Homework Functional Analysis Seminar 2005-2006

Series 2

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1. It can be concluded from the material in Chapter 5 that distinct members of the family of spaces c_0 and ℓ_p $(1 \leq p < \infty)$ do not contain any isomorphic infinite dimensional closed subspaces, which is of course much stronger than the statement in Corollary 5.9 that they are pairwise non-isomorphic. How exactly does this follow?

2. (Exercise 1 of Chapter 6) In view of Schur's theorem 6.2, to show that L_1 and ℓ_1 are not isomorphic it is clearly sufficient to find a norm 1 sequence (f_n) in L_1 that converges weakly to zero. Establish such a sequence.

3. (Generalization of Exercise 6 of Chapter 6) Prove that every bounded linear operator from a reflexive space into ℓ_1 is compact. Hints: what does the Banach-Alaoglu theorem tell you about the unit ball in a reflexive space? And then what does the Eberlein-Smulian theorem (look this up) imply? Of course, you also have to use Theorem 6.2.

4. (Slight extension of Exercise 1 of Chapter 8) Let $1 \leq p, q \leq \infty$ and define $r \in [\frac{1}{2}, \infty]$ by $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. If $f \in L_p(\mu)$ and $g \in L_q(\mu)$, show that $fg \in L_r(\mu)$ and that $||fg||_r \leq |||f||_p||g||_q$. Here $||\,.\,||_r$ is defined in the obvious fashion for $0 < r < 1$, analogous to the case $r \in [1, \infty)$ (but it is then not a norm).

5. (Exercise 1 of Chapter 7) Let (e_n) denote the usual basis of ℓ_1. We know that $(e_n^*) \subset \ell_\infty$ is a basis for $[e_n^*]$. Determine this subspace $[e_n^*]$ of ℓ_∞.

6. Prove that a reflexive Banach space which has a Schauder basis must have a separable dual. Using this, prove that $C[0, 1]$ is not reflexive.

7. (Exercise 8 and 9 of Chapter 8) In the notation of Chapter 8, if $\mathcal{B}_0$ is a sub-σ-algebra of $\mathcal{B}$, prove that $\mathbb{E}(\,\cdot \mid \mathcal{B}_0) : L_1(\mathcal{B}) \to L_1(\mathcal{B}_0)$ is both linear and positive and show how it follows from this that $|\mathbb{E}(f \mid \mathcal{B}_0)| \leq \mathbb{E}(|f| \mid \mathcal{B}_0)$ for every $f \in L_1(\mathcal{B})$. In addition, prove that $\mathbb{E}(\cdot \mid \mathcal{B}_0)$ is a projection from $L_p(\mathcal{B})$ onto $L_p(\mathcal{B}_0)$ for all $p \in [1, \infty]$.
