Exercise 1 *A von Neumann algebra $\mathcal{A}$ on a Hilbert space H is called* maximal abelian *if $\mathcal{A}$ is not properly contained in any larger abelian von Neumann algebra (equivalently, $\mathcal{A}$ is not properly contained in any larger abelian $*$-algebra).*

Show that a $*$-algebra $\mathcal{A} \subseteq B(H)$ is a maximal abelian von Neumann algebra if and only if $\mathcal{A} = \mathcal{A}'$.

Exercise 2 *Let (X, Σ, μ) be a σ-finite measure space and $L_\infty(\mu)$ be the algebra of all essentially bounded μ-measurable complex valued functions on X (with the usual identification of μ-almost equal functions), equipped with the essential supremum norm $\|\cdot\|_\infty$. Note that $L_\infty(\mu)$ is a C^*-algebra (with complex conjugation as $*$-operation). For each $f \in L_\infty(\mu)$ define the linear operator m_f on $L_2(\mu)$ by setting*

$$m_f(g) = fg, \quad g \in L_2(\mu).$$

(a). Show that $m_f \in B(L_2(\mu))$ and $\|m_f\|_{B(L_2(\mu))} = \|f\|_\infty$ for all $f \in L_\infty(\mu)$.

Define the $*$-subalgebra $\mathcal{A}$ of $B(L_2(\mu))$ by setting

$$\mathcal{A} = \{m_f : f \in L_\infty(\mu)\}.$$

(b). Show that $\mathcal{A}$ is a maximal abelian von Neumann algebra.

Hint: *consider first the case that $\mu(X) < \infty$.*

Remark *Usually, the $*$-algebra $L_\infty(\mu)$ is identified with the algebra $\mathcal{A}$ of multiplication operators.*

Exercise 3 *Let H_1 and H_2 be two Hilbert spaces. The algebraic tensor product (over $\mathbb{C}$) of H_1 and H_2 is as usual denoted by $H_1 \otimes H_2$. Note that every element $\zeta \in H_1 \otimes H_2$ may be written as $\zeta = \sum_{j=1}^n \xi_j \otimes \varepsilon_j$, where $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is an orthonormal system in H_2 and $\xi_j \in H_1$ $(j = 1, \ldots, n)$. There exists a unique inner product $\langle \cdot, \cdot \rangle$ in $H_1 \otimes H_2$ satisfying*

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1, \xi_2 \rangle \langle \eta_1, \eta_2 \rangle, \quad \xi_1, \xi_2 \in H_1, \eta_1, \eta_2 \in H_2.$$

The corresponding norm on $H_1 \otimes H_2$ is denoted by $\|\cdot\|_{H_1 \otimes H_2}$. Note that $\|\xi \otimes \eta\|_{H_1 \otimes H_2} = \|\xi\|_{H_1} \|\eta\|_{H_2}$ for all $\xi \in H_1$ and $\eta \in H_2$. The completion of *the inner product space* $(H_1 \otimes H_2, \langle \cdot, \cdot \rangle)$ *is denoted by* $H_1 \overline{\otimes} H_2$ *and is called the* Hilbert space tensor product *of* H_1 *and* H_2.

If $x \in B(H_1)$ *and* $y \in B(H_2)$, *then there exists a unique linear operator*

$$x \otimes y : H_1 \otimes H_2 \to H_1 \otimes H_2$$

satisfying $(x \otimes y)(\xi \otimes \eta) = (x\xi) \otimes (y\eta)$, $\xi \in H_1$, $\eta \in H_2$. *Note that* $\mathbf{1}_{H_1} \otimes \mathbf{1}_{H_2} = \mathbf{1}_{H_1 \otimes H_2}$.

(a). Show that the operator $x \otimes y$ *is bounded with respect to* $\|\cdot\|_{H_1 \otimes H_2}$ *for all* $x \in B(H_1)$ *and* $y \in B(H_2)$, *and satisfies*

$$\|x \otimes y\|_{B(H_1 \otimes H_2)} = \|x\|_{B(H_1)} \|y\|_{B(H_2)}.$$

If $x \in B(H_1)$ *and* $y \in B(H_2)$, *then the operator* $x \otimes y$ *extends uniquely to a bounded linear operator in* $H_1 \overline{\otimes} H_2$. *This extension is also denoted by* $x \otimes y$ *and satisfies* $\|x \otimes y\|_{B\left(H_1 \overline{\otimes} H_2\right)} = \|x\|_{B(H_1)} \|y\|_{B(H_2)}$. *It should be noted that* $(x_1 \otimes y_1)(x_2 \otimes y_2) = (x_1 x_2) \otimes (y_1 y_2)$, $x_1, x_2 \in B(H_1)$, $y_1, y_2 \in B(H_2)$, *and that* $(x \otimes y)^* = x^* \otimes y^*$, $x \in B(H_1)$, $y \in B(H_2)$ *(Prove!).*

If $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are von Neumann algebras on* H_1 *and* H_2, *respectively, then the unital* $*$-*subalgebra* $\mathcal{M}_1 \otimes \mathcal{M}_2$ *of* $B(H_1 \overline{\otimes} H_2)$ *is defined by setting*

$$\mathcal{M}_1 \otimes \mathcal{M}_2 = \left\{ \sum_{j=1}^{n} x_j \otimes y_j : x_j \in \mathcal{M}_1, y_j \in \mathcal{M}_2, 1 \leq j \leq n; n \in \mathbb{N} \right\}.$$

The von Neumann algebra on $H_1 \overline{\otimes} H_2$, *generated by* $\mathcal{M}_1 \otimes \mathcal{M}_2$ *is called the* von Neumann algebra tensor product *of* $\mathcal{M}_1$ *and* $\mathcal{M}_2$, *and is denoted by* $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$.

(b). Show that $(\mathbb{C}\mathbf{1}_{H_1} \otimes B(H_2))' = B(H_1) \otimes \mathbb{C}\mathbf{1}_{H_2}$.

Hint. *The inclusion* $B(H_1) \otimes \mathbb{C}\mathbf{1}_{H_2} \subseteq (\mathbb{C}\mathbf{1}_{H_1} \otimes B(H_2))'$ *is easy. Suppose that* $z \in (\mathbb{C}\mathbf{1}_{H_1} \otimes B(H_2))'$. *Fix* $\varepsilon_0 \in H_2$ *with* $\|\varepsilon_0\|_{H_2} = 1$ *and let*

$$H_1 \otimes \varepsilon_0 = \{\xi \otimes \varepsilon_0 : \xi \in H_1\}.$$

Define $u_0 : H_1 \to H_1 \overline{\otimes} H_2$ *by* $u_0 \xi = \xi \otimes \varepsilon_0$, $\xi \in H_2$, *which is a linear isometry from* H_1 *onto* $H_1 \otimes \varepsilon_0$. *Show that* $u_0^* u_0 = \mathbf{1}_{H_1}$ *and that* $u_0 u_0^*$ *is the orthogonal projection in* $H_1 \overline{\otimes} H_2$ *onto* $H_1 \otimes \varepsilon_0$. *Note that* $u_0 u_0^* = \mathbf{1}_{H_1} \otimes e_0$, *where* e_0 *is the orthogonal projection in* H_2 *onto* $[\varepsilon_0]$ *(and hence,* $(u_0 u_0^*) z = z (u_0 u_0^*)$*). Define* $x \in B(H_1)$ *by* $x = u_0^* z u_0$ *and show that*

$$(x \otimes \mathbf{1}_{H_2})(\xi \otimes \varepsilon_0) = z(\xi \otimes \varepsilon_0), \quad \xi \in H_1.$$

Prove next that $z = x \otimes \mathbf{1}_{H_2}$*. Note that it is sufficient to show that* $z(\xi \otimes \varepsilon) = (x \otimes \mathbf{1}_{H_2})(\xi \otimes \varepsilon)$ *for all* $\xi \in H_1$ *and all* $\varepsilon \in H_2$ *with* $\|\varepsilon\|_{H_2} = 1$*.*

(c). Show that $B(H_1) \otimes \mathbb{C}\mathbf{1}_{H_2}$ *is a von Neumann algebra, that is,*

$$B(H_1) \overline{\otimes} \mathbb{C}\mathbf{1}_{H_2} = B(H_1) \otimes \mathbb{C}\mathbf{1}_{H_2}$$

(and similarly, $\mathbb{C}\mathbf{1}_{H_1} \overline{\otimes} B(H_2) = \mathbb{C}\mathbf{1}_{H_1} \otimes B(H_2)$*).*

(d). Suppose that $\mathcal{M}_1$ *is a von Neumann algebra on* H_1*. Show that*

$$(\mathcal{M}_1 \overline{\otimes} B(H_2))' = \mathcal{M}_1' \otimes \mathbb{C}\mathbf{1}_{H_2}.$$

(e). Conclude that, in particular, $B(H_1) \overline{\otimes} B(H_2) = B(H_1 \overline{\otimes} H_2)$*.*

Exercise 4 *Consider the interval* $[0,1]$ *equipped with Lebesgue measure* λ*. The corresponding spaces* $L_\infty(\lambda)$ *and* $L_2(\lambda)$ *will be denoted simply by* L_∞ *and* L_2*, respectively. Suppose that* H *is a separable Hilbert space. A function* $f : [0,1] \to H$ *is called measurable if for every* $\eta \in H$ *the function* $t \longmapsto \langle f(t), \eta \rangle$*,* $t \in [0,1]$*, is measurable. The space* $L_2([0,1], H)$ *consists of all measurable functions* $f : [0,1] \to H$ *for which*

$$\|f\|_2 = \left(\int_0^1 \|f(t)\|_H^2 \, dt \right)^{1/2} < \infty.$$

Identifying functions which are equal a.e., the space $L_2([0,1], H)$ *is a Hilbert space with respect to the inner product* $\langle \cdot, \cdot \rangle$ *given by*

$$\langle f, g \rangle = \int_0^1 \langle f(t), g(t) \rangle \, dt, \qquad f, g \in L_2([0,1], H).$$

Via the map which sends $h \otimes \xi$ *to the function* $t \longmapsto h(t)\xi$*,* $t \in [0,1]$ *(*$h \in L_2$*,* $\xi \in H$*), the Hilbert space tensor product* $L_2 \overline{\otimes} H$ *may be identified with* $L_2([0,1], H)$*.*

Definition 5 *A function* $F : [0,1] \to B(H)$ *is called* weak operator measurable *(wo-measurable) if for all* $\xi, \eta \in H$ *the scalar function* $t \longmapsto \langle F(t)\xi, \eta \rangle$ *is measurable.*

The collection of all bounded wo-measurable functions from $[0,1]$ *into* $B(H)$ *is denoted by* $\mathcal{L}_{\infty}^{wo}([0,1],B(H))$. *For* $F \in \mathcal{L}_{\infty}^{wo}([0,1],B(H))$ *define* $F^* \in \mathcal{L}_{\infty}^{wo}([0,1],B(H))$ *by setting* $F^*(t) = F(t)^*$, $t \in [0,1]$. *With respect to the pointwise operations,* $\mathcal{L}_{\infty}^{wo}([0,1],B(H))$ *is a unital* $*$*-algebra.*

A wo-measurable function $F : [0,1] \to B(H)$ *is called a* null function *if* $\langle F(t)\xi, \eta\rangle = 0$ *a.e. on* $[0,1]$ *for all* $\xi, \eta \in H$ *(equivalently,* $F(t) = 0$ *a.e. on* $[0,1]$*; prove!). The set of all null functions in* $\mathcal{L}_{\infty}^{wo}([0,1],B(H))$ *is denoted by* $\mathcal{N}_{\infty}$*, which is a* $*$*-closed two-sided ideal in* $\mathcal{L}_{\infty}^{wo}([0,1],B(H))$. *Let*

$$L_{\infty}^{wo}([0,1],B(H)) = \mathcal{L}_{\infty}^{wo}([0,1],B(H)) \diagup \mathcal{N}_{\infty}$$

and define

$$\|F\|_{\infty} = \operatorname*{ess\,sup}_{t\in[0,1]} \|F(t)\|_{B(H)}, \quad F \in L_{\infty}^{wo}([0,1],B(H)).$$

(a). Show that $L_{\infty}^{wo}([0,1],B(H))$*, equipped with the norm* $\|\cdot\|_{\infty}$*, is a* C^**-algebra.*

The algebraic tensor product $L_{\infty} \otimes B(H)$ *may be identified with a subalgebra of* $L_{\infty}^{wo}([0,1],B(H))$ *(via the map which sends the element* $h \otimes x$ *to the function* $t \longmapsto f(t)x$, $t \in [0,1]$*;* $h \in L_{\infty}$, $x \in B(H)$*).*

(b). Show that if $F : [0,1] \to B(H)$ *is wo-measurable and* $g : [0,1] \to H$ *is measurable, then the function* $t \longmapsto F(t)g(t)$, $t \in [0,1]$*, is measurable.*

If $F \in L_{\infty}^{wo}([0,1],B(H))$ *and* $g \in L_2([0,1],H)$*, then the function* $t \longmapsto F(t)g(t)$, $t \in [0,1]$*, is measurable and*

$$\int_0^1 \|F(t)g(t)\|_H^2 \, dt \leq \|F\|_{\infty}^2 \|g\|_2^2.$$

Consequently, the linear map $x_F : L_2([0,1],H) \to L_2([0,1],H)$ *is bounded and satisfies* $\|x_F\|_{B(L_2([0,1],H))} \leq \|F\|_{\infty}$.

(c). Show that if $F \in L_{\infty}^{wo}([0,1],B(H))$*, then the map* $x_F : g \longmapsto Fg$, $g \in L_2([0,1],H)$*, is a bounded linear map from* $L_2([0,1],H)$ *into itself and* $\|x_F\|_{B(L_2([0,1],H))} = \|F\|_{\infty}$.

Hint. *First show that, for every* $\xi \in H$ *and every measurable set* $A \subseteq [0,1]$*,*

$$\int_A \|F(t)\xi\|_H^2 \, dt \leq \|x_F\|_{B(L_2)}^2 \|\xi\|_H^2 \, \lambda(A)$$

and conclude from this that $\|F(t)\xi\|_H \leq \|x_F\|_{B(L_2)} \|\xi\|_H$*, a.e. on* $[0,1]$ *(where the exceptional set may depend on* ξ*). Now use that* H *is separable.*

The map $F \longmapsto x_F$, $F \in L_\infty^{wo}([0,1], B(H))$, identifies the C^*-algebra $L_\infty^{wo}([0,1], B(H))$ with a C^*-subalgebra of $B(L_2([0,1], H))$.

Identifying $L_2([0,1], H)$ with the Hilbert space tensor product $L_2 \overline{\otimes} H$, the von Neumann algebra tensor product $L_\infty \overline{\otimes} B(H)$ is also a von Neumann algebra on $L_2([0,1], H)$.

(d). Prove that $L_\infty^{wo}([0,1], B(H)) \subseteq L_\infty \overline{\otimes} B(H)$.

Hint. *It is sufficient to show that $(L_\infty \overline{\otimes} B(H))' \subseteq L_\infty^{wo}([0,1], B(H))'$.*

(e). Show that if $z \in B(L_2([0,1]), H)$ satisfies $z(h \otimes \mathbf{1}_H) = (h \otimes \mathbf{1}_H) z$ for all $h \in L_\infty$, then there exists $F \in L_\infty^{wo}([0,1], B(H))$ such that $z = x_F$.

Hint. *This is a bit tricky. Let $\{\varepsilon_j\}_{j=1}^\infty$ be an orthonormal basis in H and let W be the $\mathbb{Q}+i\mathbb{Q}$-linear span of $\{\varepsilon_j\}$. Note that W is a dense countable $\mathbb{Q}+i\mathbb{Q}$-linear subspace of H. For each j, let $f_j = z(\mathbf{1} \otimes \varepsilon_j)$ and choose a specific representative for each f_j (also be denoted by f_j). If $\xi \in W$, then ξ has a unique representation as $\xi = \sum_{j=1}^n \alpha_j \varepsilon_j$, with $\alpha_j \in \mathbb{Q}+i\mathbb{Q}$. Define the function $g_\xi : [0,1] \to H$ by setting*

$$g_\xi(t) = \sum_{j=1}^n \alpha_j f_j(t), \quad t \in [0,1].$$

Observe that $g_{\lambda\xi+\mu\eta} = \lambda g_\xi + \mu g_\eta$ for all $\xi, \eta \in W$ and $\lambda, \mu \in \mathbb{Q}+i\mathbb{Q}$. Furthermore, $g_\xi = z(\mathbf{1} \otimes \xi)$ for all $\xi \in W$. If $\xi \in W$ and $h \in L_\infty$, then $hg_\xi = z(h \otimes \xi)$ and so,

$$\|hg_\xi\|_2 \leq \|z\|_{B(L_2)} \|h \otimes \xi\|_2 .$$

Conclude that

$$\|g_\xi(t)\|_H \leq \|z\|_{B(L_2)} \|\xi\|_H , \quad a.e. \text{ on } [0,1].$$

Consequently, for each $\xi \in W$ there exists a set $A_\xi \subseteq [0,1]$ such that

$$\|g_\xi(t)\|_H \leq \|z\|_{B(L_2)} \|\xi\|_H , \quad t \in A_\xi, \quad \lambda([0,1] \setminus A_\xi) = 0.$$

Defining $A = \bigcap_{\xi \in W} A_\xi$, it follows that

$$\|g_\xi(t)\|_H \leq \|z\|_{B(L_2)} \|\xi\|_H , \quad t \in A, \xi \in W, \quad \lambda([0,1] \setminus A) = 0.$$

For $t \in A$, define the map $F_0(t) : W \to H$ by setting

$$F_0(t)\xi = g_\xi(t), \quad \xi \in W.$$

The map $F_0(t)$ is $\mathbb{Q}+i\mathbb{Q}$-linear and continuous. Hence, $F_0(t)$ extends uniquely to a bounded linear map $F(t) \in B(H)$ satisfying $\|F(t)\|_{B(H)} \leq \|z\|_{B(L_2)}$. Setting $F(t) = 0$ for all $t \in [0,1] \diagdown A$, this defined a function $F : [0,1] \to B(H)$.

If $\eta \in H$, then

$$\langle F(t)\varepsilon_j, \eta\rangle = \langle F_0(t)\varepsilon_j, \eta\rangle = \langle f_j(t), \eta\rangle, \quad t \in A,$$

which shows that the function $t \longmapsto \langle F(t)\varepsilon_j, \eta\rangle$, $t \in [0,1]$, is measurable for each j. Since $\{\varepsilon_j\}$ is an orthonormal basis in H, this implies that F is wo-measurable. Consequently, $F \in L_\infty^{wo}([0,1], B(H))$ and $\|F\|_\infty \leq \|z\|_{B(L_2)}$. Finally, prove that $z = x_F$.

(f). Show that $L_\infty^{wo}([0,1], B(H)) = L_\infty \overline{\otimes} B(H)$.

Suppose that $\mathcal{M}$ is a von Neumann algebra on the Hilbert space H and define

$$L_\infty^{wo}([0,1], \mathcal{M}) = \{F \in L_\infty^{wo}([0,1], B(H)) : F(t) \in \mathcal{M} \quad \text{a.e. on } [0,1]\},$$

which is a C^*-subalgebra of $L_\infty^{wo}([0,1], B(H))$.

(g). Prove that $L_\infty^{wo}([0,1], \mathcal{M}) = L_\infty \overline{\otimes} \mathcal{M}$.

***Hint.** First show that $(L_\infty \overline{\otimes} \mathcal{M})' \subseteq L_\infty^{wo}([0,1], \mathcal{M}')$.*

Exercise 6 (Examples of type II$_1$ factors) *Let G be a group with unit element e. Let $\ell_2(G)$ be the space of all square summable functions $\xi : G \to \mathbb{C}$, that is,*

$$\ell_2(G) = \left\{\xi : G \to \mathbb{C} : \sum\nolimits_{g \in G} |\xi(g)|^2 < \infty\right\},$$

which is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle \xi, \eta\rangle = \sum\nolimits_{g \in G} \xi(g)\overline{\eta(g)}, \quad \xi, \eta \in \ell_2(G).$$

The norm in $\ell_2(G)$ is denoted by $\|\cdot\|_2$. For each $g \in G$, the element $\delta_g \in \ell_2(G)$ is defined by setting $\delta_g(g) = 1$ and $\delta_g(h) = 0$ whenever $h \in G$ and $h \neq g$. The set $\{\delta_g : g \in G\}$ is an orthonormal basis in $\ell_2(G)$. Note that $\langle \xi, \delta_g\rangle = \xi(g)$, $g \in G$, for all $\xi \in \ell_2(G)$.

(a). Show that for all $\xi, \eta \in \ell_2(G)$ and $h \in G$, the series $\sum_{g\in G} \xi(hg^{-1})\eta(g)$ is absolutely convergent and

$$\left|\sum_{g\in G} \xi\left(hg^{-1}\right)\eta(g)\right| \leq \|\xi\|_2 \|\eta\|_2 .$$

(b). It follows from (a) that for all $\xi, \eta \in \ell_2(G)$ the function $\xi * \eta : G \to \mathbb{C}$, given by

$$(\xi * \eta)(h) = \sum_{g\in G} \xi\left(hg^{-1}\right)\eta(g), \quad h \in G,$$

is well defined, $\xi * \eta \in \ell_\infty(G)$ and

$$\|\xi * \eta\|_\infty \leq \|\xi\|_2 \|\eta\|_2, \quad \xi, \eta \in \ell_2(G).$$

Note that $(\delta_g * \eta)(h) = \eta(g^{-1}h)$ and $(\eta * \delta_g)(h) = \eta(hg^{-1})$ for all $g, h \in G$.

(c). For $\xi \in \ell_2(G)$ define the linear maps $L_\xi, R_\xi : \ell_2(G) \to \ell_\infty(G)$ by setting

$$L_\xi(\eta) = \xi * \eta, \qquad R_\xi(\eta) = \eta * \xi, \quad \eta \in \ell_2(G),$$

respectively. Observe that

$$\|L_\xi\|_{\mathcal{L}(\ell_2,\ell_\infty)} \leq \|\xi\|_2, \quad \|R_\xi\|_{\mathcal{L}(\ell_2,\ell_\infty)} \leq \|\xi\|_2, \quad \xi \in \ell_2(G).$$

If $g \in G$, then L_{δ_g} and R_{δ_g} are the operators of left and right translation by g, respectively. In particular, L_{δ_g} and R_{δ_g} are unitary operators in $\ell_2(G)$. Furthermore, $L_{\alpha\xi+\beta\eta} = \alpha L_\xi + \beta L_\eta$ and $R_{\alpha\xi+\beta\eta} = \alpha R_\xi + \beta R_\eta$ for all $\xi, \eta \in \ell_2(G)$ and $\alpha, \beta \in \mathbb{C}$. Note that $L_\xi(\delta_e) = R_\xi(\delta_e) = \xi$ for all $\xi \in \ell_2(G)$.

(d). Define

$$\mathcal{L}(G) = \{L_\xi : \xi \in \ell_2(G), L_\xi(\ell_2(G)) \subseteq \ell_2(G)\}$$

and

$$\mathcal{R}(G) = \{R_\xi : \xi \in \ell_2(G), R_\xi(\ell_2(G)) \subseteq \ell_2(G)\}.$$

Show that $\mathcal{L}(G)$ and $\mathcal{R}(G)$ are linear subspaces of $B(\ell_2(G))$.

(e). Assuming that $x \in B(\ell_2(G))$ and $\xi \in \ell_2(G)$, prove the following two statements:

(i). if $\langle x\delta_g, \delta_h\rangle = \langle \xi * \delta_g, \delta_h\rangle$ for all $g, h \in G$, then $x = L_\xi$;

(ii). if $\langle x\delta_g, \delta_h\rangle = \langle \delta_g * \xi, \delta_h\rangle$ *for all* $g, h \in G$*, then* $x = R_\xi$*.*

(f). Assuming that $\xi, \eta \in \ell_2(G)$*, show that:*

(i). if $L_\xi, L_\eta \in \mathcal{L}(G)$*, then* $\xi * \eta \in \ell_2(G)$*,* $L_{\xi*\eta} \in \mathcal{L}(G)$ *and* $L_{\xi*\eta} = L_\xi L_\eta$*;*

(ii). if $R_\xi, R_\eta \in \mathcal{R}(G)$*, then* $\eta * \xi \in \ell_2(G)$*,* $R_{\eta*\xi} \in \mathcal{R}(G)$ *and* $R_{\eta*\xi} = R_\xi R_\eta$*.*

(g). For any $\xi \in \ell_2(G)$ *the function* $\xi^* \in \ell_2(G)$ *is defined by setting* $\xi^*(g) = \overline{\xi(g^{-1})}$*,* $g \in G$*. Evidently,* $\|\xi^*\|_2 = \|\xi\|_2$*. Prove that:*

(i). if $L_\xi \in \mathcal{L}(G)$*, then* $L_\xi^* = L_{\xi^*}$ *(in particular,* $L_\xi^* \in \mathcal{L}(G)$*);*

(ii). if $R_\xi \in \mathcal{R}(G)$*, then* $R_\xi^* = R_{\xi^*}$ *(in particular,* $R_\xi^* \in \mathcal{R}(G)$*).*

Consequently, $\mathcal{L}(G)$ *and* $\mathcal{R}(G)$ *are unitary* $*$*-subalgebras of* $B(\ell_2(G))$*.*

(h). Show that:

(i). if $L_\xi \in \mathcal{L}(G)$ *and* $R_\eta \in \mathcal{R}(G)$*, then* $L_\xi R_\eta = R_\eta L_\xi$*;*

(ii). if $x \in B(\ell_2(G))$ *and* $xR_{\delta_g} = R_{\delta_g}x$ *for all* $g \in G$*, then* $x \in \mathcal{L}(G)$*;*

(iii). if $x \in B(\ell_2(G))$ *and* $xL_{\delta_g} = L_{\delta_g}x$ *for all* $g \in G$*, then* $x \in \mathcal{R}(G)$*.*

Conclude that $\mathcal{L}(G) = \mathcal{R}(G)'$ *and* $\mathcal{R}(G) = \mathcal{L}(G)'$*. In particular,* $\mathcal{L}(G)$ *and* $\mathcal{R}(G)$ *are von Neumann algebras on the Hilbert space* $\ell_2(G)$*. Moreover,* $\mathcal{L}(G)$ *is the von Neumann algebra generated by the left translations* $\{L_{\delta_g} : g \in G\}$ *and* $\mathcal{R}(G)$ *is the von Neumann algebra generated by the right translations* $\{R_{\delta_g} : g \in G\}$*.*

(i). Show that δ_e *is a separating and generating vector for* $\mathcal{L}(G)$ *(and also for* $\mathcal{R}(G)$*). Show, furthermore, that* δ_e *is a trace vector for* $\mathcal{L}(G)$ *(that is,* $\langle L_\xi L_\eta \delta_e, \delta_e\rangle = \langle L_\eta L_\xi \delta_e, \delta_e\rangle$ *for all* $L_\xi, L_\eta \in \mathcal{L}(G)$*).*

Conclude that $\tau : \mathcal{L}(G)^+ \to [0, \infty)$*, defined by setting*

$$\tau(L_\xi) = \langle L_\xi \delta_e, \delta_e\rangle, \quad L_\xi \in \mathcal{L}(G)^+,$$

is a finite faithful normal trace. The unique linear extension of τ *to* $\mathcal{L}(G)$ *is given by* $\tau(L_\xi) = \langle L_\xi \delta_e, \delta_e\rangle$*,* $L_\xi \in \mathcal{L}(G)$*. In particular,* $\mathcal{L}(G)$ *is a finite von Neumann algebra.*

(j). Recall that the conjugation class $C(g)$ *of an element* $g \in G$ *is defined by*

$$C(g) = \left\{h^{-1}gh : h \in G\right\}.$$

Evidently, $C(e) = \{e\}$ *and two conjugation classes are either equal or disjoint.*

Show that if $L_\xi \in \mathcal{L}(G)$*, then* $L_\xi \in \mathcal{Z}(\mathcal{L}(G)) = \mathcal{L}(G) \cap \mathcal{L}(G)'$ *if and only if the function* ξ *is constant on all the conjugation classes of* G *(that is,* ξ *is a so-called* class function*).*

Conclude that if for every $g \neq e$ *in* G *the set* $C(g)$ *is infinite, then* $\mathcal{L}(G)$ *is a factor.*

(k). Show that if for every $g \neq e$ *in* G *the set* $C(g)$ *is infinite, then* $\mathcal{L}(G)$ *is a factor of type* II_1 *(**Hint:** factors of type* I_n*,* $n \in \mathbb{N}$*, are finite dimensional (isomorphic to* $M_n(\mathbb{C})$*)).*

(l). Let $\mathbb{F}_2$ *be the* free group on two generators*,* a_1 *and* a_2*, say. The unit element of* $\mathbb{F}_2$ *is denoted by* e*. Every element* $g \neq e$ *of* $\mathbb{F}_2$ *has a unique representation of the form*

$$g = a_{j_1}^{k_1} a_{j_2}^{k_1} \cdots a_{j_n}^{k_n},$$

where $j_i = 1, 2$ *for all* $1 \leq i \leq n$*,* $j_i \neq j_{i+1}$ *for all* $1 \leq i < n$*,* $k_i \in \mathbb{Z} \setminus \{0\}$ *for all* $1 \leq i \leq n$ *and* $n \in \mathbb{N}$*. This is called the* reduced form *of the element* g*. Multiplication in* $\mathbb{F}_2$ *is given by juxtaposition, subject to the rule that* $a_j^k a_j^l = a_j^{k+l}$ *for all* $k, l \in \mathbb{Z}$ *(where* $a_j^0 = e$*).*

Show that for every element $g \neq e$ *in* $\mathbb{F}_2$*, the conjugacy class* $C(g)$ *is infinite.*

Consequently, the von Neumann algebra $\mathcal{L}(\mathbb{F}_2)$ *is a factor of type* II_1*.*

Remark *The group* $\mathbb{F}_2$ *may be replaced by* $\mathbb{F}_n$*, the free group on* n *generators* $(2 \leq n \in \mathbb{N})$*.*

(m). Let $\mathbb{S}_\infty$ *be the group of all permutations* σ *of* $\mathbb{N}$ *satisfying* $\sigma(n) = n$ *for all except finitely many* $n \in \mathbb{N}$*. Show that for every* $g \neq e$ *the conjugacy class* $C(g)$ *is infinite.*

Consequently, the von Neumann algebra $\mathcal{L}(\mathbb{S}_\infty)$ *is a factor of type* II_1*.*

Remark *For* $n \in \mathbb{N}$*, let* $\mathbb{S}_n$ *be the subgroup of* $\mathbb{S}_\infty$ *consisting of all* $g \in \mathbb{S}_\infty$ *satisfying* $g(k) = k$ *for all* $k > n$*. Evidently,* $\mathbb{S}_n$ *is finite. Furthermore,* $\mathbb{S}_n \subseteq \mathbb{S}_{n+1}$ *and* $\mathbb{S}_\infty = \bigcup_{n=1}^\infty \mathbb{S}_n$*.*

The von Neumann algebras $\mathcal{L}_{\mathbb{S}_n}$ *are finite dimensional and* $\mathcal{L}_{\mathbb{S}_\infty}$ *is the von Neumann algebra generated by* $\bigcup_{n=1}^{\infty} \mathcal{L}_{\mathbb{S}_n}$. *Consequently,* $\mathcal{L}_{\mathbb{S}_\infty}$ *is the* (continuous) hyper-finite factor.

It should also be noted that it may be shown that the von Neumann algebras $\mathcal{L}_{\mathbb{S}_\infty}$ *and* $\mathcal{L}_{\mathbb{F}_2}$ *are* ***not isomorphic****.*

It is still an open problem (as far as we know) whether the von Neumann algebras $\mathcal{L}(\mathbb{F}_n)$ *and* $\mathcal{L}(\mathbb{F}_m)$, $n \neq m$, *are isomorphic or not. It is known that they are either all isomorphic or they are all mutually non-isomorphic (Voiculescu).*