Quotients

Notation 1. X and Y denote Banach spaces, $L \leq X$ denotes a closed subspace $L \subset X$, $\operatorname{ball} X = \{x \in X : ||x|| < 1\}$; the open unit ball of X. If $S \subset X$ is a subset and $\lambda \in \mathbb{C}$, then $\lambda S = \{\lambda s : s \in S\}$.

Proposition 2. *$T : X \to Y$ is an isometric isomorphism if and only if T is bijective and $T(\operatorname{ball} X) = \operatorname{ball} Y$.*

Proof. If $T : X \to Y$ is an isometric isomorphism, then clearly Y is bijective and $T(\operatorname{ball} X) = \operatorname{ball} Y$.
For the reverse implication, suppose that T is bijective and $T(\operatorname{ball} X) = \operatorname{ball} Y$. Suppose $x \in X, ||x|| = \lambda$, then there is a sequence $(x_n) \subset \lambda \cdot \operatorname{ball} X$ with $x_n \to x$. Since $||Tx_n|| < \lambda$ for all n, we see that $||Tx|| \leq \lambda = ||x||$. Note that $T^{-1}(\operatorname{ball} Y) = \operatorname{ball} X$, so in the same way we see that $||x|| = ||T^{-1}Tx|| \leq ||T^{-1}|| \cdot ||Tx|| \leq ||Tx||$. Hence $||x|| = ||Tx||$, i.e., T is an isometric isomorphism. □

It is easy to see the following.

Proposition 3. *If $L \leq X$, and $q : X \to X/L$ is the quotient map, then $q(\operatorname{ball} X) = \operatorname{ball} X/L$.*

This is used in the proof of the following result.

Proposition 4. *Y is isometrically isomorphic with a quotient of X if and only if there is a linear map $T : X \to Y$ with $T(\operatorname{ball} X) = \operatorname{ball} Y$.*

Proof. Suppose that Y is isometrically isomorphic with a quotient of X, then there is $L \leq X$ and an isometric isomorphism $\psi : X/L \to Y$. Let $T = \psi \circ q$. Then by Proposition 3 and Proposition 2, $T(\operatorname{ball} X) = \psi \circ q(\operatorname{ball} X) = \psi(\operatorname{ball} X/L) = \operatorname{ball} Y$.
On the other hand, suppose that there is a linear map $T : X \to Y$ with $T(\operatorname{ball} X) = \operatorname{ball} Y$. So T is surjective. Let $L = \ker T$. Let $S : X/L \to Y$ be defined through $S[x] = Tx$, then S is a well defined bijective linear operator, and $T = S \circ q$. By Proposition 3 and the assumption, $\operatorname{ball} Y = T(\operatorname{ball} X) = S \circ q(\operatorname{ball} X) = S(\operatorname{ball} X/L)$ Hence by Proposition 2, S is an isometric isomorphism. □

Theorem 5. *Suppose S is dense in $\subset \operatorname{ball} X$. Then X is isometrically isomorphic with a quotient of $\ell^1(S)$, the space of absolutely summable functions on S.*

Proof. By Proposition 4 it is enough to prove that there is a linear operator $T : \ell^1(S) \to X$, such that $T(\operatorname{ball} \ell^1(S)) = \operatorname{ball} X$.
Define $T : \ell^1(S) \to X$ through $T(l) = \sum_{s \in S} l_s \cdot s$. Since $||s|| < 1$ for all $s \in S$, it is clear that T is well defined and that $T(\operatorname{ball} \ell^1(S)) \subset \operatorname{ball} X$.
It is only left to prove that $\operatorname{ball} X \subset T(\operatorname{ball} \ell^1(S))$, and for this we establish the following auxiliary result.

Claim 6. *Let $x \in \operatorname{ball} X$ and $\varepsilon > 0$. Then there exists a sequence $s_0, s_1, s_2, .. \subset S$ such that, for every $N \geq 0$,*

$$\left|\left| x - \sum_{k=0}^{N} \varepsilon^k s_k \right|\right| < \varepsilon^{N+1}.$$

Proof. We prove this by induction. Since S is dense in $\operatorname{ball} X$, there is an $s_0 \in S$ such that $||x - s_0|| < \varepsilon$. This proves the claim for $N = 0$. Suppose that $N > 0$, and that the claim is true for $N - 1$. Then we have

$$\left|\left| x - \sum_{k=0}^{N-1} \varepsilon^k s_k \right|\right| < \varepsilon^N,$$

for certain $s_0, .., s_{N-1} \in S$. Since $\varepsilon^N S$ is dense in $\varepsilon^N \operatorname{ball} X$ and $x - \sum_{k=0}^{N-1} \varepsilon^k s_k \in \varepsilon^N \operatorname{ball} X$ there is a $s_N \in S$, such that

$$\left|\left| x - \sum_{k=0}^{N-1} \varepsilon^k s_k - \varepsilon^N s_N \right|\right| = \left|\left| x - \sum_{k=0}^{N} \varepsilon^k s_k \right|\right| < \varepsilon^{N+1}.$$

This establishes the induction step. □

Hence, if $x \in \operatorname{ball} X$ and $0 < \epsilon < 1$, one can write $x = \sum_{k=0}^{\infty} \varepsilon^k s_k$ for some sequence $\{s_k\}_{k=0}^{\infty} \subset S$ of mutually different elements. Let $l \in \ell^1(S)$ be defined through $l_s = \sum_{k:s_k=s} \varepsilon^k$. Note that l is well defined, and that $l \in \ell^1(S)$: in fact $||l|| = \sum_{s \in S} |l_s| = \sum_{k=0}^{\infty} \varepsilon^k = \frac{1}{1-\varepsilon}$. Furthermore, $T(l) = \sum_{s \in S} l_s s = \sum_{k=0}^{\infty} \varepsilon^k s_k = x$. Note that, since $\varepsilon \in (0,1)$ can be chosen freely, we can take the limit $\varepsilon \downarrow 0$. All in all we conclude that, if $x \in \operatorname{ball} X$, then for every $t > 1$ there is an $l^t \in \ell^1(S)$ with $T(l^t) = x$ and $||l^t|| < t$.
Now let $y \in \operatorname{ball} X$. Choose and fix $\lambda > 1$ such that $||\lambda y|| < 1$. Then, for every $t > 1$, there is an $l^t \in \ell^1(S)$ with $T(l^t) = \lambda y$ and $||l^t|| < t$. So

$$y = T\left(\frac{l^t}{\lambda}\right) \text{ and } \left|\left|\frac{l^t}{\lambda}\right|\right| < \frac{t}{\lambda}.$$

Hence for $1 < t < \lambda$ we have $y = T\left(\frac{l^t}{\lambda}\right)$ and $\frac{l^t}{\lambda} \in \operatorname{ball} \ell^1(S)$. Therefore $T(\operatorname{ball} \ell^1(S)) = \operatorname{ball} X$, as required. □

Corollary 7. *Suppose that X is separable. Then there exists $L \leq \ell^1$ such that X and ℓ^1/L are isometrically isomorphic.*

Corollary 8. *There exists $L \leq l^1(\operatorname{ball} X)$ such that $l^1(\operatorname{ball} X)/L$ and X are isometrically isomorphic.*