\). Consider the map \( J : X \rightarrow [x_n]^* \) defined by \( (Jx)(y^*) = y^*(x) \) for \( x \in X \) and \( y^* \in [x_n]^* \).

Then \( J \) is an isomorphism into. This isomorphism is onto precisely when the basis \((x_n)\) is boundedly complete.

In the proof of Theorem 7.4 an implicit reference is made to page 68, where it is remarked that the algebraic span of the \( x_n^* \) is weak*-dense in \( X^* \) for any space with a basis \((x_n)\). Now it is in general true that for a normed linear space the closure of a subspace (in fact: of any convex set) coincides with its weak closure. Since for a reflexive space \( X \) the weak topology on \( X^* \) coincides with the weak*-topology (this is trivial), we have that in \( X^* \) the weak* closure of a subspace if equal to its closure in the norm topology. Combining these two facts shows that \( [x_n^*] = X^* \).

**Chapter 8 (week of Monday March 27)**

This is the preliminary chapter on \( L_p \)-spaces, the study of which will be continued in the Chapters 9 and 10. The Chapter contains some basic material in the first two sections, and next two sections with some concrete results on \( L_p \)-spaces. In the first of these, “A Test for Disjointess”, it is shown that the only way to obtain an isometric copy of \( \ell_p \) in \( L_p \) is through a sequence of disjointly supported functions, as on page 35/36. The subspaces of \( L_p \) which are thus obtained are complemented by a norm one projection. The section “Conditional Expectation” shows how to create other subspaces which are complemented by a positive norm one projection and which sends 1 to one. In fact, this procedure of conditional expectations yields all such subspaces: see Theorem 8.7, which is not proved in the book.

Some detailed remarks:

In the first part on basic inequalities a number of results are collected, some of which are in itself not trivial. We will simply accepted them.

In the section on convex functions and Jensen’s inequality, it would have been informative if the result of item 111 in Hardy-Littlewood-Pólya had been included. This states that a convex function on an open interval has a right-hand side and a left-hand side derivative at every point, that at each point the right-hand side derivative is not less than the left-hand side derivative, and that both derivatives are increasing. It is then easy to see (as in the book) that there is a support line at every point—this is needed in the proof of Jensen’s inequality. (NB: if you want to look this differentiability property up in HLP, please note that their definition of a convex function is what could be called “midpoint-convex”. The definition of convexity in Carothers’ book requires more, and implies in particular that a convex function on an open interval is continuous (even locally Lipschitz), so that the hypotheses in item 111 in HLP are satisfied. A *continuous* mid-point convex function is convex—see page 95 in HLP.)
In equation (8.3), add that it is valid for all \( x \in I \).

In the proof of Lemma 8.2, both inequalities in \( ||x||_q \leq ||x||_p \leq 2^{1/p-1/q}||x||_q \) for \( 1 \leq p < q < \infty \) and \( x \in R^2 \) on page 74 are used.

On page 79, the sentence “What’s more, ... of this form.” is an incorrect reformulation of Theorem 8.7 (e.g., the positivity should be added).

Contrary to what is suggested in the Notes and Remarks on page 83, Theorem 8.5 is not the Marcinkiewicz Interpolation Theorem. You can look this theorem up in Lindenstrauss-Tzafriri II (page 149) or Wojtaszczyk (page 11) and see that this has to do with operators of weak type, which do not figure in Theorem 8.5. Furthermore, the condition in Theorem 8.5 that \( T \) is positive is superfluous. I suggest we disregard the proof of Theorem 8.5 in the book and base ourselves on the following (rather special) version of the Riesz-Thorin interpolation theorem (see Wojtaszczyk, page 11 or many other books, e.g., Rudin’s “Real and complex analysis”) which is certainly worth remembering:

**Theorem 3.** Let \((\Omega, \mu)\) be a measure space and let \( B \subset L_{p_1}(\Omega, \mu) \cap L_{p_2}(\Omega, \mu) \) be a linear set which is dense in both \( L_{p_i}(\Omega, \mu) \), \( i = 1, 2 \), with \( 1 \leq p_1 < p_2 \leq \infty \). Assume that we have a linear map \( T \) defined on \( B \) with values in the measurable functions on a measure space \((\Omega_1, \mu_1)\). Assume also that for every \( f \in B \) we have \( ||Tf||_{p_i} \leq C||f||_{p_i} \), for \( i = 1, 2 \). Then for every \( p \), \( p_1 \leq p \leq p_2 \) and \( f \in B \) we have \( ||Tf||_p \leq C||f||_p \), so \( T \) extends to a continuous linear operator from \( L_p(\Omega, \mu) \) to \( L_p(\Omega, \mu_1) \).

In our situation, we let \( T \) be the conditional expectation operator and we take \( B \) to be the span of the Borel characteristic functions. Certainly \( T \) is bounded on \( B \) for \( ||.||_1 \) and \( ||.||_\infty \) with norm one, as a consequence of the paragraphs preceding Theorem 8.5. By the Riesz-Thorin theorem, \( T \) extends for arbitrary \( p \) with norm one. To see that this extension is necessarily equal to the expectation operator as defined on \( L_1 \), now use that a convergent sequence in an \( L_p \)-space has a convergent subsequence which converges pointwise almost everywhere and which is almost everywhere dominated by a fixed \( L_p \)-function (which is then in \( L_1 \)), together with the defining equation (8.4) and the dominated convergence theorem. In this way we see that the conditional expectation operator is continuous on \( L_p \) of norm one, and since it is obviously idempotent we are done. Note that in this approach, the positivity is only used to get boundedness on \( L_1 \) and \( L_\infty \), setting the scene for Riesz-Thorin.

**Chapter 9 (week of Monday April 3)**

In this Chapter the study of \( L_p \)-spaces truly begins and the isomorphism problem for these spaces is completely solved (Corollary 9.13, see also the statement in italics below). The strategy consists of relating these spaces to \( \ell_p \)-subspaces, for which we already know quite a lot (and notably have Corollary 5.9). Of course, it is easy to find copies of \( \ell_p \) in \( L_p \) using disjointly supported sequences (see Lemma 4.3), but the surprising fact is that, loosely speaking, for \( 2 < p < \infty \), copies of \( \ell_p \) are virtually everywhere, with only \( \ell_2 \) figuring as a possible substitute (see Theorem 9.6).

The detailed remarks:

The claim on page 86 that, for the series involving Rademacher functions, pointwise almost everywhere convergence is equivalent to \( L_2 \)-convergence is not proved (but interesting).
On page 86, when hinting at a probabilistic interpretation for the Rademacher functions, it is meant that \([0, 1]\) is regarded as a measure space with Lebesgue measure as probability measure. The functions \(r_n\) then define random variables \(X_n\) on this measure space and it is these random variables that are mutually independent. As far as I can see, the only use of this fact is in giving an alternative proof for Khinchine’s inequality on page 89/90, which is then based on the unproved equality in the end of the first section which follows from this independence. I suggest that we disregard this material on independence, as well as the alternative proof of Khinchine’s inequality.

In the proof of Khinchine’s inequality, the case \(0 < p < 2\) is derived from the case \(p > 2\) using Liapounov’s inequality on page 74, which is stated there for the case \(1 \leq p, q < \infty\). In the proof it is, however, needed for all positive \(p\) and \(q = 4\). Fortunately, Liapounov’s inequality is actually valid in the wider range \(0 < p, q < \infty\), so that this application is valid after all.

I am not satisfied with the second part of the proof of Corollary 9.2, covering the case \(1 < p < 2\). What is meant is that one considers first \(P\) on \(L_q\), where the construction has already shown to be valid, and then dualizes to obtain \(P^*\) on the dual of \(L_q\), i.e., on \(L_p\). Surely \(P^*\) is a projection with the same norm as \(P\), but on what subspace does it project? Here the book is not clear. What has to be done is first note that \(P^*\) leaves each \(r_n\) fixed, so that the range of \(P^*\) contains \([r_n]\). Next, one shows that for each \(f \in L_p\), \(P^*f\) is the weak*-limit of elements of the algebraic span of the \(r_n\). Since \(L_q\) is reflexive, the weak*-closure and norm closure for convex sets in its dual \(L_p\) are the same, so that in fact \(P^*f\) is in \([r_n]\). That completes the proof that \(P^*\) projects onto \([r_n]\).

Corollary 9.3 should be strengthened, stating that for \(1 \leq p < \infty\) and \(p \neq 2\), \(L_p\) does not embed isomorphically into \(\ell_p\). The proof remains unchanged.

In the second and third part of Lemma 9.4, I prefer to assume that \(1 \leq q < p\), so that we have the inequalities on page 73 available, in particular \(||f||_q \leq ||f||_p\).

The \(\varepsilon\delta\)-statement in the begin of the proof of Lemma 9.5 is equivalent to the usual definition of absolute continuity in terms of sets of measure zero—look it up in, e.g., Rudin’s “Real and complex analysis” (Theorem 6.11). On page 94, it is used that the coordinate functionals \((x_n^*)\) for a basis \((x_n)\) with basis constant \(K\) satisfy \(||x_n^*|| \leq 2K/||x_n||\) (in this case \(K = 1\) since the \(g_n\) are disjointly supported).

In the second part of Theorem 9.6, the statement about being complemented follows from the fact that disjointly supported sequences in \(l_p\) span complemented subspaces (Lemma 4.3), together with the fact that being complemented is preserved under sufficiently small perturbations (part (ii) of Theorem 4.7).

In the proof of Corollary 9.8, it is remarked that \(X\) can be considered as a subspace of \(L_2\). While this is accurate it is also trivial since \(L_p \subset L_2\) for \(p \geq 2\). In fact, one needs this inclusion to understand why the projection \(P\) is defined on the whole of \(L_p\). The real point is that \(X\) is a closed subspace of \(L_2\) and that this enables one to define this projection in terms of an orthonormal basis of \(X\), which exists precisely because \(X\) is closed.

The content of Lemma 9.10 was already stated in the last paragraph on page 19.

Corollary 9.13 is true for all \(1 \leq p, q \leq \infty\). Obviously, \(L_1\) and \(L_\infty\) are not isomorphic to any \(L_p\) for any \(1 < p < \infty\), nor are they mutually isomorphic, for reasons of reflexivity and separability. For the remaining cases with finite \(p\) and \(q\) both larger than 1, the result follows from Corollary 9.12, Lemma 4.3 and the fact that different \(\ell_p\)-spaces do not contain any isomorphic subspaces (our improved
version of Corollary 5.9).

Note that the following picture has now emerged: among the spaces \( L_p \) and \( \ell_q \), for \( 1 \leq p, q \leq \infty \), the only isomorphisms are the isomorphisms between \( \ell_2 \) and \( L_2 \) and \( \ell_\infty \) and \( L_\infty \), the latter pairs in fact being isometric. Indeed, \( \ell_2 \) and \( L_2 \) are obviously isometric, and that the same holds for \( \ell_\infty \) and \( L_\infty \) is stated on page 91. That these are the only possible isomorphisms follows from our improved version of Corollary 5.9, Corollary 9.3, Corollary 9.13 and Lemma 4.3 (and the fact that \( \ell_\infty \) and \( L_\infty \) are not separable). Make a table and convince yourself!

Chapter 10 (week of Monday April 10)

This is the last chapter on the detailed study of \( L_p \)-spaces. Now that the isomorphism problem for spaces \( \ell_p \) and \( L_q \) has been solved, the embedding problem between these spaces is considered: which spaces \( \ell_q \) or \( L_q \) embed into \( \ell_p \) or \( L_p \) (four embedding problems in all)? We have seen a number of embeddings constructed in the book and also many possibilities being ruled out. With the material in this chapter also added, these four questions can now be answered, except for one not-at-all-trivial type of embedding, which is contained in the first part of the following result (see e.g. the recent book by Albiac and Kalton "Topics in Banach spaces theory", Theorem 6.4.19 and Proposition 11.1.9):

**Theorem 4.**

1. For \( 1 \leq p \leq 2 \), \( L_q \) embeds in \( L_p \) if and only if \( p \leq q \leq 2 \).

2. For \( 2 < p < \infty \), \( L_q \) embeds in \( L_p \) if and only if \( q = 2 \) or \( q = p \).

Moreover, if \( L_q \) embeds in \( L_p \) then it embeds isometrically and the same statements all hold for \( L_q \) replaced with \( \ell_q \).

Taking this into account, we have:

**Theorem 5.** The answer to the question which of the spaces \( \ell_q \) or \( L_q \) \((1 \leq q \leq \infty)\) embed into \( \ell_p \) or \( L_p \) \((1 \leq p \leq \infty)\) is as follows:

1. All spaces \( \ell_q \) and \( L_q \) \((1 \leq q \leq \infty)\) embed into both \( \ell_\infty \) and \( L_\infty \).

2. \( \ell_2 \) and \( L_2 \) embed into each other and into all spaces \( L_p \) \((1 \leq p \leq \infty)\).

3. \( \ell_p \) embeds into \( L_p \) for all \( 1 \leq p \leq \infty \).

4. For \( 1 \leq p \leq q \leq 2 \), \( L_q \) and \( \ell_q \) both embed in \( L_p \).

These possibilities exhaust all non-trivial embeddings. If an embedding exists, then there is an isometric embedding.

Some detailed remarks:

On page 101, to see why the supremum in (10.1) is finite, use that

\[
\sup_{\epsilon_i = \pm 1} \left\| \sum_{i=1}^{\infty} \epsilon_i x_i \right\| < \infty
\]

as a consequence of the compactness of \( \{-1, 1\}^\mathbb{N} \), together with (iv) in Theorem 10.1.
It is not so trivial that (v) on page 101 is equivalent to the other conditions in Theorem 10.1. If you want you can look it up in Megginson’s book (item 4.2.8)—we won’t need it however.

Note that Proposition 10.1 specializes to Khinchine’s inequality in Theorem 9.1 again if the \( f_i \) are constant.

In the proof of Theorem 10.6, the inequality in the first line follows from the fact that \( ||f||_1 \leq ||f||_p \). In that same proof, when concluding the case \( 1 \leq p \leq 2 \) it is used that for all non-negative \( x_1, \ldots, x_n \) and all \( 0 \leq \alpha \leq 1 \) one has \( (\sum_{i=1}^n x_i)^\alpha \leq \sum_{i=1}^n x_i^\alpha \), which can first be proved for \( n=2 \) using elementary calculus and then follows by induction.

Chapter 11 (week of Monday April 17)

This chapter starts afresh and concentrates on properties of the space which are tied up with the norm itself and not just with the topology. These properties (strict and uniform convexity—also called rotundity—and approximation properties) are therefore criteria for spaces to be not isometric, which of course still leaves open the possibility that they are isomorphic.

On page 109 under (c) it is mentioned that \( K \) is the intersection of two closed convex sets, hence closed and convex. If you want to understand why \( \{ f \in L_1 \mid f \geq 0 \} \) is closed, use that this set is equal to

\[
\{ f \in L_1 \mid \int_0^1 fg \, dx \geq 0 \text{ for all } g \in L_\infty \text{ such that } g \geq 0 \}.
\]

The example on page 110 under (d) of a normed space that is not strictly convex has no added value to the example of \( \mathbb{R}^2 \) under \( \| \cdot \|_\infty \) that was already given under (b) on page 108.

On page 111, the definition of extreme points is unusual. Generally, \( x \) is an extreme point of a set \( S \) if \( x \in S \) and \( x \) cannot be written as a non-trivial convex combination of elements of \( S \). It is only if one already knows that \( S \) is convex that this “midpoint”-definition in the book is equivalent to the correct definition.

Contrary to what is suggested, the last paragraph on page 111 does not contain the proof that an extreme point is necessarily exposed. It shows that the reverse implication holds, i.e., that (v) implies (iv) in Theorem 11.2.

So far, I have not found a reference for the statement on page 112 that there is always a nearest point in a weakly compact set.

In the proof of Theorem 116, Goldstine’s theorem is uses: the unit ball of \( X \) is dense in that of \( X^{**} \) in the weak*-topology. This provides the net as required. The rest of the proof is not too detailed; you will find more precision in Megginson’s book, item 5.2.15.

Theorem 11.12 follows immediately from Lemma 11.11 which provides the necessary pointwise inequality.

I don’t agree that the section on the dual of \( L_p \) contains an elementary proof of this standard result. If you read this section carefully, you will see that it is imperative to know that every functional attains its norm. In the “elementary” proof in the book, this follows from the fact that \( L_p \) is uniformly convex, which is a non-trivial result of Clarkson’s in itself. In addition, one has to show in Lemma 1.18 that functionals attain their norm in a unique point on the unit sphere, the proof
of which rests on Corollary 11.7 in turn. I would not say this is not elementary. It avoids measure theory, but at a substantial price.

Contrary to what is stated directly below Lemma 11.17, the limits are not directional derivatives, since \( f \) is not assumed to be of unit length. In the proof, one has to assume that \(|\langle T \rangle| \neq 0\) since otherwise the application of L'Hôpital's rule cannot be validated. Since the lemma is trivial in that case, this can indeed be done.

At the bottom of page 120, a reference is made to one of James' deep results on reflexivity, namely, that a Banach spaces is reflexive precisely when every continuous functional attains its norm on the unit ball. See item 1.13.16 in Megginson's book for a summary of a number of properties that are equivalent to reflexivity.

In the proof of Lemma 11.18, it should be noted first that \( T \) does in fact attain its norm on the unit sphere, since we know \( X \) to be reflexive (this easy part of James' equivalence follows from the Hahn-Banach theorem).

The proof of Lemma 11.19 is imprecise, since the function \( \varphi \) is of course not differentiable in \( t = 0 \) if \( a = 0 \). What has to be done first is to single out the set where \( g \) vanishes. On that part of the space, the limit in the integral is trivial and one has only to check that \( 0 = 0 \). On the remaining part of the space one works with the dominated convergence theorem. However, note that the condition which is stated to ensure that \( F \) is differentiable is not strong enough. The proof works though, although I did not quite see the reason for the final inequality on page 121. I used that
\[
|g(x) + tf(x)| \leq 2 \max(|f(x), g(x)|)
\]
and that this maximum is then again in \( L_p \), since this is a lattice.

Chapter 12 (week of Monday April 24)

The main prize in this chapter is the Banach-Mazur theorem in Corollary 12.14: every separable normed space is isometric to a subspace of \( C[0,1] \). Although this is an honest Banach space theory statement, the proof is mainly topological (have you noted the appendix on topology on page 166?), with only a small portion of functional analytic ingredients, as will become apparent when reading the chapter. The strategy for proving this result is as follows:

- Every normed space is isometric to a subspace of some space \( C(K) \) for a compact Hausdorff space \( K \) (Corollary 12.12). This is a trivial consequence of the Banach-Alaoglu theorem (Theorem 12.1).
- If the space is separable, then \( K \) can be chosen to be metrizable (Theorem 12.13).
- Now comes the surprising and relatively deep topological part: every compact metric space \( K \) is a quotient of the ternary Cantor set \( \Delta \) (Theorem 12.6).
- Pulling the functions back from \( K \) to \( \Delta \), we now know that every separable normed space is isometric to a subspace of \( C(\Delta) \).
- In the final and easy step, \( C(\Delta) \) is embedded isometrically in \( C[0,1] \) (Lemma 12.8).

Some detailed remarks:

In Corollary 12.1 is based on the fact that \( \Delta \simeq [0,1]^N \). Since \( N \) can be partitioned into countably infinite subsets, each of which is again countably infinite, one has
\[
\Delta \simeq ([0,1]^N)^N \simeq \Delta^N.
\]
Composing this homeomorphism with the \( N \)-fold product

12
of the surjection of $\Delta$ onto $[0, 1]$ from Lemma 12.1, the conclusion of the Lemma follows.

On page 126 in line 3, $e(x)$ should read $e(y)$.

In the proof of Theorem 12.4, it is stated that the Embedding Lemma shows that (b) implies (d). This is not correct, since the cube in (b) need not be related to $C(X; [0, 1])$. However, as is observed preceding Theorem 12.4, it is true that (a) implies (d), and as one checks this implication is also sufficient for the proof.

In the proof of Theorem 12.6, the point $y \in F$ that realizes the distance is unique because metric spheres consist of at most one point, as stated on page 124.

In Corollary 12.10, it is concluded from the embedding that $C(K)$ is separable if $K$ is compact metric. Although correct, this is not the way to prove this. The obvious and direct proof consists of introducing the functions $f_n(x) = d(x, x_n)$ where the $x_n$ range over a dense set. Together with the constant function one obtains an algebra to which the Stone-Weierstraß theorem applies, so that the algebra (with 1) which is generated over $\mathbb{Q}$ by the $f_n$ is dense in $C(K)$. This algebra is countable.

On page 131, just before Corollary 12.15, it is mentioned that $C[0, 1]^*$ can not be separable since $C[0, 1]$ contains a copy of $\ell_1$. While this conclusion would be valid if we knew that $C[0, 1]$ contained a complemented copy of $\ell_1$ (so that $C[0, 1]^*$ would contain a copy of $\ell_1^* = \ell_\infty$, which is not separable), I fail to see the implication as in the book. Fortunately, the non-separability of $C[0, 1]^*$ is completely elementary, so a possible proof based on the Banach-Mazur theorem would be somewhat suboptimal at any rate.

Corollary 12.16 is indeed an application of the Banach-Mazur theorem, together with Corollary 5.3 (see the penultimate paragraph on page 46).

Chapter 13 (week of Monday May 1)

After the first chapter on $C(K)$-spaces now follow two intermediate chapters. The first one of these, the current chapter, answers the question which subsets of an $L_1$-space are weakly compact. This is Theorem 13.6, in the proof of which the Eberlein-Smulian theorem is used. As an important by-product we obtain the Vitali-Hahn-Saks Theorem, Corollary 13.7, which states that the set-wise limit of measures is again a measure (i.e., is again countably additive).

A few words on the Eberlein-Smulian theorem: as we know, compactness and sequential compactness are unrelated notions in general topology. For metric spaces these are known to coincide and the same is true (and this is the content of the theorem) for subsets of a normed linear space in the weak topology. Likewise, relative compactness (i.e., compactness of the closure) and relative sequential compactness are also the same notions in the weak topology. You can look this up in many books, e.g., in Megginson’s as item 2.8.6.

Some detailed remarks:

In Example 1, mention is made of Chebyshev’s inequality. This inequality relates the measure of certain sets to the variance of a random variable, and personally I do not see what this has to do with this example. At any rate, it is immediate from the dominated convergence theorem that

$$\lim_{a \to \infty} \int \chi_{\{|f| > a\}} |f| \, dt = 0,$$

and this is precisely what we want. Likewise, mentioning this inequality in the last paragraph on page 137 seems superfluous, since the first inequality in the displayed
equation at that point is completely obvious from
\[ \int |f| \, dt \geq \int_{\{|f| > a\}} |f| \, dt. \]

On page 138, the sentence “In this setting, \( (\tilde{B}, d) \)” is not correct. Quite obviously, for a measure to descend to \( \tilde{B} \) in the first place, is has to be independent of the chosen representative and this is equivalent to requiring it to be absolutely continuous. If this condition is satisfied, so that the measure is well-defined on \( \tilde{B} \), then it is automatically continuous on \( (\tilde{B}, d) \) — see the reading guide concerning Chapter 9 for a remark on two equivalent characterizations of absolute continuity, from which we now use the other one.

In the proof of Theorem 13.6, when showing that uniform integrability of \( F \) implies weak compactness, it is remarked that it is sufficient to show that the weak*-closure of \( F \) is contained in \( L_1 \), when \( F \) is considered as a subset of the second dual of \( L_1 \) via the canonical embedding. When understanding this, it is helpful to realize two things:

- The topology that any subset \( S \) of a normed space \( X \) inherits from the weak topology of \( X \), is precisely the same as the topology it inherits, via the canonical embedding, from the weak*-topology of \( X^{**} \). Convince yourself that this is obvious!
- The weak*-closure of any norm-bounded set in \( X^{**} \) is always weak*-compact, according to Banach-Alaoglu.

With these ingredients and assuming that one has shown that the weak*-closure of \( F \) is contained in \( L_1 \), the argumentation is then complete.

Chapter 14 (week of Monday May 8)

The Dunford-Pettis property is concerned with simultaneous sequential continuity of the natural pairing between \( X \) and \( X^* \), when both spaces are supplied with the respective weak topologies. A noteworthy result is Corollary 14.4, which can be used to show that infinite dimensional spaces are not reflexive or that reflexive subspaces are not complemented.

Some detailed remarks:
In the proof of Theorem 14.4 I am not sure that that the first equality is correct. Fortunately, for the proof it is sufficient that
\[ \overline{T(B_X)}^{Y^{**}, \text{weak}^*} \subset \overline{T(B_X)}^{Y, \text{weak}} \]
and this is certainly correct, because the right hand side is weak*-compact, hence weak*-closed, and contains \( T(B_X) \).

The statement in the last two lines on page 143 about the existence of a subsequence \( (g_{n_k}) \) follows from the Eberlein-Smulian theorem, applied to \( X^* \) in its weak topology.

On page 145, in the proof of Theorem 14.6 it is claimed that the \( (\mu_n) \) are uniformly absolutely continuous with respect to \( \lambda \). To see this, I used not only the Propositions 14.3 and 14.4, but also the Riesz representation theorem and the Radon Nikodym theorem. If you think that the argumentation in the book is not complete at this point then I agree.
Here is Egorov’s theorem, which states that pointwise convergence implies uniform convergence on the complement of arbitrarily small sets. It is used in the proof of Theorem 14.6.

**Theorem 6.** Let \( X \) be a measurable space and \( \mu \) be a finite measure. If \((f_n)\) is a sequence of complex measurable functions which converges pointwise at every point of \( X \), and if \( \epsilon > 0 \), then there is a measurable set \( E \subset X \) with \( \mu(X - E) < \epsilon \), such that \((f_n)\) converges uniformly on \( E \).

Corollary 14.7 is formulated inaccurately. It should state that \( X \) embeds isometrically into \( C[0,1] \), but that there do not exist complemented isomorphic copies of \( X \) in \( C[0,1] \).

Here is a version of Lusin’s theorem, used in the proof of Proposition 14.8:

**Theorem 7.** Let \( X \) be a locally compact Hausdorff space and suppose \( \mu \) is a finite regular Borel measure on \( X \). If \( f \) is a complex measurable function, then for every \( \epsilon > 0 \) there exists a continuous and compactly supported function \( g \) such that \( f(x) = g(x) \) outside a set of measure at most \( \epsilon \).

Here one picks a continuous \( \tilde{g}_n \) such that \( \tilde{g}_n \) and \( g \) agree outside a set of measure at most \( \epsilon/2^n \) and then collects all exceptional sets into \( B \) (so that the \( g_n \) are not just continuous on \( B \), as stated on page 146, but on the whole of \([0,1] \), and equality holds everywhere on \( B \)).

The Lebesgue density theorem states that for any Lebesgue measurable set \( B \) on the real line one has

\[
\lim_{r \to 0} \frac{m((x-r, x+r) \cap B)}{2r} = 1
\]

for almost all \( x \) in \( B \) and that the limit is zero for almost all \( x \) not in \( B \). Of course, if this is true for all \( x \) in \( B \) (as one may assume), then \( B \) has no isolated points as is remarked in the book, but this is irrelevant. What is relevant is that one has

\[
\left| \sum_{k=1}^{n} a_k \tilde{g}_k(x) \right| \leq \text{ess. sup}_B \left| \sum_{k=1}^{n} a_k \tilde{g}_k(x) \right|
\]

for all \( x \) in \( B \). Indeed, denote the right hand side by \( L \) and suppose that the left hand side exceeds \( L \) for some \( x \in B \). By continuity, the left hand side still exceeds \( L \) in an open neighbourhood of \( x \), but this neighborhood has an intersection with \( B \) of positive measure by Lebesgue’s density theorem, contradicting the fact that \( L \) is an essential upper bound of the sum on \( B \).

**Chapter 15 (week of Monday May 15)**

Compact spaces are nicer to work with than non-compact spaces, so there is some relevance in studying the possibility of finding a compact space containing a copy of a given space. If the given space is, e.g., locally compact and Hausdorff, then the one-point compactification is a possible example, but there are many more “larger” spaces which are compact. Theorem 15.1 shows that for a completely regular space the situation is very satisfactory: there is a “largest” compactification (the Stone–Čech-compactification), where “largest” does not mean that other compactifications are a subspace, but that they are a quotient. This theorem and Corollary 15.2, are the main results in the chapter.
Some detailed remarks:

In both parts of Theorem 15.1 it should be added that the extension is unique. Note that part (a) follows from (b).

In Lemma 15.3 it should be required that $T$ is completely regular (else it does not make sense to speak about $\beta T$).

I don’t see the relevance of the examples 3 and 4 on page 152, but I trust and hope that they are correct.

Example 7 on page 153 is not only curious, it is circular: one cannot really prove a version of Tychonov’s theorem as a consequence of results on $\beta X$ when the compactness of this space follows from Tychonov’s theorem.

The first part of Corollary 15.5 follows immediately from the discussion on norming sets on page 60 (take point masses in a countable dense subset). Therefore, invoking the Stone-Cech compactification in this case is a valid way to conclude the existence of the embedding, but not the most efficient. The second part uses Banach-Alaoglu and the metrizability of the unit ball in the dual space in the weak* topology when the space is separable.

As to the discussion on page 154 and 155: since the decomposition of a functional into a positive and negative part is not proved, my advice is to simply accept the decomposition, disregard the remarks on Banach lattices for the time being and move on to the final chapter.

Chapter 16 (week of Monday May 22)

This chapter provides a not so well known type of proof for the Riesz representation theorem. I am not too happy with the presentation, so below follows my own outline of the proof.

First of all, disregard the material in the section “A Few Facts About $\beta N$”. It is irrelevant for the Riesz representation theorem. Second, the material between the proof of Lemma 16.5 and “The Dual of $\ell_\infty$” is useful to understand better the difference between the Baire and the Borel $\sigma$-algebras, but this also is not needed for the proof of Riesz’ theorem.

What is relevant will hopefully become clear from the following outline. It consists of first proving the theorem for $C(\beta D)$ with $D$ discrete and then deriving the general case from this special case.

To start with, let us improve our understanding of clopen sets in a completely regular space and it Stone-Cech compactification.

**Lemma 8.** Let $X$ be a completely regular space. If $A$ is clopen in $X$, let $\chi_A$ be its continuous characteristic function on $X$ with its unique continuous extension $\chi_{A,ext}$ to $\beta X$. Then $\chi_{A,ext}$ is the characteristic function of $cl_{\beta X}A$. In particular, $cl_{\beta X}A$ is clopen.

Try to prove this yourself or look it up (it should be somewhere in the literature).

From this, it is not too difficult to show that the algebra of clopen sets on a completely regular space $X$ is isomorphic to the algebra of clopen sets on $\beta X$. The isomorphism is given by sending $A$ to $cl_{\beta X}A$; its inverse is given by sending $S$ to $S \cap X$.

As a next step, if $D$ is discrete, then the simple functions based on clopen sets in $\beta X$ are dense in $C(\beta X)$. The reason for this is that this is true in $C(D) = \ell_\infty(D)$ and that $C(D)$ and $C(\beta D)$ are isometrically isomorphic by extension and restriction.
— note that, by the previous paragraphs, the simple functions based on clopen sets in $D$ have as extensions precisely the simple functions based on clopen sets in $\beta D$.

Now suppose $y^*$ is a positive continuous functional on $C(\beta D)$ with $D$ discrete. If $S \subset \beta D$ is clopen, define $\mu(S) = y^*(\chi_S)$ (the book applies this construction on $D$, but in this outline it is done without this detour). Then clearly $\mu$ is a finitely additive set function on the algebra of clopen subsets of $\beta D$.

The remarkable point is that, if $S$ is clopen and $S = \bigcup_{i=1}^{\infty} S_i$ for mutually disjoint clopen subsets $S_i$ of $\beta D$, then by compactness of $S$ the right hand is actually a finite union. So $\sigma$-additivity and finite additivity are the same notions on the algebra of clopen subsets of $\beta D$ and we conclude that $\mu$ is $\sigma$-additive on the halfring (even: algebra) of clopen subsets of $\beta D$. Therefore, the Carathéodory procedure can be applied and yields an extension of $\mu$ (denoted by $\mu$ again) to some $\sigma$-algebra of measurable subsets of $\beta D$. This $\sigma$-algebra contains the clopen sets by construction, so that it certainly contains the $\sigma$-algebra generated by the clopen sets. This, however, is precisely the Baire $\sigma$-algebra of $\beta D$, as is remarked on page 159 (stating this as a theorem would not have been inappropriate).

All in all, we now have a measure $\mu$ on (at least) the Baire $\sigma$-algebra of $\beta D$, such that

$$y^*(f) = \int_{\beta D} f(x) \, d\mu(x)$$

for all $f$ which are simple functions based on the clopen sets in $\beta D$. But the right hand side also makes sense now for arbitrary $f$ in $C(\beta D)$, since we have managed to extend the measure to the Baire $\sigma$-algebra, and then by continuity and density we have equality for arbitrary continuous functions on $\beta D$.

By now, we have the Riesz representation theorem for positive functionals on $\beta D$ with $D$ discrete. If $X$ is an arbitrary compact Hausdorff space, then Lemma 15.4 provides a discrete space $D$ and a continuous surjection $\phi : \beta D \to X$. The map $f \mapsto f \circ \phi$ gives an isometric injection of $C(X)$ into $C(\beta D)$, so that we may view a positive functional $x^*$ on $C(X)$ as a positive functional on this isometric copy. Let $y^*$ be a Hahn-Banach extension of this positive functional. It is a standard result (and not too difficult to prove) that $y^*$ is necessarily positive, and by the Riesz representation theorem for positive functionals on $C(\beta D)$ we see that there exists a measure $\mu$ on the Baire $\sigma$-algebra of $\beta D$ such that

$$x^*(f) = y^*(f \circ \phi) = \int_{\beta D} (f \circ \phi)(x) \, d\mu(x)$$

Now we push the restriction of the measure $\mu$ to the Baire $\sigma$-algebra of $\beta D$ forward to $X$ (this is a standard construction) by considering the $\sigma$-algebra $Z$ of all subsets $A$ of $X$ such that $\phi^{-1}(A)$ is in the Baire $\sigma$-algebra of $\beta D$ and defining $\nu(A) = \mu(\phi^{-1}(A))$ for such $A$. It is then a standard result that a $Z$-measurable function $f$ is integrable if and only if $f \circ \phi$ is $\mu$-integrable, and that in that case

$$\int_{\beta D} (f \circ \phi)(x) \, d\mu(x) = \int_{X} f(y) \, d\nu(y).$$

(This equality is true by construction for characteristic functions of elements of $Z$ and the rest follows from this by routine checks).

Taking the last two equations together, this proves the Riesz representation theorem for positive functionals on $X$, provided that we know that $Z$ contains the Baire $\sigma$-algebra (this is not checked in the book). However, by the third part of
Lemma 16.5, the Baire $\sigma$-algebra of $X$ is generated by the compact $G_\delta$’s and these certainly belong to $\mathcal{Z}$ since the preimage of a compact $G_\delta$ under $\phi$ is again a compact $G_\delta$, hence in the Baire $\sigma$-algebra of $\beta D$.

In this way one proves the representation theorem for positive functionals on $C(X)$. To get the full Riesz representation theorem, one has to decompose an arbitrary functional on $C(X)$ into a positive and negative part (preceded by taking its real and imaginary part in the complex case) and then combine the result. Also, the equality of the norm and the total variation is not obvious (the book does this only for the dual of $\ell_\infty(D)$ but does not remark that it is not trivial that this equality remains valid through the various following steps). Also, there is a uniqueness issue for regular measures to be taken care of. But all this is relatively easy to do, the hardest and surprising part of the Riesz representation theorem is to show that there is a Baire measure representing a positive functional on $C(X)$. The approach to this result as presented in this chapter is not the most efficient way to prove it, but it is certainly an interesting and remarkable one.