A review of some relevant operator theory

Contents

1 Bounded Hilbert space operators

In this section, some notation and terminology concerning Hilbert space operators which will be used throughout this book, are collected together. Most of the results are stated without proofs. Let H be a *complex Hilbert space* equipped with an *inner product* $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|_H$. The inner product is linear in the first and conjugate linear in the second variable. The elements of H will usually be denoted by small Greek letters $\xi, \eta, \zeta, \ldots$. The space of all bounded linear operators in H is denoted by $B(H)$. The elements of $B(H)$ will be denoted by small Latin letters $x, y, u, v, \ldots$. The identity operator on H is denoted by $\mathbf{1} = \mathbf{1}_H$ (which is the unit element in the algebra $B(H)$). Equipped with the operator norm $\|\cdot\|_{B(H)}$, given by $\|x\|_{B(H)} = \sup_{\|\xi\|_H \leq 1} \|x\xi\|_H$, the space $B(H)$ is a Banach algebra. The closed unit ball in $B(H)$ is denoted by $B(H)_1$. For any $x \in B(H)$, its *adjoint* is denoted by x^*, so $\langle x\xi, \eta \rangle = \langle \xi, x^*\eta \rangle$, $\xi, \eta \in H$. The mapping $x \longmapsto x^*$ is a conjugate linear involution in $B(H)$, satisfying $\|x^*\|_{B(H)} = \|x\|_{B(H)}$ and $\|x\|^2_{B(H)} = \|x^*x\|_{B(H)}$ for all $x \in B(H)$ (so, $B(H)$ is an example of a C^*-algebra, which will be discussed in Section 10).

An operator $x \in B(H)$ satisfying $x^* = x$ is called *self-adjoint* (or, *hermitian*). The collection of all self-adjoint operators is denoted by $B_h(H)$, which is a real linear subspace of $B(H)$. An operator $x \in B(H)$ is called *normal* if $xx^* = x^*x$. Furthermore, if $u \in B(H)$ satisfies $u^*u = uu^* = \mathbf{1}$ (equivalently, $u^{-1} = u^*$), then u is called *unitary.* An operator $p \in B(H)$ is said to be an *orthogonal projection* (or simply, a *projection*) if $p^2 = p = p^*$ and the set of all projections is denoted by $P(B(H))$. If $p \in P(B(H))$, then it is clear that $\mathbf{1} - p \in P(B(H))$, which is called the *complement* of p and this projection will also be denoted by $p^\perp$.

A self-adjoint operator $a \in B_h(H)$ is called *positive* if $\langle a\xi, \xi \rangle \geq 0$ for all $\xi \in H$. The collection of all positive elements of $B_h(H)$ is denoted by $B_h(H)^+$. This set is a proper closed cone in $B_h(H)$ and it induces a partial ordering in $B_h(H)$ by defining $a \leq b$ whenever $b - a \in B_h(H)^+$, which turns $B_h(H)$ into a partially ordered vector space.

2 Topologies on the space $B(H)$

In addition to the norm topology, generated by the operator norm $\|\cdot\|_{B(H)}$, there are a number of other important topologies on $B(H)$. For every $\xi \in H$, the semi-norm ρ_ξ on $B(H)$ is defined by setting $\rho_\xi(x) = \|x\xi\|_H$, $x \in B(H)$. The locally convex Hausdorff topology on $B(H)$ generated by the family of semi-norms $\{\rho_\xi : \xi \in H\}$ is called the *strong operator topology* (briefly, *so-topology*). A net $\{x_\alpha\}$ in $B(H)$ so-converges to an operator $x \in B(H)$, denoted by $x_\alpha \overset{so}{\to} x$, if and only if $\|x_\alpha\xi - x\xi\|_H \to 0$ for all $\xi \in H$. Clearly, the so-topology is weaker than the topology generated by the norm $\|\cdot\|_{B(H)}$. Multiplication in $B(H)$ is continuous with respect to the so-topology in each factor separately, but in general, not jointly so-continuous in both factors (however, multiplication is jointly so-continuous when restricted to norm bounded sets). The mapping $x \longmapsto x^*$ is not so-continuous (unless H is finite dimensional).

For $\xi, \eta \in H$, the semi-norm $\rho_{\xi,\eta}$ is defined by setting $\rho_{\xi,\eta}(x) = |\langle x\xi, \eta\rangle|$, $x \in B(H)$. The locally convex Hausdorff topology on $B(H)$ generated by the family $\{\rho_{\xi,\eta} : \xi, \eta \in H\}$ of semi-norms is called the *weak operator topology* (briefly, *wo-topology*). A net $\{x_\alpha\}$ in $B(H)$ wo-converges to an operator $x \in B(H)$, denoted by $x_\alpha \overset{wo}{\to} x$, if and only if $\langle x_\alpha \xi, \eta\rangle \to \langle x\xi, \eta\rangle$ for all $\xi, \eta \in H$. Obviously, the wo-topology is weaker than the so-topology and coincides with the latter only if H is finite dimensional. Multiplication is wo-continuous in each factor separately, but is not jointly wo-continuous (unless H is finite dimensional). The mapping $x \longmapsto x^*$ is evidently wo-continuous.

For $\xi, \eta \in H$, the linear functional $\omega_{\xi,\eta}$ on $B(H)$ is defined by setting $\omega_{\xi,\eta}(x) = \langle x\xi, \eta\rangle$, $x \in B(H)$. Evidently, $\omega_{\xi,\eta}$ belongs to the Banach dual space $B(H)^*$ of $B(H)$. Let Ω_H be the linear subspace of $B(H)^*$ generated by $\{\omega_{\xi,\eta} : \xi, \eta \in H\}$. It is clear that the wo-topology on $B(H)$ coincides with $\sigma(B(H), \Omega_H)$, the weak topology in $B(H)$ generated by Ω_H. From this it follows that the wo-dual space $B(H)'_{wo}$ of $B(H)$ is precisely Ω_H. Denoting the so-dual space of $B(H)$ by $B(H)'_{so}$, it is evident that $B(H)'_{wo} \subseteq B(H)'_{so}$. It turns out that the reverse inclusion also holds.

Theorem 2.1 *A linear functional on $B(H)$ is wo-continuous if and only if it is so-continuous, that is, $B(H)'_{so} = B(H)'_{wo}$. Consequently, convex subsets of $B(H)$ have the same closures with respect to the so- and wo-topology and for every $\varphi \in B(H)'_{so}$ there exist $\xi_1, \ldots, \xi_n \in H$ and $\eta_1, \ldots, \eta_n \in H$ such that $\varphi = \sum_{j=1}^n \omega_{\xi_j,\eta_j}$.*

Another useful property of the wo-topology is given in the next theorem.

Theorem 2.2 *The closed unit ball $B(H)_1$ of $B(H)$ is wo-compact.*

Consider next the locally convex Hausdorff topology on $B(H)$ generated by the semi-norms $\rho_{\{\xi_i\},\{\eta_i\}}$, given by $\rho_{\{\xi_i\},\{\eta_i\}}(x) = |\sum_{i=1}^\infty \langle x\xi_i, \eta_i\rangle|$, where $\{\xi_i\}_{i=1}^\infty$are $\{\eta_i\}_{i=1}^\infty$ sequences in H satisfying $\sum_{i=1}^\infty \|\xi_i\|_H^2 < \infty$ and $\sum_{i=1}^\infty \|\eta_i\|_H^2 < \infty$. This topology is called the *ultra-weak operator topology* (briefly, *uwo-topology*). The ultra-weak operator topology is stronger than the wo-topology. On norm bounded subsets of $B(H)$ the uwo- and wo-topology coincide. In particularly, $B(H)_1$ is uwo-compact. Convergence of a net $\{x_\alpha\}$ to an element x in $B(H)$ with respect to the uwo-topology is denoted by $x_\alpha \overset{uwo}{\to} x$.

Given a sequence $\{\xi_i\}_{i=1}^{\infty}$ in H satisfying $\sum_{i=1}^{\infty} \|\xi_i\|_H^2 < \infty$, the semi-norm $\rho_{\{\xi_i\}}$ on $B(H)$ is defined by setting $\rho_{\{\xi_i\}}(x) = \left(\sum_{i=1}^{\infty} \|x\xi_i\|_H^2\right)^{1/2}$, $x \in B(H)$. The Hausdorff locally convex topology on $B(H)$ generated by these semi-norms $\rho_{\{\xi_i\}}$ is called the *ultra-strong operator topology* (briefly, *uso-topology*). The uso-topology is stronger than the so- and uwo-topologies and is weaker than the norm topology. On norm bounded subsets of $B(H)$, the uso- and so-topology coincide. Convergence of a net $\{x_\alpha\}$ to an element x in $B(H)$ with respect to the uso-topology is denoted by $x_\alpha \stackrel{uso}{\to} x$.

The uwo-dual and uso-dual spaces of $B(H)$ are denoted by $B(H)'_{uwo}$ and $B(H)'_{uso}$ respectively. The following result is similar to Theorem 2.1.

Theorem 2.3 *A linear functional on $B(H)$ is uwo-continuous if and only if it is uso-continuous, that is, $B(H)'_{uso} = B(H)'_{uwo}$. Consequently, convex subsets of $B(H)$ have the same closures with respect to the uso- and uwo-topology and for every $\varphi \in B(H)'_{uso}$ there exist $\{\xi_j\}_{j=1}^{\infty}$ and $\{\eta_j\}_{j=1}^{\infty}$ in H such that $\sum_{j=1}^{\infty} \|\xi_j\|_H^2 < \infty$, $\sum_{j=1}^{\infty} \|\eta_j\|_H^2 < \infty$ and $\varphi = \sum_{j=1}^{\infty} \omega_{\xi_j,\eta_j}$, as a convergent series in $B(H)^*$.*

3 The lattice of projections

In the sequel, the following notation concerning *partial ordering* shall be used frequently. Let $(X, \leq)$ be a partially ordered set. If D is a non-empty subset of X for which the least upper bound (or, supremum) exists, then this least upper bound is denoted by $\sup D$ or $\bigvee D$. Similarly, $\inf D$ or $\bigwedge D$ denotes the greatest lower bound (or, infimum) of D whenever it exists. If $D = \{x, y\}$, then the notation $\sup D = x \vee y$ and $\inf D = x \wedge y$ is also used. A net $\{x_\alpha\}_{\alpha\in\Lambda}$ (where Λ is a directed index set) in X is called *increasing* (or, *upwards directed*) if $x_\alpha \leq x_\beta$ whenever $\alpha \leq \beta$ in Λ. This is sometimes written as $x_\alpha \uparrow$. If $\{x_\alpha\}_{\alpha\in\Lambda}$ is increasing and $x = \sup_\alpha x_\alpha$ exists, then this is denoted by $x_\alpha \uparrow x$. *Decreasing nets* are defined analogously and $x_\alpha \downarrow x$ means that the decreasing net $\{x_\alpha\}$ has infimum x.

A bijection $\varphi : X \to X$ is called an order isomorphism if $x \leq y$ in X if, and only if, $\varphi(x) \leq \varphi(y)$. The following simple observation is included for later reference.

Lemma 3.1 *Suppose that D is a non-empty subset of the partially ordered set $(X, \leq)$ such that $x = \sup D$ exists. If Φ is a collection of order isomorphisms in X such that $\varphi(D) = D$ for all $\varphi \in \Phi$, then $\varphi(x) = x$ for all $\varphi \in \Phi$.*

A partially ordered set $(X, \leq)$ in which the supremum and infimum exist for any pair of elements, is called a *lattice*. If the supremum and infimum exist for any non-empty subset of X, then X is called a *complete lattice*.

Given a Hilbert space H, the collection of all closed linear subspaces of H is denoted by $\mathrm{Lat}\,(H)$. The set $\mathrm{Lat}\,(H)$ is partially ordered by inclusion, that is, if $L_1, L_2 \in \mathrm{Lat}\,(H)$ then $L_1 \leq L_2$ whenever $L_1 \subseteq L_2$. With respect to this partial ordering, $\mathrm{Lat}\,(H)$ is a complete lattice. Indeed, given any non-empty subset $D \subseteq \mathrm{Lat}\,(H)$, the infimum of D is given by $\bigwedge D = \bigcap \{L : L \in D\}$ and the supremum of D is given by $\bigvee D = \overline{\mathrm{span}}\,\{L : L \in D\}$. The smallest and largest element of $\mathrm{Lat}\,(H)$ are $\{0\}$ and H, respectively. It should be pointed out that the lattice $\mathrm{Lat}\,(H)$ is *not* associative. If $\{L_i : i \in I\}$ is a collection of pairwise orthogonal closed subspaces of H (that is, $L_i \perp L_j$ whenever $i \neq j$ in I), then $\bigvee_{i \in I} L_i$ is also denoted by $\sum_{i \in I} L_i$.

The set $P\,(B\,(H))$ of all projections in H is a subset of $B_h\,(H)$ and so, $P\,(B\,(H))$ may be equipped with the partial ordering inherited from $B_h\,(H)$, that is, if $p, q \in P\,(B\,(H))$, then $p \leq q$ if and only if $\langle p\xi, \xi \rangle \leq \langle q\xi, \xi \rangle$ for all $\xi \in H$. As is easily verified, the relation $p \leq q$ in $P\,(B\,(H))$ is equivalent to $p\,(H) \leq q\,(H)$ in $\mathrm{Lat}\,(H)$. Hence, the mapping $p \longmapsto p\,(H)$ is an order isomorphism from $P\,(B\,(H))$ onto $\mathrm{Lat}\,(H)$. Consequently, $P\,(B\,(H))$ is a complete lattice with smallest element 0 and largest element $\mathbf{1}$. Note that $p \leq q$ in $P\,(B\,(H))$ is also equivalent to saying that $p = pq$ (or, $p = qp$). For every $p \in P\,(B\,(H))$, the projection $p^{\perp} = \mathbf{1} - p$ is a *complement* of p, that is, $p \vee p^{\perp} = \mathbf{1}$ and $p \wedge p^{\perp} = 0$. The mapping $p \longmapsto p^{\perp}$ reverses the ordering in $P\,(B\,(H))$, that is, $p \leq q$ if and only if $q^{\perp} \leq p^{\perp}$, and so, $(p \vee q)^{\perp} = p^{\perp} \wedge q^{\perp}$ and $(p \wedge q)^{\perp} = p^{\perp} \vee q^{\perp}$ for all $p, q \in P\,(B\,(H))$. If $p, q \in P\,(B\,(H))$ commute, then $p \wedge q = pq$ and $p \vee q = p + q - pq$. The supremum of an upwards directed system in $P\,(B\,(H))$ can be characterized as follows.

Proposition 3.2 *If $\{p_\alpha\}$ is an increasing net in $P\,(B\,(H))$ and $p \in P\,(B\,(H))$, then $p_\alpha \uparrow p$ in $P\,(B\,(H))$ if and only if $p_\alpha \xrightarrow{so} p$.*

Two projections $p, q \in P\,(B\,(H))$ are called mutually orthogonal if $pq = 0$ (equivalently, the ranges $p\,(H)$ and $q\,(H)$ are mutually orthogonal subspaces). Suppose that $\{p_i : i \in I\}$ is a pairwise orthogonal collection in $P\,(B\,(H))$ (that is, $p_i p_j = 0$ whenever $i \neq j$ in I). For each finite subset F of I, the projection p_F is defined by setting $p_F = \sum_{i \in F} p_i$. It is clear that $\{p_F\}$ is an increasing net (with respect to the inclusion ordering of the finite subsets of I). Hence, there exists $p \in P\,(B\,(H))$ such that $p_F \uparrow p$. This projection p is denoted by $\sum_{i \in I} p_i$. It follows from Proposition 3.2 that this series is actually so-convergent in $B\,(H)$.

The set $P(B(H))$ is not a sublattice of $B_h(H)$. Actually, if $a, b \in B_h(H)$, then $a \vee b$ exists in $B_h(H)$ if and only if $a \leq b$ or $b \leq a$, by the *anti-lattice theorem* of R.V. Kadison. However, for increasing nets the following important result holds (an analogous statement holds for decreasing nets).

Theorem 3.3 *If $\{a_\alpha\}$ is an increasing net in $B_h(H)$ which is bounded above, then there exists $a \in B_h(H)$ such that $a_\alpha \xrightarrow{so} a$ and $a_\alpha \uparrow a$ in $B_h(H)$.*

It follows, in particular, from Proposition 3.2 and Theorem 3.3 that, if $\{p_\alpha\}$ is an increasing net in $P(B(H))$ such that $p_\alpha \uparrow p$ in $P(B(H))$, then $p_\alpha \uparrow p$ in $B_h(H)$.

Given a unitary operator $u \in B(H)$, the mapping $a \longmapsto uau^*$, $a \in B_h(H)$, is an order isomorphism in $B_h(H)$. Similarly, the mapping $p \longmapsto upu^*$, $p \in P(B(H))$, is an order isomorphism in $P(B(H))$. Hence, the following observation is an immediate consequence of Lemma 3.1.

Lemma 3.4 *Suppose that $\mathcal{U}_0$ is a collection of unitary operators on H.*

(i). Let D be a non-empty subset of $B_h(H)$ for which $a_0 = \bigvee D$ exists in $B_h(H)$. If $uDu^* = D$ for all $u \in \mathcal{U}_0$ (in particular, if $uau^* = a$ for all $u \in \mathcal{U}_0$), then $ua_0u^* = a_0$ for all $u \in \mathcal{U}_0$.

(ii). Let D be a non-empty subset of $P(B(H))$ and $p_0 = \bigvee D$ in $P(B(H))$. If $uDu^* = D$ for all $u \in \mathcal{U}_0$ (in particular, if $upu^* = a$ for all $u \in \mathcal{U}_0$), then $up_0u^* = p_0$ for all $u \in \mathcal{U}_0$.

Note that similar statements hold for infima in $B_h(H)$ and $P(B(H))$.

4 Closed linear operators

Many of the linear operators that will be encountered, are not bounded and are only defined on a (dense) subspace of the Hilbert space H. Here, the necessary notions to deal with such operators will be introduced. A linear operator x in H is a linear mapping from its *domain* $\mathfrak{D}(x)$, which is a linear subspace of H, into the space H. Given two such linear operators x and y in H, the operator y is said to be an *extension* of x (or, x is a *restriction* of y), if $\mathfrak{D}(x) \subseteq \mathfrak{D}(y)$ and $x\xi = y\xi$ for all $\xi \in \mathfrak{D}(x)$. This is denoted as $x \subseteq y$. If $x \subseteq y$ as well as $y \subseteq x$, then, by definition, $x = y$. The *range* and *kernel* of a linear operator x are denoted by $\operatorname{Ran}(x)$ and $\operatorname{Ker}(x)$, respectively, that is

$$\begin{aligned} \operatorname{Ran}(x) &= \{x\xi : \xi \in \mathfrak{D}(x)\} \\ \operatorname{Ker}(x) &= \{\xi \in \mathfrak{D}(x) : x\xi = 0\}. \end{aligned}$$

In the collection of linear operators, the algebraic operations of scalar multiplication, addition and multiplication may be introduced as follows. Given linear operators x, y in H and $\lambda \in \mathbb{C}$, define:

- λx by setting $\mathfrak{D}(\lambda x) = \mathfrak{D}(x)$ and $(\lambda x)\xi = \lambda(x\xi)$ for all $\xi \in \mathfrak{D}(\lambda x)$;
- $x + y$ by setting $\mathfrak{D}(x+y) = \mathfrak{D}(x) \cap \mathfrak{D}(y)$ and $(x+y)\xi = x\xi + y\xi$ for all $\xi \in \mathfrak{D}(x+y)$;
- xy by setting $\mathfrak{D}(xy) = \{\xi \in \mathfrak{D}(y) : y\xi \in \mathfrak{D}(x)\}$ and $(xy)\xi = x(y\xi)$ for all $\xi \in \mathfrak{D}(xy)$;
- the inverse operator x^{-1}, whenever x is injective, by setting $\mathfrak{D}(x^{-1}) = \operatorname{Ran}(x)$ and $x^{-1}\xi = \eta$ whenever $\xi = x\eta$ for some $\eta \in \mathfrak{D}(x)$.

It should be noted that, in general, it may happen that $\mathfrak{D}(x+y) = \{0\}$ or $\mathfrak{D}(xy) = \{0\}$. Furthermore, with respect to these algebraic operations the set of all linear operators is *not* a vector space. However, the following relations for arbitrary linear operators x, y and z hold:

(a). $(x+y)+z = x+(y+z)$;

(b). $(xy)z = x(yz)$;

(c). $(x+y)z = xz + yz$;

(d). $zx + zy \subseteq z(x+y)$.

It follows from (a) and (b) above that, without ambiguity, sums and products of an arbitrary number of linear operators may be formed. In particular, polynomials in linear operators are well defined.

For a linear operator x in H the *graph* $\Gamma(x)$ is defined to be the linear subspace of $H \times H$ given by $\Gamma(x) = \{(\xi, x\xi) : \xi \in \mathfrak{D}(x)\}$. Note that $x \subseteq y$ is equivalent to $\Gamma(x) \subseteq \Gamma(y)$. A linear operator x is called *closed* if $\Gamma(x)$ is a closed subspace of $H \times H$ (equipped with the product topology). In other words, x is closed if and only if it follows from $\{\xi_n\}_{n=1}^{\infty} \subseteq \mathfrak{D}(x)$, $\xi, \eta \in H$, $\xi_n \to \xi$ and $x\xi_n \to \eta$ as $n \to \infty$, that $\xi \in \mathfrak{D}(x)$ and $x\xi = \eta$. If x is closed, then $\operatorname{Ker}(x)$ is a closed subspace of H. It is clear that any bounded linear operator in H is closed. Conversely, if x is a closed linear operator and if the domain $\mathfrak{D}(x)$ is a closed subspace of H, then it follows from the Closed Graph Theorem that x is bounded on its domain $\mathfrak{D}(x)$. This applies in particular if $\mathfrak{D}(x) = H$. Furthermore, if x is a closed injective linear operator in H, then its inverse x^{-1} is also closed. Consequently, if x is in addition surjective, then $\mathfrak{D}(x^{-1}) = \operatorname{Ran}(x) = H$ and so, $x^{-1} \in B(H)$.

The linear operator x in H is called *pre-closed* (or, *closable*) if the closure $\overline{\Gamma(x)}$ of the graph $\Gamma(x)$ in $H \times H$ is the graph of some linear operator in H. Note that $\overline{\Gamma(x)}$ is the graph of a linear operator if and only if $(0, \eta) \in \overline{\Gamma(x)}$ implies that $\eta = 0$. In other words, x is pre-closed if and only if it follows from $\{\xi_n\}_{n=1}^{\infty} \subseteq \mathfrak{D}(x)$, $\eta \in H$, $\xi_n \to 0$ and $x\xi_n \to \eta$ as $n \to \infty$, that $\eta = 0$. If a linear operator x is pre-closed, then the linear operator whose graph is $\overline{\Gamma(x)}$ is denoted by $\bar{x}$ and is called the *closure* of x. Evidently, $x \subseteq \bar{x}$.

If x is a linear operator in H and $\mathfrak{D}$ is a linear subspace of $\mathfrak{D}(x)$ satisfying $\Gamma(x) \subseteq \overline{\Gamma\left(x_{|\mathfrak{D}}\right)}$, where $x_{|\mathfrak{D}}$ denotes the restriction of x to $\mathfrak{D}$, then $\mathfrak{D}$ is called a *core* of x. In other words, $\mathfrak{D}$ is a core of x if and only if for every $\xi \in \mathfrak{D}(x)$ there exists a sequence $\{\xi_n\}_{n=1}^{\infty} \subseteq \mathfrak{D}$ such that $\xi_n \to \xi$ and $x\xi_n \to x\xi$ as $n \to \infty$. It is easily verified that two pre-closed operators which coincide on a common core, have identical closures.

A linear operator x in H is called *densely defined* if its domain $\mathfrak{D}(x)$ is a dense subspace of H. Note that a closed and densely defined operator x is bounded if and only if $\mathfrak{D}(x) = H$.

Suppose now that x is a densely defined operator in H and consider the linear subspace $\mathfrak{D}$ of H given by

$$\mathfrak{D} = \{\eta \in H : \exists \zeta \in H \text{ such that } \langle x\xi, \eta \rangle = \langle \xi, \zeta \rangle \ \forall \xi \in \mathfrak{D}(x)\}.$$

If $\eta \in \mathfrak{D}$, then the element $\zeta \in H$ satisfying $\langle x\xi, \eta \rangle = \langle \xi, \zeta \rangle$ for all $\xi \in \mathfrak{D}(x)$ is uniquely determined by η, as $\mathfrak{D}(x)$ is dense in H. Therefore, the mapping $x^* : \eta \longmapsto \zeta$ from $\mathfrak{D}$ into H is well defined. It is readily verified that x^* is linear. Hence, x^* is a linear operator in H with domain $\mathfrak{D}(x^*) = \mathfrak{D}$. The operator x^* is called the *adjoint* of x. Note that, by definition,

$$\langle x\xi, \eta \rangle = \langle \xi, x^*\eta \rangle, \quad \xi \in \mathfrak{D}(x), \eta \in \mathfrak{D}(x^*). \tag{1}$$

From (1) it is immediately clear that x^* is closed. If x is densely defined and pre-closed, then $\bar{x}^* = x^*$. The following theorem lists some elementary properties of adjoint operators.

Theorem 4.1 *Let x and y be densely defined linear operators in H.*

(i). If $x \subseteq y$, then $y^* \subseteq x^*$.

(ii). $(\lambda x)^* = \bar{\lambda} x^*$ for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$.

(iii). If $x + y$ is densely defined, then $x^* + y^* \subseteq (x + y)^*$.

(iv). If xy is densely defined, then $y^* x^* \subseteq (xy)^*$.

(v). If $y \in B(H)$, then $(x+y)^* = x^* + y^*$ and $(yx)^* = x^* y^*$.

(vi). If $u \in B(H)$ is unitary, then $(ux)^* = x^* u^*$, $(xu)^* = u^* x^*$ and $(uxu^*)^* = ux^* u^*$.

(vii). If x is injective and $\mathrm{Ran}(x)$ is dense, then $(x^{-1})^* = (x^*)^{-1}$.

Given the densely defined linear operator x in H, the adjoint $x^* : \mathfrak{D}(x^*) \to H$ is closed. If x^* is densely defined, then $(x^*)^* = x^{**}$ is well defined and it is easy to see that $x \subseteq x^{**}$. Hence, x is pre-closed. Conversely, if x is pre-closed, then it can be shown that x^* is densely defined. This is contained in the next theorem.

Theorem 4.2 *If x is a densely defined linear operator in H, then x^* is densely defined if, and only if, x is pre-closed. Moreover, if x is pre-closed, then $\bar{x} = x^{**}$. In particular, $x^{**} = x$ holds for every densely defined closed operator x.*

A densely defined linear operator a in H is called *self-adjoint* if $a = a^*$. A self-adjoint operator a in H is called *positive* if $\langle a\xi, \xi \rangle \geq 0$ for all $\xi \in \mathfrak{D}(a)$. This is denoted by $a \geq 0$. Furthermore, as in the case of bounded linear operators on H, a closed and densely defined linear operator x in H is called *normal* when $x^* x = xx^*$.

There are a number of important projections associated with a closed, densely defined operator in H. First, the following simple result should be recalled.

Proposition 4.3 *If x is a densely defined closed linear operator in H, then $\mathrm{Ran}(x)^\perp = \mathrm{Ker}(x^*)$ and $\mathrm{Ker}(x)^\perp = \overline{\mathrm{Ran}(x^*)}$. In particular, if a is a self-adjoint operator in H, then $\mathrm{Ran}(a)^\perp = \mathrm{Ker}(a)$ and $\mathrm{Ker}(a)^\perp = \overline{\mathrm{Ran}(a)}$.*

Definition 4.4 *Let x be a closed, densely defined linear operator x in H.*

(i). The projection $n(x)$ onto $\mathrm{Ker}(x)$ is called the null projection *of x.*

(ii). The projection $r(x)$ onto $\overline{\mathrm{Ran}(x)}$ is called the range projection *of x.*

(iii). The projection $\mathbf{1} - n(x)$, which is the projection onto $\overline{\mathrm{Ran}(x^*)}$, is called the support projection *of x and is denoted by $s(x)$.*

Observe that $n(x^*) = \mathbf{1} - r(x)$, $r(x^*) = s(x)$ and $s(x^*) = r(x)$. It is easy to see that $x = r(x)x = xs(x)$. In particular, if a is a self-adjoint operator in H, then $s(a) = r(a)$ and $a = s(a)a = as(a)$. The following theorem presents alternative descriptions of $s(x)$ and $r(x)$.

Theorem 4.5 *Let x be a closed and densely defined linear operator in H.*

(i). $s(x)$ is the smallest projection $p \in P(B(H))$ satisfying $x = xp$.

(ii). $r(x)$ is the smallest projection $p \in P(B(H))$ satisfying $x = px$.

In view of these characterizations of $s(x)$ and $r(x)$, these projections are sometimes called the *right* and *left support projections* of x, respectively.

Next, recall the definition of the spectrum of a linear operator. Suppose that x is a densely defined closed linear operator. A complex number λ belongs to the *resolvent set* $\rho(x)$ of x if the operator $\lambda\mathbf{1} - x : \mathfrak{D}(x) \to H$ is bijective. If $\lambda \in \rho(x)$, then the inverse $r(\lambda, x) = (\lambda\mathbf{1} - x)^{-1}$, which is called the *resolvent* of x at the point λ, is a bounded linear operator on H, as follows from the Closed Graph Theorem. The *spectrum* $\sigma(x)$ of x is defined by $\sigma(x) = \mathbb{C} \diagdown \rho(x)$. In contrast to the case of bounded operators, it may happen that $\sigma(x) = \emptyset$ or $\sigma(x) = \mathbb{C}$. The next theorem lists some properties of the resolvent and the spectrum.

Theorem 4.6 *If x is a closed and densely defined linear operator in H, then:*

(i). the resolvent set $\rho(x)$ is an open subset of $\mathbb{C}$ and so, the spectrum is closed;

(ii). the mapping $\lambda \longmapsto r(\lambda, x)$ is an analytic function from $\rho(x)$ into $B(H)$.

Next, the construction of closed, densely defined linear operators in H as direct sums of bounded linear operators will be discussed. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of pairwise orthogonal projections in H satisfying $\sum_{n=1}^{\infty} p_n = \mathbf{1}$ and suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence in $B(H)$ satisfying $x_n = p_n x_n = x_n p_n$ for all n. Define

$$\mathfrak{D}(x) = \left\{ \xi \in H : \sum_{n=1}^{\infty} \|x_n \xi\|^2 < \infty \right\}$$

and $x\xi = \sum_{n=1}^{\infty} x_n \xi$ for all $\xi \in \mathfrak{D}(x)$. The linear operator $x : \mathfrak{D}(x) \to H$ is denoted by $\oplus_{n=1}^{\infty} x_n$ and is called the *direct sum* of $\{x_n\}_{n=1}^{\infty}$. Some of its properties are collected together in the following proposition.

Proposition 4.7 *Given the sequence $\{x_n\}_{n=1}^{\infty}$ in $B(H)$ as above, the following statements hold:*

(i). the operator $\oplus_{n=1}^{\infty} x_n$ is closed and densely defined; $\oplus_{n=1}^{\infty} x_n$ is bounded if and only if $\sup_n \|x_n\|_{B(H)} < \infty$;

(ii). the adjoint of $\oplus_{n=1}^{\infty} x_n$ *is equal to* $\oplus_{n=1}^{\infty} x_n^*$*;*

(iii). if the operators x_n *(*$n = 1, 2, \ldots$*) are normal, then* $\oplus_{n=1}^{\infty} x_n$ *is normal;*

(iv). if the operators x_n *(*$n = 1, 2, \ldots$*) are self-adjoint, then* $\oplus_{n=1}^{\infty} x_n$ *is self-adjoint.*

5 The spectral theorem

As before, $(H, \langle \cdot, \cdot \rangle)$ is a complex Hilbert space. Suppose that Ω is a non-empty set and that $\mathcal{A}$ is a σ-algebra of subsets of Ω, so $(\Omega, \mathcal{A})$ is a measurable space.

Definition 5.1 *A* spectral measure *on* $(\Omega, \mathcal{A})$ *is a mapping* $e : \mathcal{A} \to B(H)$ *such that:*

(i). $e(\delta)$ *is a projection in* H *for each* $\delta \in \mathcal{A}$*;*

(ii). $e(\emptyset) = 0$ *and* $e(\Omega) = \mathbf{1}$*;*

(iii). $e(\delta_1 \cap \delta_2) = e(\delta_1) e(\delta_2)$ *for all* $\delta_1, \delta_2 \in \mathcal{A}$*;*

(iv). if $\delta_j \in \mathcal{A}$ *(*$j = 1, 2, \ldots$*) are pairwise disjoint, then*

$$e\left(\cup_{j=1}^{\infty} \delta_j\right) = \sum_{j=1}^{\infty} e(\delta_j),$$

where the series is strongly convergent in $B(H)$*.*

Observe that condition (iii), in particular, implies that the projections $e(\delta_1)$ and $e(\delta_2)$ commute for any two $\delta_1, \delta_2 \in \mathcal{A}$. If condition (iv) is satisfied only for finite disjoint collections $\delta_1, \ldots, \delta_n$ ($n \in \mathbb{N}$) in $\mathcal{A}$, then e is referred to as a *finitely additive spectral measure.* A set $\delta \in \mathcal{A}$ is called an e*-null set* if $e(\delta) = 0$. The collection of all e-null sets, denoted by $\mathcal{N}_e$, is a σ-ideal in $\mathcal{A}$.

Given a spectral measure e on $(\Omega, \mathcal{A})$ and $\xi, \eta \in H$, the σ-additive measure $e_{\xi,\eta} : \mathcal{A} \to \mathbb{C}$ is defined by setting $e_{\xi,\eta}(\delta) = \langle e(\delta) \xi, \eta \rangle$, $\delta \in \mathcal{A}$. The variation $|e_{\xi,\eta}|$ of $e_{\xi,\eta}$ satisfies

$$|e_{\xi,\eta}|(\delta) \leq \|e(\delta) \xi\|_H \|e(\delta) \eta\|_H, \quad \delta \in \mathcal{A}.$$

In particular, the total variation of $e_{\xi,\eta}$, which is denoted by $\|e_{\xi,\eta}\|$, satisfies $\|e_{\xi,\eta}\| \leq \|\xi\|_H \|\eta\|_H$.

The space of all $\mathcal{A}$-simple complex valued functions on Ω is denoted by $\operatorname{sim}(\Omega, \mathcal{A})$, that is,

$$\operatorname{sim}(\Omega, \mathcal{A}) = \left\{ \sum_{j=1}^{n} \alpha_j \chi_{\delta_j} : \alpha_j \in \mathbb{C}, \delta_j \in \mathcal{A}, j = 1, \ldots, n; n \in \mathbb{N} \right\}.$$

Clearly, $\operatorname{sim}(\Omega, \mathcal{A})$ is an algebra with respect to the pointwise operations. Given a spectral measure $e : \mathcal{A} \to B(H)$ and $s = \sum_{j=1}^{n} \alpha_j \chi_{\delta_j}$ in $\operatorname{sim}(\Omega, \mathcal{A})$, define

$$\int_{\Omega} s de = \sum_{j=1}^{n} \alpha_j e(\delta_j).$$

The integration map $s \longmapsto \int_{\Omega} s de$ is an algebra homomorphism from $\operatorname{sim}(\Omega, \mathcal{A})$ into $B(H)$ satisfying for all $s \in \operatorname{sim}(\Omega, \mathcal{A})$ and all $\xi, \eta \in H$:

(a). $\int_{\Omega} \bar{s} de = \left(\int_{\Omega} s de\right)^*$;

(b). $\left\|\int_{\Omega} s de\right\|_{B(H)} \leq \|s\|_{\infty}$, where $\|s\|_{\infty} = \max_{\omega \in \Omega} |s(\omega)|$.

(c). $\left\langle \left(\int_{\Omega} s de\right) \xi, \eta \right\rangle = \int_{\Omega} s de_{\xi,\eta}$.

Let $B_b(\Omega, \mathcal{A})$ be the algebra of all bounded complex valued $\mathcal{A}$-measurable functions on Ω. For $f \in B_b(\Omega, \mathcal{A})$ let $\|f\|_{\infty} = \sup_{\omega \in \Omega} |f(\omega)|$. With respect to the norm $\|\cdot\|_{\infty}$, the space $B_b(\Omega, \mathcal{A})$ is a Banach algebra and $\operatorname{sim}(\Omega, \mathcal{A})$ is a subalgebra. For any $f \in B_b(\Omega, \mathcal{A})$ there exists a sequence $\{s_n\}_{n=1}^{\infty}$ in $\operatorname{sim}(\Omega, \mathcal{A})$ such that $\|f - s_n\|_{\infty} \to 0$ as $n \to \infty$. From the estimate given in (b) above it follows that $\left\{\int_{\Omega} s_n de\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $B(H)$. Consequently, this sequence is convergent in $B(H)$ and it is readily verified that its limit depends only on the function f and not on the choice of the particular sequence $\{s_n\}_{n=1}^{\infty}$. This justifies the following definition.

Definition 5.2 *Let $e : \mathcal{A} \to B(H)$ be a spectral measure on the measurable space $(\Omega, \mathcal{A})$. For $f \in B_b(\Omega, \mathcal{A})$ the integral with respect to e is defined by $\int_{\Omega} f de = \lim_{n \to \infty} \int_{\Omega} s_n de$ (norm convergence in $B(H)$), where $\{s_n\}_{n=1}^{\infty}$ is any sequence in $\operatorname{sim}(\Omega, \mathcal{A})$ such that $\|f - s_n\|_{\infty} \to \infty$ as $n \to \infty$.*

It is clear that for $f \in \operatorname{sim}(\Omega, \mathcal{A})$ the above definition agrees with Definition 5.1. In the next theorem the basic properties of the integration map are collected together.

Theorem 5.3 *If $e : \mathcal{A} \to B(H)$ is a spectral measure on the measurable space $(\Omega, \mathcal{A})$, then the following statements hold.*

(i). *The integration map* $f \longmapsto \int_{\Omega} f de$ *is an algebra homomorphism from* $B_b(\Omega, \mathcal{A})$ *into* $B(H)$.

(ii). $\int_{\Omega} \bar{f} de = \left(\int_{\Omega} f de\right)^*$ *for all* $f \in B_b(\Omega, \mathcal{A})$.

(iii). $\left\|\int_{\Omega} f de\right\|_{B(H)} \leq \|f\|_{\infty}$ *for all* $f \in B_b(\Omega, \mathcal{A})$.

(iv). *For every* $f \in B_b(\Omega, \mathcal{A})$, *the operator* $\int_{\Omega} f de$ *is normal and if* f *is real valued, then* $\int_{\Omega} f de$ *is self-adjoint.*

(v). $\left\langle \left(\int_{\Omega} f de\right) \xi, \eta \right\rangle = \int_{\Omega} f de_{\xi,\eta}$ *for all* $\xi, \eta \in H$ *and all* $f \in B_b(\Omega, \mathcal{A})$.

(vi). $\left\|\left(\int_{\Omega} f de\right) \xi\right\|_H^2 = \int_{\Omega} |f|^2 de_{\xi,\xi}$ *for all* $\xi \in H$ *and all* $f \in B_b(\Omega, \mathcal{A})$.

(vii). *If* $f \geq 0$ *in* $B_b(\Omega, \mathcal{A})$, *then* $\int_{\Omega} f de \geq 0$ *in* $B(H)$.

(viii). *If* $f, g \in B_b(\Omega, \mathcal{A})$ *are such that* $|f| \leq |g|$, *then*

$$\left\|\left(\int_{\Omega} f de\right) \xi\right\|_H \leq \left\|\left(\int_{\Omega} g de\right) \xi\right\|_H, \quad \xi \in H.$$

In particular,

$$\left\|\int_{\Omega} f de\right\|_{B(H)} \leq \left\|\int_{\Omega} g de\right\|_{B(H)}.$$

(ix). *If* $\{f_n\}_{n=1}^{\infty}$ *is a uniformly bounded sequence in* $B_b(\Omega, \mathcal{A})$ *and if* $f \in B_b(\Omega, \mathcal{A})$ *such that* $f_n(\omega) \to f(\omega)$ *as* $n \to \infty$ *for all* $\omega \in \Omega$, *then* $\int_{\Omega} f_n de \to \int_{\Omega} f de$ *strongly as* $n \to \infty$.

The integral of a function over a subset of Ω is defined in the usual manner.

Definition 5.4 *If* $f \in B_b(\Omega, \mathcal{A})$ *and* $\Delta \in \mathcal{A}$, *then* $\int_{\Delta} f de = \int_{\Omega} f \chi_{\Delta} de$.

From the multiplicativity of the integration map it is obvious that

$$\int_{\Delta} f de = e(\Delta) \int_{\Omega} f de = \left(\int_{\Omega} f de\right) e(\Delta).$$

The next objective is to extend the integration map to all measurable functions on $(\Omega, \mathcal{A})$. Assume that $e : \mathcal{A} \to B(H)$ is a spectral measure. The space of all complex valued $\mathcal{A}$-measurable functions on Ω is denoted by $B(\Omega, \mathcal{A})$. If $f \in B(\Omega, \mathcal{A})$ and $\Delta \in \mathcal{A}$ is such that $f\chi_{\Delta} \in B_b(\Omega, \mathcal{A})$, then the integral $\int_{\Omega} f \chi_{\Delta} de$ is defined in Definition 5.2. In this situation, this is also denoted by $\int_{\Delta} f de = \int_{\Omega} f \chi_{\Delta} de$. Note that this agrees with Definition 5.4 in case f is bounded. Moreover, $\int_{\Delta} f de = e(\Delta) \int_{\Delta} f de = \left(\int_{\Delta} f de\right) e(\Delta)$.

Definition 5.5 *Given $f \in B(\Omega, \mathcal{A})$, a countable collection $\{\Delta_n\}_{n=1}^{\infty}$ in $\mathcal{A}$ is called* admissible *for the function f if:*

(i). $\{\Delta_n\}_{n=1}^{\infty}$ consists of mutually disjoint sets and $\cup_{n=1}^{\infty} \Delta_n = \Omega$;

(ii). $f\chi_{\Delta_n} \in B_b(\Omega, \mathcal{A})$ for all $n = 1, 2, \ldots$.

Note that if follows from (i) that $\sum_{n=1}^{\infty} e(\Delta_n) = I$ strongly in $B(H)$. Let $f \in B(\Omega, \mathcal{A})$ be given and let $\{\Delta_n\}_{n=1}^{\infty}$ be any admissible sequence for f (evidently, such a sequence always exists). For each $n = 1, 2, \ldots$ define $x_n = \int_{\Delta_n} f de$. Then $x_n \in B(H)$ is a normal operator satisfying $x_n = e(\Delta_n) x_n = x_n e(\Delta_n)$. Consequently, $x = \oplus_{n=1}^{\infty} x_n$ may be defined as in the discussion preceding Proposition 4.7, that is,

$$\mathfrak{D}(x) = \left\{ \xi \in H : \sum_{n=1}^{\infty} \|x_n \xi\|_H^2 < \infty \right\}$$

and $x\xi = \sum_{n=1}^{\infty} x_n \xi$ for all $\xi \in \mathfrak{D}(x)$. The operator $x : \mathfrak{D}(x) \to H$ is normal in H. It is not too difficult to show that the operator x does not depend on the choice of the admissible sequence. This justifies the following definition.

Definition 5.6 *Let $e : \mathcal{A} \to B(H)$ be a spectral measure on the measurable space $(\Omega, \mathcal{A})$. Let $\{\Delta_n\}_{n=1}^{\infty}$ be an admissible sequence for the function $f \in B(\Omega, \mathcal{A})$. The integral of f with respect to e is defined by*

$$\int_{\Omega} f de = \bigoplus_{n=1}^{\infty} \int_{\Delta_n} f de.$$

The next theorem collects together the basic properties of the integration map. It will be convenient to write

$$\Phi_e(f) = \int_{\Omega} f de, \quad f \in B(\Omega, \mathcal{A}).$$

Hence, Φ_e maps $B(\Omega, \mathcal{A})$ into $\mathcal{C}(H)$, the collection of all closed, densely defined linear operators on H. If $x, y \in \mathcal{C}(H)$ are such that the operator $x + y$ is pre-closed, then $x \hat{+} y$ denotes the closure of $x + y$. Similarly, if xy is pre-closed, then $x \hat{\cdot} y$ denotes the closure of xy.

Theorem 5.7 *Let $e : \mathcal{A} \to B(H)$ be a spectral measure on the measurable space $(\Omega, \mathcal{A})$ and define $\Phi_e : B(\Omega, \mathcal{A}) \to \mathcal{C}(H)$ by $\Phi_e(f) = \int_{\Omega} f de$. For all $f, g \in B(\Omega, \mathcal{A})$ the following statements hold:*

(i). $\mathfrak{D}(\Phi_e(f)) = \left\{\xi \in H : \int_\Omega |f|^2 \, de_{\xi,\xi} < \infty\right\}$ *and*

$$\|\Phi_e(f)\xi\|_H^2 = \int_\Omega |f|^2 \, de_{\xi,\xi}, \quad \xi \in \mathfrak{D}(\Phi_e(f));$$

(ii). *if* $\xi \in \mathfrak{D}(\Phi_e(f))$ *and* $\eta \in H$*, then* $f \in L_1(|e_{\xi,\eta}|)$ *and* $\langle \Phi_e(f)\xi, \eta\rangle = \int_\Omega f de_{\xi,\eta}$*;*

(iii). *if* $\{\Delta_n\}_{n=1}^\infty$ *is an admissible sequence for* f*, then the algebraic direct sum* $\oplus_{n=1}^\infty e(\Delta_n) H$ *is a core for the operator* $\Phi_e(f)$*;*

(iv). $\Phi_e(\bar{f}) = \Phi_e(f)^*$*;*

(v). *if* f *is real valued, then* $\Phi_e(f)$ *is self-adjoint and if* $f \geq 0$*, then* $\Phi_e(f) \geq 0$*;*

(vi). $\Phi_e(f+g) = \Phi_e(f) \hat{+} \Phi_e(g)$*;*

(vii). $\Phi_e(fg) = \Phi_e(f) \hat{\cdot} \Phi_e(g)$*. Moreover,* $\mathfrak{D}(\Phi_e(f)\Phi_e(g)) = \mathfrak{D}(\Phi_e(g)) \cap \mathfrak{D}(\Phi_e(fg))$*;*

(viii). *if* $g \in B_b(\Omega, \mathcal{A})$*, then* $\Phi_e(fg) = \Phi_e(f)\Phi_e(g)$*;*

(ix). $\Phi_e(p(f)) = p(\Phi_e(f))$ *for all polynomials* p*. In particular,* $p(\Phi_e(f))$ *is closed for any polynomial* p*;*

(x). *if* $\Delta \in \mathcal{A}$ *such that* $f\chi_\Delta \in B_b(\Omega, \mathcal{A})$*, then* $e(\Delta) H \subseteq \mathfrak{D}(\Phi_e(f))$ *and* $\Phi_e(f\chi_\Delta) = \Phi_e(f) e(\Delta)$*.*

If $f \in B(\Omega, \mathcal{A})$ and $\Delta \in \mathcal{A}$, then the integral $\int_\Delta f de$ is defined by setting

$$\int_\Delta f de = \int_\Omega f\chi_\Delta de.$$

Note that it follows from (viii) in the above theorem that

$$\int_\Delta f de = \left(\int_\Omega f de\right) e(\Delta).$$

Furthermore, it is easy to see that if the spectral measure e is supported by a set $\Delta \in \mathcal{A}$ (that is, $\Omega \diagdown \Delta$ is an e-null set), then $\int_\Omega f de = \int_\Delta f de$ for all $f \in B(\Omega, \mathcal{A})$. Therefore, in this situation, the integration map may be considered to be defined on the space $B(\Delta, \mathcal{A}_\Delta)$, where $\mathcal{A}_\Delta = \{\delta \in \mathcal{A} : \delta \subseteq \Delta\}$.

Next, a *change of measure formula* for spectral measures will be discussed. Assume that $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are two measurable spaces and that $e :$

$\mathcal{A}_1 \to B(H)$ is a spectral measure. Suppose that $\psi : \Omega_1 \to \Omega_2$ is an $(\mathcal{A}_1, \mathcal{A}_2)$-measurable mapping, that is, $\psi^{-1}(\delta) \in \mathcal{A}_1$ whenever $\delta \in \mathcal{A}_2$. Defining $\psi(e)(\delta) = e\left(\psi^{-1}(\delta)\right)$, $\delta \in \mathcal{A}_2$, the map $\psi(e) : \mathcal{A}_2 \to B(H)$ is a spectral measure which is called the *image measure* of e under ψ. The following theorem relates integrals with respect to $\psi(e)$ to integrals with respect to e.

Theorem 5.8 *Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be two measurable spaces and $\psi : \Omega_1 \to \Omega_2$ be an $(\mathcal{A}_1, \mathcal{A}_2)$-measurable mapping. If $e : \mathcal{A}_1 \to B(H)$ is a spectral measure, then*

$$\int_{\Omega_2} g d\psi(e) = \int_{\Omega_1} g \circ \psi de, \quad g \in B(\Omega_2, \mathcal{A}_2).$$

Spectral measures on the complex plane and spectral measures associated with normal operators will be discussed next. The σ-algebra of all Borel subsets of $\mathbb{C}$ is denoted by $\mathcal{B}(\mathbb{C})$ and the space of all complex valued Borel functions on $\mathbb{C}$ is denoted by $B(\mathbb{C})$, that is, $B(\mathbb{C}) = B(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. Similarly, $B_b(\mathbb{C}) = B_b(\mathbb{C}, \mathcal{B}(\mathbb{C}))$.

Suppose that $e : \mathcal{B}(\mathbb{C}) \to B(H)$ is a spectral measure. For every $f \in B(\mathbb{C})$ the integral $\int_{\mathbb{C}} f de$ defines a normal operator in H. In particular, $x = \int_{\mathbb{C}} \lambda de(\lambda)$ is a normal operator in H. It turns out that the spectral measure e is uniquely determined by the normal operator x.

Proposition 5.9 *Suppose that $e_1 : \mathcal{B}(\mathbb{C}) \to B(H)$ and $e_2 : \mathcal{B}(\mathbb{C}) \to B(H)$ are two spectral measures such that $\int_{\mathbb{C}} \lambda de_1(\lambda) = \int_{\mathbb{C}} \lambda de_2(\lambda)$. Then $e_1 = e_2$.*

Therefore, if $x : \mathfrak{D}(x) \to H$ is a normal operator and $e : \mathcal{B}(\mathbb{C}) \to B(H)$ is a spectral measure such that $x = \int_{\mathbb{C}} \lambda de(\lambda)$, then e is called *the spectral measure of* x and is denoted by e^x. The fact that for every normal operator such a spectral measure actually exists can be obtained in many different ways and for a proof the reader is referred to the literature. For later reference, the above discussion is summarized in the next theorem.

Theorem 5.10 *If $x : \mathfrak{D}(x) \to H$ is a normal operator then there exists a uniquely determined spectral measure $e^x : \mathcal{B}(\mathbb{C}) \to B(H)$ such that $x = \int_{\mathbb{C}} \lambda de^x(\lambda)$.*

Definition 5.11 *Given a normal operator $x : \mathfrak{D}(x) \to H$ with spectral measure $e^x : \mathcal{B}(\mathbb{C}) \to B(H)$, define*

$$f(x) = \int_{\mathbb{C}} f(\lambda) de^x(\lambda)$$

for any $f \in B(\mathbb{C})$.

So, for every $f \in B(\mathbb{C})$ the operator $f(x)$ is normal. Using the notation introduced in Theorem 5.7, it is clear that

$$f(x) = \Phi_{e^x}(f), \quad f \in B(\mathbb{C}).$$

Therefore, the basic properties of the mapping $f \longmapsto f(x)$ follow from Theorems 5.3 and 5.7, and they will not be repeated. The map $f \longmapsto f(x)$ is called the *Borel functional calculus of the operator* x. An important additional property is given in the following theorem.

Theorem 5.12 *Suppose that* $x : \mathfrak{D}(x) \to H$ *is a normal operator. For* $y \in B(H)$ *the following statements are equivalent:*

(i). $ye^x(\delta) = e^x(\delta)y$ *for all* $\delta \in \mathcal{B}(\mathbb{C})$*;*

(ii). $yf(x) \subseteq f(x)y$ *for all* $f \in B(\mathbb{C})$*;*

(iii). $yx \subseteq xy$.

It should be noted that one of the ingredients in the proof that (iii) implies (i) is Fuglede's theorem, stating that $yx \subseteq xy$ implies $yx^* \subseteq x^*y$ for any normal operator x and bounded operator y on H. Furthermore, observe that, given $y \in B(H)$, the collection of all sets $\delta \in \mathcal{B}(\mathbb{C})$ satisfying $ye^x(\delta) = e^x(\delta)y$ is a σ-subalgebra of $\mathcal{B}(\mathbb{C})$. Therefore, in (i) of the above theorem, $\mathcal{B}(\mathbb{C})$ may be replaced by any subcollection which generates $\mathcal{B}(\mathbb{C})$ as a σ-algebra.

Let $e : \mathcal{B}(\mathbb{C}) \to B(H)$ be a spectral measure and let V be the union of all open subsets δ of $\mathbb{C}$ satisfying $e(\delta) = 0$. Via a standard argument, it follows that $e(V) = 0$. The support of e is defined by $\operatorname{supp}(e) = \mathbb{C} \diagdown V$. Hence, $\operatorname{supp}(e)$ is the smallest closed subset of $\mathbb{C}$ with $e(\operatorname{supp}(e)) = I$. Moreover, if U is any open set in $\mathbb{C}$ such that $U \cap \operatorname{supp}(e) \neq \emptyset$, then $e(U) \neq 0$. Note that $\int_{\mathbb{C}} f de = \int_{\operatorname{supp}(e)} f de$ for all $f \in B(\mathbb{C})$. For all $\xi, \eta \in H$, the support of the Borel measure $e_{\xi,\eta}$ is contained in $\operatorname{supp}(e)$. The support of the spectral measure of a normal operator can be identified precisely.

Theorem 5.13 *If* $x : \mathfrak{D}(x) \to H$ *is a normal operator, then* $\operatorname{supp}(e^x) = \sigma(x)$.

From this result it follows that $f(x) = \int_{\sigma(x)} f(\lambda)\, de(\lambda)$ for all $f \in B(\mathbb{C})$ and so, for the Borel functional calculus of x, only functions $f \in B(\sigma(x))$ need to be considered.

Some consequences of the change of measure formula (see Theorem 5.8) will be discussed next.

Theorem 5.14 *Given the spectral measure* $e : \Omega \to B(H)$ *on the measurable space* $(\Omega, \mathcal{A})$ *and* $f \in B(\Omega, \mathcal{A})$*, define* $x = \int_\Omega f de$*. The spectral measure* e^x *of* x *is given by* $e^x = f(e)$*, that is,* $e^x(\delta) = e(f^{-1}(\delta))$ *for all* $\delta \in \mathcal{B}(\mathbb{C})$*. Moreover,* $g(x) = \int_\Omega g \circ f de$ *for all* $g \in B(\mathbb{C})$*.*

Corollary 5.15 *Let* $x : \mathfrak{D}(x) \to H$ *be a normal operator and* $f, g \in B(\mathbb{C})$*. Then* $e^{f(x)} = f(e^x)$ *and* $g(f(x)) = (g \circ f)(x)$*.*

The following corollary is a useful consequence of the above result.

Corollary 5.16 *Assume that* x *and* y *are two normal operators in* H*. If* $f, g \in B(\mathbb{C})$ *are such that* $g(f(\lambda)) = \lambda$ *for all* $\lambda \in \sigma(x) \cup \sigma(y)$*, then* $f(x) = f(y)$ *implies that* $x = y$*.*

6 Self-adjoint operators

A densely defined linear operator x in H is called *symmetric* if $x \subseteq x^*$. Equivalently, x is symmetric whenever $\langle x\xi, \eta \rangle = \langle \xi, x\eta \rangle$ for all $\xi, \eta \in \mathfrak{D}(x)$. In particular, $\langle x\xi, \xi \rangle \in \mathbb{R}$ for all $\xi \in \mathbb{R}$. Clearly, a symmetric operator is pre-closed, $\bar{x} \subseteq x^*$ and $\bar{x}$ is also symmetric. Recall that a densely defined linear operator a in H is *self-adjoint* if $a = a^*$. Obviously, any self-adjoint operator is closed and symmetric. However, a closed symmetric operator is not necessarily self-adjoint. The following theorem gives a useful criterion to verify whether a given symmetric operator is self-adjoint.

Theorem 6.1 *For a symmetric operator* x *in* H *the following conditions are equivalent:*

(i). x *is self-adjoint;*

(ii). x *is closed and* $\operatorname{Ker}(x^* \pm i\mathbf{1}) = \{0\}$*;*

(iii). x *is closed and* $\operatorname{Ran}(x \pm i\mathbf{1})^\perp = \{0\}$*;*

(iv). $\operatorname{Ran}(x \pm i\mathbf{1}) = H$*.*

In connection with the above theorem, it is of some interest to note that a symmetric operator x is closed if and only if $\operatorname{Ran}(x + i\mathbf{1})$ (or, equivalently, $\operatorname{Ran}(x - i\mathbf{1})$) is a closed subspace of H. Furthermore, it should be noted that a self-adjoint operator is *maximal symmetric*. Indeed, suppose that a is a self-adjoint operator in H and that b is a symmetric operator such that $a \subseteq b$. Then $b \subseteq b^* \subseteq a^* = a$ and hence, $a = b$.

For self-adjoint operators, the following important result holds.

Theorem 6.2 *If a is a self-adjoint operator in H, then $\sigma(a) \subseteq \mathbb{R}$.*

Consequently, if a is a self-adjoint operator on H, then the Borel functional calculus of a is given by

$$f(a) = \int_{\mathbb{R}} f(\lambda)\, de^a(\lambda), \quad f \in B(\mathbb{R}).$$

Furthermore, for a self-adjoint operator a the null projection is given by $n(a) = e^a(\{0\})$ and the support projection by $s(a) = e^a(\mathbb{R} \setminus \{0\})$.

Let $a : \mathfrak{D}(a) \to H$ be a self-adjoint operator in H. Recall that a is called *positive*, denoted by $a \geq 0$, if $\langle a\xi, \xi \rangle \geq 0$ for all $\xi \in \mathfrak{D}(a)$. A self-adjoint operator a is positive if and only if $\sigma(a) \subseteq [0, \infty)$. This result can be easily obtained by via the properties of the Borel calculus. Indeed, assume first that $\sigma(a) \subseteq [0, \infty)$ and take $\xi \in \mathfrak{D}(a)$. Then it follows from (ii) in Theorem 5.7 that

$$\langle a\xi, \xi \rangle = \int_{\sigma(a)} \lambda de^a_{\xi,\xi}(\lambda) \geq 0,$$

since $e^a_{\xi,\xi}$ is a positive measure and $\lambda \geq 0$ on $\sigma(a)$. Hence, $a \geq 0$. For the proof of the converse implication, suppose that $\sigma(a) \cap (-\infty, 0) \neq \emptyset$. Since $\sigma(a) = \operatorname{supp}(e^a)$, this implies that $e^a(-\infty, 0) \neq 0$ and hence, there exist $\alpha < \beta < 0$ such that $e^a(\alpha, \beta) \neq 0$. Take $\xi \in H$ such that $e^a(\alpha, \beta)\xi = \xi$ and $\|\xi\|_H = 1$. By Theorem 5.7 (x), $\xi \in \mathfrak{D}(a)$ and

$$ae^a(\alpha, \beta) = \int_{\mathbb{R}} \lambda \chi_{(\alpha,\beta)}(\lambda)\, de^a(\lambda).$$

Using once again (ii) of the same theorem, it follows that

$$\langle a\xi, \xi \rangle = \int_{\mathbb{R}} \lambda \chi_{(\alpha,\beta)}(\lambda)\, de^a_{\xi,\xi}(\lambda) \leq \beta e^a_{\xi,\xi}(\alpha, \beta) = \beta \langle e^a(\alpha, \beta)\xi, \xi \rangle = \beta < 0.$$

This shows that a is not positive and the statement is proved. From these observations the next proposition follows immediately.

Proposition 6.3 *Let a be a self-adjoint operator in H If $m, M \in [-\infty, \infty]$ are defined by setting*

$$m = \inf \{\langle a\xi, \xi \rangle : \xi \in \mathfrak{D}(a), \|\xi\|_H = 1\}$$

and

$$M = \sup \{\langle a\xi, \xi \rangle : \xi \in \mathfrak{D}(a), \|\xi\|_H = 1\},$$

then $m = \inf \sigma(a)$ and $M = \sup \sigma(a)$.

As an application of the functional calculus yields also obtain the following result.

Proposition 6.4 *If a is a positive self-adjoint operator in the Hilbert space H, then for each $n \in \mathbb{N}$, there exists a unique positive self-adjoint operator b in H such that $b^n = a$.*

Proof. Since $\sigma(a) \subseteq [0, \infty)$, the self-adjoint operator b may be defined by $b = \int_{[0,\infty)} \lambda^{1/n} de^a(\lambda)$. From (ii) and (ix) in Theorem 5.7 it follows that $b \geq 0$ and $b^n = a$. Suppose that b_1 and b_2 are positive self-adjoint operators in H such that $b_1^n = b_2^n = a$. Define the functions $f, g : \mathbb{C} \to \mathbb{C}$ by setting $f(\lambda) = \lambda^n$, $\lambda \in \mathbb{C}$, and $g(\lambda) = \lambda^{1/n}$ if $\lambda \in [0, \infty)$ and $g(\lambda) = 0$ otherwise. Since $\sigma(b_1), \sigma(b_1) \subseteq [0, \infty)$, it is clear that $g(f(\lambda)) = \lambda$ whenever $\lambda \in \sigma(b_1) \cup \sigma(b_1)$. Hence, it follows from Corollary 5.16 that $b_1 = b_2$. ∎

If a is a positive self-adjoint operator in H, then the unique positive self-adjoint operator b satisfying $b^n = a$ is denoted by $a^{1/n}$ or $\sqrt[n]{a}$. In particular, a has a unique *positive square root* $a^{1/2} = \sqrt{a}$. Note furthermore, it follows from Theorem 5.7 (i) that $\mathfrak{D}(a) \subseteq \mathfrak{D}\left(a^{1/2}\right)$ and, by (iii) of the same theorem, $\mathfrak{D}(a)$ is a core of $a^{1/2}$.

The *positive part* a^+ and *negative part* a^- of a self-adjoint operator a are defined by

$$a^+ = \int_{\mathbb{R}} \lambda^+ de^a(\lambda) \quad \text{and} \quad a^- = \int_{\mathbb{R}} \lambda^- de^a(\lambda)$$

respectively, where $\lambda^+ = \max(\lambda, 0)$ and $\lambda^- = \max(-\lambda, 0)$. By Theorem 5.7 (v), $a^+, a^- \geq 0$ and it follows from (vi) of the same theorem that $a = a^+ \hat{-} a^-$. Actually, using Theorem 5.7 (i), it follows that $\mathfrak{D}(a) = \mathfrak{D}(a^+) \cap \mathfrak{D}(a^-)$ and so, $a = a^+ - a^-$. Observe that it follows from Corollary 5.15 that $s(a^+) = e^{a^+}(\mathbb{R} \setminus \{0\}) = e^a(0, \infty)$ and $s(a^-) = e^{a^-}(\mathbb{R} \setminus \{0\}) = e^a(-\infty, 0)$. Therefore, $s(a^+) s(a^-) = 0$. The above decomposition in positive and negative part is unique in the following sense.

Proposition 6.5 *Suppose that a is a self-adjoint operator in the Hilbert space H. If b and c are self-adjoint positive operators on H such that $a = b - c$ and $s(b) s(c) = 0$, then $b = a^+$ and $c = a^-$.*

The *absolute value* (or, *modulus*) $|a|$ of a self-adjoint operator a is defined by

$$|a| = \int_{\mathbb{R}} |\lambda| \, de^a(\lambda).$$

From Theorem 5.7 it follows that $|a| \geq 0$, $\mathfrak{D}(|a|) = \mathfrak{D}(a)$ and $a = a^+ + a^-$. Note that $\frac{1}{2}(|a| + a) \subseteq a^+$ and $\frac{1}{2}(|a| - a) \subseteq a^-$. Moreover, by (ix) of Theorem 5.7, it follows that

$$|a|^2 = \int_{\mathbb{R}} |\lambda|^2 de^a(\lambda) = \int_{\mathbb{R}} \lambda^2 de^a(\lambda) = a^2.$$

Therefore, the uniqueness of the positive square root implies that $|a| = (a^2)^{1/2}$.

Two self-adjoint operators a and b in H are called *resolvent commuting* if $r(\lambda, a) r(\mu, b) = r(\mu, b) r(\lambda, a)$ for all $\lambda, \mu \in \mathbb{C} \diagdown \mathbb{R}$.

Proposition 6.6 *If a and b are self-adjoint operators in H, then the following two statements are equivalent:*

(i). a and b are resolvent commuting;

(ii). $e^a(\delta_1) e^b(\delta_2) = e^b(\delta_2) e^a(\delta_1)$ for all Borel sets $\delta_1, \delta_2 \subseteq \mathbb{R}$.

Proof. For the proof that (i) implies (ii), it should be observed first that if $x \in B(H)$ and $\lambda \in \mathbb{C} \diagdown \mathbb{R}$ are such that $r(\lambda, a) x = xr(\lambda, a)$, then $xa \subseteq ax$. Indeed, given $\xi \in \mathfrak{D}(a)$ there exists $\eta \in H$ such that $\xi = r(\lambda, a) \eta$ and so, $x\xi = xr(\lambda, a) \eta = r(\lambda, a) x\eta$, which shows that $x\xi \in \mathfrak{D}(a)$. Furthermore,

$$\begin{aligned} xa\xi &= xar(\lambda, a) \eta = x\{\lambda r(\lambda, a) \eta - \eta\} = \lambda xr(\lambda, a) \eta - x\eta \\ &= \lambda r(\lambda, a) x\eta - x\eta = ar(\lambda, a) x\eta = ax\xi. \end{aligned}$$

This proves that $xa \subseteq ax$.

Now suppose that $\lambda, \mu \in \mathbb{C} \diagdown \mathbb{R}$ and $r(\lambda, a) r(\mu, b) = r(\mu, b) r(\lambda, a)$. From the first part of the proof it follows that $r(\mu, b) a \subseteq ar(\mu, b)$. By Theorem 5.12, this implies that $r(\mu, b) e^a(\delta_1) = e^a(\delta_1) r(\mu, b)$ for all Borel sets δ_1 in $\mathbb{R}$. Keeping the Borel set δ_1 fixed for the moment, it follows from the first part of the proof that $e^a(\delta_1) b \subseteq be^a(\delta_1)$. Using Theorem 5.12 once more, it follows that $e^a(\delta_1) e^b(\delta_2) = e^b(\delta_2) e^a(\delta_1)$ for all Borel sets δ_2 in $\mathbb{R}$. Since δ_1 is an arbitrary Borel set in $\mathbb{R}$, this shows that (i) implies (ii).

If condition (ii) is satisfied, then it follows immediately from Definition 5.2 that the operators $\int_{\mathbb{R}} f de^a$ and $\int_{\mathbb{R}} g de^b$ commute for any two functions $f, g \in B_b(\mathbb{R})$. Since, $r(\lambda, a) = \int_{\mathbb{R}} (\lambda - t)^{-1} de^a(t)$ and $r(\mu, b) = \int_{\mathbb{R}} (\mu - t)^{-1} de^b(t)$ for all $\lambda, \mu \in \mathbb{C} \diagdown \mathbb{R}$, this implies that (i) holds. ■

7 Polar decomposition

The following important result should be mentioned first.

Theorem 7.1 *If x is a closed, densely defined linear operator in H, then the operator x^*x is self-adjoint and positive. Moreover, $\mathfrak{D}(x^*x)$ is a core for the operator x.*

The self-adjoint positive operator x^*x has a unique positive square root $(x^*x)^{1/2}$.

Definition 7.2 *For every closed, densely defined operator x in H the absolute value $|x|$ of x is defined by $|x| = (x^*x)^{1/2}$.*

It should be noted that $\mathfrak{D}(x^*x)$ is a core for the operators $|x|$ and x.

Recall next that an operator $v \in B(H)$ is called a *partial isometry* if $\|v\xi\|_H = \|\xi\|_H$ for all $\xi \in \mathrm{Ker}(v)^\perp$. In this case, $\mathrm{Ran}(v)$ is a closed subspace of H. The projection $p = s(v)$ onto $\mathrm{Ker}(v)^\perp$ is called the *initial projection* of v and the projection $q = r(v)$ onto $\mathrm{Ran}(v)$ is called the *final projection* of v. It is easy to see that $p = v^*v$ and $q = vv^*$. Note that $p = v^*qv$, $q = vqv^*$ and $v = vp = qv$.

Conversely, if $v \in B(H)$ is such that v^*v is a projection, then v is a partial isometry with initial projection v^*v (and final projection vv^*).

Theorem 7.3 (polar decomposition) *Let $x : \mathfrak{D}(x) \to H$ be a closed and densely defined operator. There exists a partial isometry v, with initial projection $r(|x|)$ and final projection $r(x)$, such that $x = v|x|$. Moreover, if $x = wa$, where a is positive self-adjoint and w is a partial isometry with initial projection $r(a)$, then $a = |x|$ and $w = v$.*

The factorization $x = v|x|$ in the above theorem is called the *polar decomposition* of x. The last statement in this theorem is usually referred to as *the uniqueness of the polar decomposition.* Since $v^*v = r(|x|)$, it is also clear that $|x| = v^*x$. There are a number of statements implicit in the above theorem. In particular, $\mathfrak{D}(|x|) = \mathfrak{D}(x)$ and $\||x|\xi\|_H = \|x\xi\|_H$ for all $\xi \in \mathfrak{D}(x)$. Hence, $\mathrm{Ker}(|x|) = \mathrm{Ker}(x)$, $n(|x|) = n(x)$ and $r(|x|) = r(x^*) = s(x)$. This implies that $x^* = r(|x|)x^* = v^*vx^*$. It follows from $x = v|x|$ that $x^* = |x|v^*$ (see Theorem 4.1 (v)) and so,

$$x^* = v^*vx^* = v^*(v|x|v^*). \tag{2}$$

Since $v|x|v^* \geq 0$ and $vv^* = r(x) = r(v|x|v^*)$, it follows from the uniqueness of the polar decomposition that $v|x|v^* = |x^*|$ and that $x^* = v^*|x^*|$ is the polar decomposition of x^*.

Another application of the uniqueness of the polar decomposition is given in the following proposition.

Proposition 7.4 *If $x : \mathfrak{D}(x) \to H$ is a closed, densely defined operator with polar decomposition $x = v|x|$ and if $u \in B(H)$ is unitary, then uxu^* is closed and its polar decomposition is given by*

$$uxu^* = (uvu^*)(u|x|u^*).$$

In particular, $uxu^* = x$ if and only if $uvu^* = v$ and $u|x|u^* = |x|$.

8 Quadratic forms

As before, H is a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. If D is a linear subspace of H and $\mathfrak{q} : D \times D \to \mathbb{C}$ is a map which is linear in the first variable and conjugate linear in the second, then $\mathfrak{q}$ is called a *sesquilinear form* on H with *domain* D. The domain of $\mathfrak{q}$ is usually denoted by $\mathfrak{D}(\mathfrak{q})$. Given a sesquilinear form $\mathfrak{q}$, the corresponding *quadratic form*, also denoted by $\mathfrak{q}$, is defined by setting $\mathfrak{q}(\xi) = \mathfrak{q}(\xi, \xi)$, $\xi \in \mathfrak{D}(\mathfrak{q})$. It follows from the *polarization identity*

$$\mathfrak{q}(\xi, \eta) = \frac{1}{4} \sum_{k=0}^{3} i^k \mathfrak{q}\left(\xi + i^k \eta, \xi + i^k \eta\right), \quad \xi, \eta \in \mathfrak{D}(\mathfrak{q}),$$

that a sesquilinear form is uniquely determined by its quadratic form. From now on, both sesquilinear forms and quadratic forms will be simply called *forms*.

A form $\mathfrak{q}$ is called *symmetric* if $\mathfrak{q}(\xi, \eta) = \overline{\mathfrak{q}(\eta, \xi)}$ for all $\xi, \eta \in \mathfrak{D}(\mathfrak{q})$; note that this implies that $\mathfrak{q}(\xi) \in \mathbb{R}$ for all $\xi \in \mathfrak{D}(\mathfrak{q})$. Furthermore, $\mathfrak{q}$ is called *positive* if $\mathfrak{q}$ is symmetric and $\mathfrak{q}(\xi) \geq 0$ for all $\xi \in \mathfrak{D}(\mathfrak{q})$. From now on, only positive forms will be considered. Note that any positive form satisfies the Cauchy-Schwarz inequality, that is,

$$|\mathfrak{q}(\xi, \eta)| \leq \mathfrak{q}(\xi)^{1/2} \mathfrak{q}(\eta)^{1/2}, \quad \xi, \eta \in \mathfrak{D}(\mathfrak{q}),$$

and the map $\xi \longmapsto \mathfrak{q}(\xi)^{1/2}$ is a seminorm on $\mathfrak{D}(\mathfrak{q})$. This implies in particular that $\mathfrak{q}(\xi_n) \to \mathfrak{q}(\xi)$ for any sequence $\{\xi_n\}_{n=1}^{\infty}$ in $\mathfrak{D}(\mathfrak{q})$ and $\xi \in \mathfrak{D}(\mathfrak{q})$ satisfying $\mathfrak{q}(\xi_n - \xi) \to 0$.

Definition 8.1 *A positive form $\mathfrak{q}$ on H is called closed if for any sequence $\{\xi_n\}_{n=1}^{\infty}$ in $\mathfrak{D}(\mathfrak{q})$ and $\xi \in H$ the conditions* *(i).* $\xi_n \to \xi$ *in* H,

(ii). $\mathfrak{q}(\xi_n - \xi_m) \to 0$ *as* $n, m \to \infty$,

imply that $\xi \in \mathfrak{D}(\mathfrak{q})$ *and* $\mathfrak{q}(\xi_n - \xi) \to 0$ *as* $n \to \infty$.

Definition 8.2 *Let* $\mathfrak{q}$ *be a positive form on* H. *A linear subspace* D *of* $\mathfrak{D}(\mathfrak{q})$ *is called a core of* $\mathfrak{q}$ *if for each* $\xi \in \mathfrak{D}(\mathfrak{q})$ *there exists a sequence* $\{\xi_n\}_{n=1}^{\infty}$ *in* D *such that* $\xi_n \to \xi$ *in* H *and* $\mathfrak{q}(\xi_n - \xi) \to 0$.

Suppose that $\mathfrak{q}$ is a positive form on H and define

$$\langle \xi, \eta \rangle_{\mathfrak{q}} = \mathfrak{q}(\xi, \eta) + \langle \xi, \eta \rangle, \quad \xi, \eta \in \mathfrak{D}(\mathfrak{q}).$$

Evidently, $\langle \cdot, \cdot \rangle_{\mathfrak{q}}$ is an inner product in $\mathfrak{D}(\mathfrak{q})$. The corresponding norm is denoted by $\|\cdot\|_{\mathfrak{q}}$ and satisfies

$$\|\xi\|_{\mathfrak{q}}^2 = \mathfrak{q}(\xi) + \|\xi\|_H^2, \quad \xi \in \mathfrak{D}(\mathfrak{q}). \tag{3}$$

Note that convergence of a sequence with respect to $\|\cdot\|_{\mathfrak{q}}$ always implies convergence in H and $\mathfrak{q}$ is continuous on $\mathfrak{D}(\mathfrak{q})$ with respect to $\|\cdot\|_{\mathfrak{q}}$. The following facts are easily verified:

(a). A positive form $\mathfrak{q}$ is closed if and only if $\mathfrak{D}(\mathfrak{q})$ is a Hilbert space with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{q}}$;

(b). A linear subspace D of $\mathfrak{D}(\mathfrak{q})$ is a core of $\mathfrak{q}$ if and only if D is dense in $\mathfrak{D}(\mathfrak{q})$ with respect to the norm $\|\cdot\|_{\mathfrak{q}}$;

(c). Let $\mathfrak{q}_1$ and $\mathfrak{q}_2$ be closed positive forms on H and suppose that D is a subspace of $\mathfrak{D}(\mathfrak{q}_1) \cap \mathfrak{D}(\mathfrak{q}_2)$ which is a core for both $\mathfrak{q}_1$ and $\mathfrak{q}_2$. If $\mathfrak{q}_1(\xi) = \mathfrak{q}_2(\xi)$ for all $\xi \in D$, then $\mathfrak{q}_1 = \mathfrak{q}_2$.

In the set of all positive forms a partial ordering is introduced as follows.

Definition 8.3 *Given two positive forms* $\mathfrak{q}_1$ *and* $\mathfrak{q}_2$ *on* H, *then* $\mathfrak{q}_1 \leq \mathfrak{q}_2$ *whenever* $\mathfrak{D}(\mathfrak{q}_2) \subseteq \mathfrak{D}(\mathfrak{q}_1)$ *and* $\mathfrak{q}_1(\xi) \leq \mathfrak{q}_2(\xi)$ *for all* $\xi \in \mathfrak{D}(\mathfrak{q}_2)$.

Lemma 8.4 *If* $\{\mathfrak{q}_\alpha\}$ *is an increasing net of positive forms on* H *and if* $\mathfrak{q}$ *is defined by setting*

$$\begin{aligned} D &= \left\{ \xi \in \bigcap_\alpha \mathfrak{D}(\mathfrak{q}_\alpha) : \sup_\alpha \mathfrak{q}_\alpha(\xi) < \infty \right\}, \\ \mathfrak{q}(\xi) &= \sup_\alpha \mathfrak{q}_\alpha(\xi) = \lim_\alpha \mathfrak{q}_\alpha(\xi), \quad \xi \in D, \end{aligned}$$

then $\mathfrak{q}$ *is a positive form on* H *with domain* $\mathfrak{D}(\mathfrak{q}) = D$.

Proof. Observe that it follows from (3) that D is also given by

$$D = \left\{ \xi \in \bigcap_{\alpha} \mathfrak{D}(\mathfrak{q}_\alpha) : \sup_\alpha \|\xi\|_{\mathfrak{q}_\alpha} < \infty \right\},$$

from which it is immediately clear that D is a linear subspace of H. From the polarization identity it follows that $\lim_\alpha \mathfrak{q}_\alpha(\xi, \eta)$ exists in $\mathbb{C}$ for all $\xi, \eta \in D$. Therefore, $\mathfrak{q} : D \times D \to \mathbb{C}$ may be defined by setting $\mathfrak{q}(\xi, \eta) = \lim_\alpha \mathfrak{q}_\alpha(\xi, \eta)$, $\xi, \eta \in D$. Evidently, $\mathfrak{q}$ is a sesquilinear form with domain $\mathfrak{D}(\mathfrak{q}) = D$ and the corresponding quadratic form is given by

$$\mathfrak{q}(\xi) = \mathfrak{q}(\xi, \xi) = \lim_\alpha \mathfrak{q}_\alpha(\xi, \xi) = \sup_\alpha \mathfrak{q}_\alpha(\xi), \quad \xi \in D.$$

It is also clear that $\mathfrak{q}$ is symmetric and positive. ■

It is easily verified that the positive form $\mathfrak{q}$ constructed in the lemma above, is actually the supremum of the net $\{\mathfrak{q}_\alpha\}$ in the partially ordered set of all positive forms on H. Therefore, this form $\mathfrak{q}$ is denoted by $\sup_\alpha \mathfrak{q}_\alpha$.

Lemma 8.5 *If $\{\mathfrak{q}_\alpha\}$ is an increasing net of closed positive forms on H, then $\mathfrak{q} = \sup_\alpha \mathfrak{q}_\alpha$ is also closed.*

Proof. Note that the norm $\|\cdot\|_{\mathfrak{q}}$ in $\mathfrak{D}(\mathfrak{q})$ satisfies $\|\xi\|_{\mathfrak{q}} = \sup_\alpha \|\xi\|_{\mathfrak{q}_\alpha}$ for all $\xi \in \mathfrak{D}(\mathfrak{q})$. Suppose that $\{\xi_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathfrak{D}(\mathfrak{q})$ with respect to $\|\cdot\|_{\mathfrak{q}}$. Since $\{\xi_n\}_{n=1}^{\infty}$ is a Cauchy sequence in H, there exists $\xi \in H$ such that $\|\xi_n - \xi\|_H \to 0$. Furthermore, each $\mathfrak{q}_\alpha$ is closed and $\{\xi_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathfrak{D}(\mathfrak{q}_\alpha)$ with respect to $\|\cdot\|_{\mathfrak{q}_\alpha}$. This implies that $\xi \in \mathfrak{D}(\mathfrak{q}_\alpha)$ and $\|\xi_n - \xi\|_{\mathfrak{q}_\alpha} \to 0$ for each α. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|\xi_n - \xi_m\|_{\mathfrak{q}} \leq \varepsilon$ for all $n, m \geq N$ and hence, $\|\xi_n - \xi_m\|_{\mathfrak{q}_\alpha} \leq \varepsilon$ for all $n, m \geq N$ and all α. Since $\|\xi_m - \xi\|_{\mathfrak{q}_\alpha} \to 0$ for each α, it follows that $\|\xi_n - \xi\|_{\mathfrak{q}_\alpha} \leq \varepsilon$ for all $n \geq N$ and all α and so, $\sup_\alpha \|\xi_n - \xi\|_{\mathfrak{q}_\alpha} \leq \varepsilon$ for all $n \geq N$. This implies that $\xi_n - \xi \in \mathfrak{D}(\mathfrak{q})$ for all $n \geq N$. Hence $\xi \in \mathfrak{D}(\mathfrak{q})$, and $\|\xi_n - \xi\|_{\mathfrak{q}} \leq \varepsilon$ for all $n \geq N$. Consequently, $\xi_n \to \xi$ in $\mathfrak{D}(\mathfrak{q})$ with respect to $\|\cdot\|_{\mathfrak{q}}$. This shows that $\mathfrak{D}(\mathfrak{q})$ is complete with respect to $\|\cdot\|_{\mathfrak{q}}$ and so, $\mathfrak{q}$ is closed. ■

If the domain $\mathfrak{D}(\mathfrak{q})$ is dense in H, then the form $\mathfrak{q}$ is called *densely defined.* If two positive forms $\mathfrak{q}_1$ and $\mathfrak{q}_2$ satisfy $\mathfrak{q}_1 \leq \mathfrak{q}_2$ and if $\mathfrak{q}_2$ is densely defined, then $\mathfrak{q}_1$ is also densely defined, as $\mathfrak{D}(\mathfrak{q}_2) \subseteq \mathfrak{D}(\mathfrak{q}_1)$.

The set of all closed positive densely defined forms on H is denoted by $\mathfrak{Q}^+$. The set $\mathfrak{Q}^+$ is equipped with the partial ordering inherited from the set of all positive forms.

Corollary 8.6 *If* $\{\mathfrak{q}_\alpha\}$ *is an increasing net in* $\mathfrak{Q}^+$ *and if* $\mathfrak{q}_0 \in \mathfrak{Q}^+$*is such that* $\mathfrak{q}_\alpha \le \mathfrak{q}_0$ *for all* α*, then* $\sup \mathfrak{q}_\alpha \in \mathfrak{Q}^+$ *(and* $\sup \mathfrak{q}_\alpha$ *is the supremum of* $\{\mathfrak{q}_\alpha\}$ *in* $\mathfrak{Q}^+$*).*

Let $a : \mathfrak{D}(a) \to H$ be a positive self-adjoint operator in H. Define the form $\mathfrak{q}_a$ by setting $\mathfrak{D}(\mathfrak{q}_a) = \mathfrak{D}\left(a^{1/2}\right)$ and $\mathfrak{q}_a(\xi,\eta) = \left\langle a^{1/2}\xi, a^{1/2}\eta\right\rangle$, $\xi,\eta \in \mathfrak{D}(\mathfrak{q}_a)$. Evidently, $\mathfrak{q}_a$ is a positive, densely defined form on H and $\mathfrak{q}_a(\xi) = \left\|a^{1/2}\xi\right\|_H^2$, $\xi \in \mathfrak{D}(\mathfrak{q}_a)$. Using that $a^{1/2}$ is a closed operator, it follows that $\mathfrak{q}_a$ is also closed.

The form $\mathfrak{q}_a$ is called the quadratic form corresponding to the positive self-adjoint operator a. It is easily verified that the operator a is uniquely determined by its quadratic form in the following sense: if a and b are two positive self-adjoint operators in H such that $\mathfrak{q}_a = \mathfrak{q}_b$, then $a = b$. Observe, furthermore, that a linear subspace D of $\mathfrak{D}(\mathfrak{q}_a)$ is a core of $\mathfrak{q}_a$ if and only if D is a core of $a^{1/2}$. In particular, $\mathfrak{D}(a)$ is a core of $\mathfrak{D}(\mathfrak{q}_a)$.

The set of all positive self-adjoint operators in H is denoted by $\mathfrak{H}^+$. By the above observation, the map $a \longmapsto \mathfrak{q}_a$ from $\mathfrak{H}^+$ into $\mathfrak{Q}^+$ is injective. The next proposition shows that this map is also surjective (and hence, it is a bijection).

Proposition 8.7 *If* $\mathfrak{q}$ *is a closed densely defined positive form on* H*, then there exists a unique self-adjoint positive operator* a *in* H *such that* $\mathfrak{D}(\mathfrak{q}) = \mathfrak{D}\left(a^{1/2}\right)$ *and* $\mathfrak{q}(\xi) = \left\|a^{1/2}\xi\right\|_H^2$ *for all* $\xi \in \mathfrak{D}(\mathfrak{q})$ *(in other words,* $\mathfrak{q} = \mathfrak{q}_a$*).*

Proof. Since $\mathfrak{q}$ is closed, $\mathfrak{D}(\mathfrak{q})$ is a Hilbert space with respect to $\langle\cdot,\cdot\rangle_{\mathfrak{q}}$. For $\xi \in H$, define the linear functional φ_ξ on $\mathfrak{D}(\mathfrak{q})$ by $\varphi_\xi(\eta) = \langle \eta, \xi\rangle$. Since

$$\left|\varphi_\xi(\eta)\right| = |\langle \eta,\xi\rangle| \le \|\xi\|_H \|\eta\|_H \le \|\xi\|_H \|\eta\|_{\mathfrak{q}}$$

for all $\eta \in \mathfrak{D}(\mathfrak{q})$, it follows that φ_ξ is bounded with respect to $\|\cdot\|_{\mathfrak{q}}$ and $\left\|\varphi_\xi\right\| \le \|\xi\|_H$. Consequently, there exists a unique element $b\xi \in \mathfrak{D}(\mathfrak{q})$ such that $\varphi_\xi(\eta) = \langle \eta, b\xi\rangle_{\mathfrak{q}}$ for all $\eta \in \mathfrak{D}(\mathfrak{q})$ and $\|b\xi\|_{\mathfrak{q}} = \left\|\varphi_\xi\right\| \le \|\xi\|_H$. The map $\xi \longmapsto b\xi$ is linear from H into $\mathfrak{D}(\mathfrak{q})$. Composing this map with the embedding of $\mathfrak{D}(\mathfrak{q})$ into H, yields a linear operator $b : H \to H$, which is bounded and $\|b\|_{B(H)} \le 1$. From the definition of $b\xi$ it follows that

$$\langle \eta,\xi\rangle = \langle \eta, b\xi\rangle_{\mathfrak{q}} = \mathfrak{q}(\eta, b\xi) + \langle \eta, b\xi\rangle, \quad \eta \in \mathfrak{D}(\mathfrak{q}), \xi \in H, \tag{4}$$

and so,

$$\mathfrak{q}(\eta, b\xi) = \langle \eta, \xi - b\xi\rangle, \quad \eta \in \mathfrak{D}(\mathfrak{q}), \xi \in H. \tag{5}$$

Observe the following facts:

(i). b is self-adjoint. Indeed, if $\xi, \zeta \in H$, then $b\zeta \in \mathfrak{D}(\mathfrak{q})$ and so, it follows from (4) that

$$\langle b\zeta, \xi \rangle = \langle b\zeta, b\xi \rangle_{\mathfrak{q}} = \overline{\langle b\xi, b\zeta \rangle_{\mathfrak{q}}} = \overline{\langle b\xi, \zeta \rangle} = \langle \zeta, b\xi \rangle .$$

(ii). b is injective. Indeed, if $\xi \in H$ satisfies $b\xi = 0$, then $\langle \eta, \xi \rangle = \langle \eta, b\xi \rangle_{\mathfrak{q}} = 0$ for all $\eta \in \mathfrak{D}(\mathfrak{q})$. Since $\mathfrak{D}(\mathfrak{q})$ is dense in H, this implies that $\xi = 0$.

(iii). $\mathrm{Ran}(b)$ is dense in $\mathfrak{D}(\mathfrak{q})$ with respect to $\|\cdot\|_{\mathfrak{q}}$ (and hence, $\mathrm{Ran}(b)$ is dense in H). Indeed, if $\eta \in \mathfrak{D}(\mathfrak{q})$ satisfies $\langle \eta, b\xi \rangle_{\mathfrak{q}} = 0$ for all $\xi \in H$, then $\langle \eta, \xi \rangle = \langle \eta, b\xi \rangle_{\mathfrak{q}} = 0$ for all $\xi \in H$ and hence, $\eta = 0$.

Consider the inverse operator $b^{-1} : \mathfrak{D}(b^{-1}) \to H$, where $\mathfrak{D}(b^{-1}) = \mathrm{Ran}(b)$. From the above observations it follows that b^{-1} is closed, densely defined and self-adjoint (recall that $(b^{-1})^* = (b^*)^{-1} = b^{-1}$). Equation (5) can then be written as

$$\mathfrak{q}(\eta, \zeta) = \langle \eta, b^{-1}\zeta - \zeta \rangle = \langle \eta, (b^{-1} - \mathbf{1})\zeta \rangle, \quad \eta \in \mathfrak{D}(\mathfrak{q}), \zeta \in \mathfrak{D}(b^{-1}).$$

The operator $a = b^{-1} - \mathbf{1}$ is closed, densely defined and self-adjoint with $\mathfrak{D}(a) = \mathfrak{D}(b^{-1}) = \mathrm{Ran}(b)$ and satisfies

$$\mathfrak{q}(\eta, \zeta) = \langle \eta, a\zeta \rangle, \quad \eta \in \mathfrak{D}(\mathfrak{q}), \zeta \in \mathfrak{D}(a).$$

In particular, if $\xi, \eta \in \mathfrak{D}(a)$, then

$$\mathfrak{q}(\xi, \eta) = \langle \xi, a\eta \rangle = \langle a\xi, \eta \rangle$$

and $\langle a\xi, \xi \rangle = \mathfrak{q}(\xi, \xi) = \mathfrak{q}(\xi) \geq 0$ for all $\xi \in \mathfrak{D}(a)$, which shows that a is positive.

Finally, it is shown that $\mathfrak{D}(\mathfrak{q}) = \mathfrak{D}(a^{1/2})$ and $\mathfrak{q}(\xi) = \|a^{1/2}\xi\|_H^2$ for all $\xi \in \mathfrak{D}(\mathfrak{q})$. Let $\mathfrak{q}_a$ be the closed positive form corresponding to a, that is, $\mathfrak{D}(\mathfrak{q}_a) = \mathfrak{D}(a^{1/2})$ and $\mathfrak{q}_a(\xi) = \|a^{1/2}\xi\|_H^2$ for all $\xi \in \mathfrak{D}(\mathfrak{q}_a)$. Since $\mathfrak{D}(a)$ is a core of the operator $a^{1/2}$, it follows that $\mathfrak{D}(a)$ is a core for $\mathfrak{q}_a$. As has been observed above, $\mathfrak{D}(a) = \mathrm{Ran}(b)$ is dense in $\mathfrak{D}(\mathfrak{q})$ with respect to $\|\cdot\|_{\mathfrak{q}}$, which implies that $\mathfrak{D}(a)$ is a core for $\mathfrak{q}$. Furthermore,

$$\mathfrak{q}(\xi) = \langle a\xi, \xi \rangle = \langle a^{1/2}\xi, a^{1/2}\xi \rangle = \|a^{1/2}\xi\|_H^2 = \mathfrak{q}_a(\xi), \quad \xi \in \mathfrak{D}(a).$$

This implies that $\mathfrak{q} = \mathfrak{q}_a$ and the proof is complete. ■

It follows from the above results that the map $a \longmapsto \mathfrak{q}_a$ is a bijection from $\mathfrak{H}^+$ onto $\mathfrak{Q}^+$. Via this map, the partial ordering in $\mathfrak{Q}^+$ may be transferred to

$\mathfrak{H}^+$: if a and b are two self-adjoint positive operators, then $a \leq b$ whenever $\mathfrak{q}_a \leq \mathfrak{q}_b$. This partial ordering in $\mathfrak{H}^+$ is called the *quadratic form ordering* in $\mathfrak{H}^+$. By definition, the map $a \longmapsto \mathfrak{q}_a$ is an order isomorphism from $\mathfrak{H}^+$ onto $\mathfrak{Q}^+$. Therefore, the following result is now an immediate consequence of Corollary 8.6 and Lemma 8.4.

Proposition 8.8 *If $\{a_\beta\}$ is an increasing net in $\mathfrak{H}^+$ and $b \in \mathfrak{H}^+$ is such that $a_\beta \leq b$ for all β, then $a = \sup_\beta a_\beta$ exists in $\mathfrak{H}^+$. Moreover, this supremum a is characterized by:*

$$\begin{aligned} \mathfrak{D}\left(a^{1/2}\right) &= \left\{\xi \in \bigcap_\beta \mathfrak{D}\left(a_\beta^{1/2}\right) : \sup_\beta \left\|a_\beta^{1/2}\xi\right\|_H < \infty\right\}, \\ \left\|a^{1/2}\xi\right\|_H &= \sup_\beta \left\|a_\beta^{1/2}\xi\right\|_H, \quad \xi \in \mathfrak{D}\left(a^{1/2}\right). \end{aligned}$$

If $a \in \mathfrak{H}^+$ and $u \in B(H)$ is unitary, then $u^*au \in \mathfrak{H}^+$ and so, the map $\varphi_u : \mathfrak{H}^+ \to \mathfrak{H}^+$ may be defined by $\varphi_u(a) = u^*au$. It is straightforward to show that $\varphi_u : \mathfrak{H}^+ \to \mathfrak{H}^+$ is an order isomorphism. Therefore, the next result follows immediately from Lemma 3.1.

Corollary 8.9 *Let $\mathcal{U}_0$ be a non-empty collection of unitary operators on H. Suppose that $\{a_\beta\}$ is an increasing net in $\mathfrak{H}^+$ and $a \in \mathfrak{H}^+$ such that $a = \sup_\beta a_\beta$. If $u^*a_\beta u = a_\beta$ for all $u \in \mathcal{U}_0$ and all β, then $u^*au = a$ for all $u \in \mathcal{U}_0$.*

Remark 8.10 *Another useful consequence of the above considerations should be pointed out. Suppose that a is a positive self-adjoint operator in the Hilbert space H. If $f : \sigma(a) \to [0, \infty)$ is a Borel function, then $f(a) \in \mathfrak{H}^+$ and it follows from Theorem 5.7 (i) that the corresponding quadratic form $\mathfrak{q}_{f(a)}$ is given by*

$$\mathfrak{D}\left(\mathfrak{q}_{f(a)}\right) = \left\{\xi \in H : \int_{\sigma(a)} f(\lambda)\, de_{\xi,\xi}(\lambda) < \infty\right\}$$

and

$$\mathfrak{q}_{f(a)}(\xi) = \int_{\sigma(a)} f(\lambda)\, de_{\xi,\xi}(\lambda), \qquad \xi \in \mathfrak{D}\left(\mathfrak{q}_{f(a)}\right).$$

A combination of Proposition 7 and the monotone convergence theorem immediately yields the following result: if $f, f_n : \sigma(a) \to [0, \infty)$ $(n = 1, 2, \ldots)$ are Borel functions such that $0 \leq f_n(\lambda) \uparrow f(\lambda)$ for all $\lambda \in \sigma(a)$, then $f_n(a) \uparrow f(a)$ in $\mathfrak{H}^+$.

9 Algebras with an involution

In this and the next section, some aspects of the theory of abstract operator algebras, which will be convenient to have available, are discussed. First, the general concept of a $*$-algebra.

Let $\mathcal{A}$ be an algebra over the complex numbers. A map $x \longmapsto x^*$, $x \in \mathcal{A}$, from $\mathcal{A}$ into itself is said to be an *involution* if

(i). $(x+y)^* = x^* + y^*$;

(ii). $(\lambda x)^* = \bar{\lambda} x^*$;

(iii). $(xy)^* = y^* x^*$;

(iv). $(x^*)^* = x$,

whenever $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. An algebra equipped with an involution is called a *$*$-algebra*. If $\mathcal{A}$ is a $*$-algebra, then an element $x \in \mathcal{A}$ is called *self-adjoint* (or, *hermitian*) if $x^* = x$. The set of all self-adjoint elements in $\mathcal{A}$ is denoted by $\mathcal{A}_h$, which is clearly a real linear subspace of $\mathcal{A}$. If $x, y \in \mathcal{A}_h$, then $xy \in \mathcal{A}_h$ if and only if $xy = yx$. Note furthermore that x^*x and xx^* belong to $\mathcal{A}_h$ for every $x \in \mathcal{A}$. For $x \in \mathcal{A}$, set

$$\operatorname{Re}(x) = \frac{1}{2}(x + x^*), \quad \operatorname{Im}(x) = \frac{1}{2i}(x - x^*).$$

Clearly, $\operatorname{Re}(x), \operatorname{Im}(x) \in \mathcal{A}_h$ and $x = \operatorname{Re}(x) + i \operatorname{Im}(x)$ for all $x \in \mathcal{A}$. Conversely, if $x \in \mathcal{A}$ and $x = x_1 + ix_2$ with $x_1, x_2 \in \mathcal{A}_h$, then necessarily $x_1 = \operatorname{Re}(x)$ and $x_2 = \operatorname{Im}(x)$.

A $*$-algebra $\mathcal{A}$ is called *unital* if it possesses a multiplicative identity, a *unit element*, denoted by $\mathbf{1} = \mathbf{1}_{\mathcal{A}}$. Note that $\mathbf{1}^* = \mathbf{1}$. An element x in the unital algebra $\mathcal{A}$ is said to be *invertible* if there exists $y \in \mathcal{A}$ such that $xy = yx = \mathbf{1}$; in this case, the element y is unique and denoted by x^{-1}, the *inverse* of x. It is easy to see that $x \in \mathcal{A}$ is invertible if and only if x^* is invertible and, in this case, $(x^{-1})^* = (x^*)^{-1}$.

An element $x \in \mathcal{A}$ is called *normal* if $x^*x = xx^*$. Furthermore, $u \in \mathcal{A}$ is said to be *unitary* if $u^*u = uu^* = \mathbf{1}$ (equivalently, u is invertible and $u^{-1} = u^*$). All unitary elements in $\mathcal{A}$ form a (multiplicative) group, which is denoted by $U(\mathcal{A})$. An element $p \in \mathcal{A}$ is said to be a *projection* if $p^2 = p$ and $p^* = p$. The set of all projections in $\mathcal{A}$ is denoted by $P(A)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras. The map $\phi : \mathcal{A} \to \mathcal{B}$ is called a *$*$-homomorphism* whenever ϕ is an algebra homomorphism (that is, ϕ is linear and $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathcal{A}$) satisfying $\phi(x^*) = \phi(x)^*$ for all

$x \in \mathcal{A}$. If, in addition, $\mathcal{A}$ and $\mathcal{B}$ are unital and $\phi(\mathbf{1}_{\mathcal{A}}) = \mathbf{1}_{\mathcal{B}}$, then ϕ is called *unital* $*$*-homomorphism.* If $\phi : \mathcal{A} \to \mathcal{B}$ is a injective $*$-homomorphism, then ϕ is said to be a $*$*-isomorphism.* The algebras $\mathcal{A}$ and $\mathcal{B}$ are said to be $*$*-isomorphic* if there exists a $*$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. If $\phi : \mathcal{A} \to \mathcal{B}$ is a $*$-isomorphism onto and $\mathcal{A}$ is unital, then $\mathcal{B}$ is also unital and ϕ is a unital $*$-isomorphism. A $*$-isomorphism from $\mathcal{A}$ onto itself is called a $*$*-automorphism* of $\mathcal{A}$.

A subset $\mathcal{S}$ of a $*$-algebra $\mathcal{A}$ is called self-adjoint if $x^* \in \mathcal{S}$ whenever $x \in \mathcal{S}$. A self-adjoint subalgebra $\mathcal{S}$ of $\mathcal{A}$ is said to be a $*$*-subalgebra* and, in this case, $\mathcal{S}$ itself is a $*$-algebra with respect to the algebraic operations and involution inherited from $\mathcal{A}$.

10 C^*-algebras

An algebra $\mathcal{A}$ equipped with a norm $\|\cdot\|_{\mathcal{A}}$ such that $\mathcal{A}$ is a Banach space and

(i). $\|xy\|_{\mathcal{A}} \le \|x\|_{\mathcal{A}} \|y\|_{\mathcal{A}}$ for all $x, y \in \mathcal{A}$,

is called a *Banach algebra.* If $\mathcal{A}$ has a unit element $\mathbf{1}$, then it will be assumed that $\|\mathbf{1}\|_{\mathcal{A}} = 1$. Furthermore, if $\mathcal{A}$ is a $*$-algebra and its norm satisfies in addition

(ii). $\|x^*\|_{\mathcal{A}} = \|x\|_{\mathcal{A}}$ for all $x \in \mathcal{A}$,

then $\mathcal{A}$ is called a *Banach* $*$*-algebra.*

A C^**-algebra* is a $*$-algebra $\mathcal{A}$ equipped with a norm $\|\cdot\|_{\mathcal{A}}$ such that $\mathcal{A}$ is a Banach algebra and

(iii). $\|x^*x\|_{\mathcal{A}} = \|x\|_{\mathcal{A}}^2$ for all $x \in \mathcal{A}$.

If $\mathcal{A}$ is a C^*-algebra, then it is easy to see that the norm also satisfies condition (ii). So, any C^*-algebra is a Banach $*$-algebra. Moreover, if a C^*-algebra has a unit element $\mathbf{1}$, then the equality $\|\mathbf{1}\|_{\mathcal{A}} = 1$ is automatically satisfied.

If $\mathcal{A}$ is a C^*-algebra and $\mathcal{A}_1$ is a closed $*$-subalgebra of $\mathcal{A}$, then $\mathcal{A}_1$ is a C^*-algebra with respect to the structure inherited from $\mathcal{A}$ and $\mathcal{A}_1$ is called a C^*-subalgebra of $\mathcal{A}$.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two C^*-algebras and that $\phi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism. Then, $\|\phi(x)\|_{\mathcal{B}} \le \|x\|_{\mathcal{A}}$ for all $x \in \mathcal{A}$ and so, in particular, ϕ is continuous. Moreover, if ϕ is a $*$-isomorphism, then $\|\phi(x)\|_{\mathcal{B}} = \|x\|_{\mathcal{A}}$ for all $x \in \mathcal{A}$. Consequently, two $*$-isomorphic C^*-algebras are isometrically isomorphic as well.

A few simple examples of C^*-algebras are listed in the following example.

Example 10.1 *1. Let X be an arbitrary non-empty set. Let $\ell_\infty(X)$ be the $*$-algebra of all bounded complex-valued functions $f : X \to \mathbb{C}$, with pointwise algebraic operations and involution given by pointwise complex conjugation. Equipped with the* sup-*norm given by $\|f\|_\infty = \sup_{t\in X}|f(t)|$ for all $f \in \ell_\infty(X)$, the algebra $\ell_\infty(X)$ is a commutative unital C^*-algebra.*

2. *Let X be a locally compact Hausdorff space. A continuous function $f : X \to \mathbb{C}$ is said to vanish at infinity if for every $\varepsilon > 0$ the set $\{t \in X : |f(t)| \geq \varepsilon\}$ is compact. Let $C_0(X)$ be the $*$-algebra of all continuous complex-valued functions on X vanishing at infinity (which is a $*$-subalgebra of $\ell_\infty(X)$), equipped with the* sup-*norm $\|\cdot\|_\infty$. Clearly, $C_0(X)$ is a commutative C^*-algebra, which has a unit element if and only if X is compact.*

   By the Gelfand-Naimark theorem, *every commutative C^*-algebra $\mathcal{A}$ is $*$-isomorphic to an algebra $C_0(X)$ for some suitable locally compact Hausdorff space X (which is compact if, and only if, $\mathcal{A}$ is unital).*

3. *Let (Ω, Σ, μ) be a measure space and $L_\infty(\Omega, \Sigma, \mu) = L_\infty(\mu)$ be the space of all μ-essentially bounded complex-valued Σ-measurable functions on Ω, with identification of functions which coincide μ-almost everywhere. The space $L_\infty(\mu)$, equipped with the norm given by*

$$\|f\|_\infty = \operatorname*{esssup}_{t\in\Omega}|f(t)|, \quad f \in L_\infty(\mu),$$

   is a commutative C^*-algebra with respect to the pointwise algebraic operations and the involution given by complex conjugation.

   All measure spaces considered will be assumed to be so-called Maharam measure spaces. A measure space (Ω, Σ, μ) is said to be a Maharam measure space *if it is semi-finite (that is, for every $E \in \Sigma$ with $\mu(E) > 0$ there exists $F \in \Sigma$ such that $F \subseteq E$ and $0 < \mu(F) < \infty$) and its measure algebra (that is, the quotient Boolean algebra of Σ by the ideal of μ-null sets) is complete. Clearly, all σ-finite measure spaces are Maharam. It should be noted that a measure space (Ω, Σ, μ) is Maharam if and only if the space $L_\infty^{\mathbb{R}}(\mu)$ of all real-valued functions in $L_\infty(\mu)$ is a Dedekind complete vector lattice. Another equivalent condition is that $L_\infty(\mu)$ is (isometrically isomorphic to) the Banach dual space of $L_1(\mu)$ (equivalently, the Radon-Nikodym theorem is valid for the measure space (Ω, Σ, μ)).*

4. *If H is a Hilbert space, then the algebra $B(H)$ of all bounded linear operators on H is a C^*-algebra, as observed already in Section 1. This*

C^* algebra is non-commutative whenever $\dim H > 1$. *Any closed self-adjoint subalgebra of $B(H)$ is also a C^*-algebra.*

The Gelfand-Naimark-Segal theorem *asserts that any C^*-algebra $\mathcal{A}$ is $*$-isomorphic to a C^*-subalgebra of $B(H)$ for some Hilbert space H.*

For the remaining part of this section it will be assumed that $\mathcal{A}$ is a unital C^*-algebra, although most of the results mentioned can be adjusted to the non-unital situation via adjunction of a unit element.

As in any unital Banach algebra, for $x \in \mathcal{A}$ the set $\sigma(x) = \sigma_{\mathcal{A}}(x)$ of all $\lambda \in \mathbb{C}$ for which $\lambda\mathbf{1} - x$ is not invertible in $\mathcal{A}$, is called the *spectrum* of x. The complement $\rho(x) = \mathbb{C} \diagdown \sigma(x)$ is called the *resolvent set* of x. The spectrum $\sigma(x)$ is a non-empty compact subset of $\mathbb{C}$. If $u \in U(\mathcal{A})$, then $\sigma(u) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and if $a \in \mathcal{A}_h$, then $\sigma(a) \subseteq \mathbb{R}$. In general, if $\mathcal{B}$ is a Banach subalgebra of the Banach algebra $\mathcal{A}$ and $x \in \mathcal{B}$, then $\sigma_{\mathcal{A}}(x) \subsetneq \sigma_{\mathcal{B}}(x)$. However, for C^*-algebras the following important observation holds: if $\mathcal{B}$ is a C^*-subalgebra of the C^*-algebra $\mathcal{A}$, then $\sigma_{\mathcal{A}}(x) = \sigma_{\mathcal{B}}(x)$ for all $x \in \mathcal{B}$.

The following theorem, which is a consequence of the Gelfand-Naimark theorem for commutative C^*-algebras, gives the functional calculus for normal elements.

Theorem 10.2 *Let $\mathcal{A}$ be a unital C^*-algebra and x be a normal element of $\mathcal{A}$. There exists a unique unital $*$-isomorphism $\phi : C(\sigma(x)) \to \mathcal{A}$ satisfying $\phi(\iota) = x$, where $\iota(\lambda) = \lambda$ for all $\lambda \in \mathbb{C}$. Actually, ϕ is a $*$-isomorphism from $C(\sigma(x))$ onto the C^*-subalgebra of $\mathcal{A}$ generated by x and $\mathbf{1}$.*

In the situation of the above theorem, for any $f \in C(\sigma(x))$ the corresponding element $\phi(f)$ is denoted by $f(x)$ and the mapping $f \longmapsto f(x)$ is called the *functional calculus* of the normal element x. The *spectral mapping theorem* holds for this functional calculus, that is,

$$\sigma(f(x)) = f(\sigma(x)) = \{f(\lambda) : \lambda \in \sigma(x)\} \tag{6}$$

for every $f \in C(\sigma(x))$. Moreover, if $f \in C(\sigma(x))$ and $g \in C(\sigma(f(x)))$, then $(g \circ f)(x) = g(f(x))$.

Remark 10.3 *Suppose that H is a Hilbert space and $x \in B(H)$ is a normal operator. Let $\phi : C(\sigma(x)) \to B(H)$ be the $*$-isomorphism which exists according to Theorem 10.2. On the other hand, it follows from Theorems 5.10 and 5.13, that there exists a unique spectral measure $e^x : \mathcal{B}(\sigma(x)) \to B(H)$ for the operator x. Let $B_b(\sigma(x))$ denote the C^*-algebra of all bounded Borel functions on $\sigma(x)$. For $f \in B_b(\sigma(x))$ define $\Phi(f) \in B(H)$ by*

$$\Phi(f) = \int_{\sigma(x)} f(\lambda)\, de^x(\lambda).$$

It follows from Theorem 5.3 that $\Phi : B_b(\sigma(x)) \to B(H)$ is a unital $*$-homomorphism satisfying $\Phi(\iota) = x$ (by the definition of the spectral measure e^x). Consequently, if p is a polynomial in λ and $\bar{\lambda}$, then $\phi(p) = p(x, x^*) = \Phi(p)$. Since the subalgebra of $C(\sigma(x))$ consisting of all such polynomials is dense in $C(\sigma(x))$ with respect to $\|\cdot\|_\infty$, the continuity of ϕ and Φ implies that $\phi(f) = \Phi(f)$ for all $f \in C(\sigma(x))$. Hence,

$$f(x) = \phi(f) = \int_{\sigma(x)} f(\lambda)\, de^x(\lambda)$$

for all $f \in C(\sigma(x))$. So, the notation

$$f(x) = \int_{\sigma(x)} f(\lambda)\, de^x(\lambda)$$

may be used for all $f \in B_b(\sigma(x))$ (and also for $f \in B(\sigma(x))$; see Definition 5.11) without any danger of confusion.

Actually, the existence of the spectral measure of a normal operator $x \in B(H)$ may be obtained via Theorem 10.2 (a method which is analogous to the Riesz representation theorem for positive functionals on a $C(K)$-space).

Let $\mathcal{A}$ be a (unital) C^*-algebra. An element $x \in \mathcal{A}$ is said to be *positive* if $x^* = x$ and $\sigma(x) \subseteq [0, \infty)$. This is denoted by $x \geq 0$. The set of all positive elements in $\mathcal{A}$ is denoted by $\mathcal{A}^+$. If $x \in \mathcal{A}^+$, then it follows from Theorem 10.2 that there exists an element $y \in \mathcal{A}^+$ such that $y^2 = x$ (indeed, take $y = f(x)$, where $f(\lambda) = \lambda^{1/2}$ for $\lambda \in \sigma(x) \subseteq [0, \infty)$ and use (6) to see that $y \geq 0$). The element $y \in \mathcal{A}^+$ with $y^2 = x$ is unique and denoted by $\sqrt{x}$ or $x^{1/2}$, the *positive square root* of x. The following proposition describes the basic properties of positive elements.

Proposition 10.4 *(i). $\mathcal{A}^+$ is a cone in $\mathcal{A}$ (that is, $\lambda x + \mu y \in \mathcal{A}^+$ whenever $x, y \in \mathcal{A}^+$ and $\lambda, \mu \in \mathbb{R}_+$) which is closed and proper (that is, $\mathcal{A}^+ \cap (-\mathcal{A}^+) = \{0\}$);*

(ii). $\mathcal{A}^+ = \{x^* x : x \in \mathcal{A}\} = \{a^2 : a \in \mathcal{A}_h\}$;

(iii). if $\mathcal{A}$ is a C^*-subalgebra of $B(H)$, for some Hilbert space H, and if $x \in \mathcal{A}$, then $x \in \mathcal{A}^+$ if and only if $\langle x\xi, \xi \rangle \geq 0$ for all $\xi \in H$ (that is, x is a positive self-adjoint operator on H).

In the real vector space $\mathcal{A}_h$, a partial ordering is defined by setting $a \leq b$ whenever $a, b \in \mathcal{A}_h$ satisfy $b - a \in \mathcal{A}^+$. In the next proposition, some of the properties of this partial ordering are collected together.

Proposition 10.5 *(i). $\mathcal{A}_h$ is a partially ordered vector space (that is, $a + c \leq b + c$ and $\lambda a \leq \lambda b$ whenever $a \leq b$ in $\mathcal{A}_h$, $c \in \mathcal{A}_h$ and $\lambda \in \mathbb{R}_+$);*

(ii). if $a \leq b$ in $\mathcal{A}_h$, then $x^*ax \leq x^*bx$ for all $x \in \mathcal{A}$;

(iii). if $a, b \in \mathcal{A}^+$ and $ab = ba$, then $ab \in \mathcal{A}^+$;

(iv). if $0 \leq a \leq b$ in $\mathcal{A}_h$, then $\|a\|_{\mathcal{A}} \leq \|b\|_{\mathcal{A}}$ and $a^{1/2} \leq b^{1/2}$; moreover, if in addition a is invertible, then b is also invertible and $0 \leq b^{-1} \leq a^{-1}$;

(v). if $x \in \mathcal{A}$ is normal and $f \in C(\sigma(x))$ satisfies $f(\lambda) \geq 0$ for all $\lambda \in \sigma(x)$, then $f(x) \geq 0$ in $\mathcal{A}_h$.

Define the functions f_0, f_1 and f_2 on $\mathbb{R}$ by $f_0(\lambda) = |\lambda|$, $f_1(\lambda) = \lambda^+$ and $f_2(\lambda) = \lambda^-$ respectively, for all $\lambda \in \mathbb{R}$. For $a \in \mathcal{A}_h$, set $|a| = f_0(a)$, the *absolute value* of a, $a^+ = f_1(a)$, the *positive part* of a, and $a^- = f_2(a)$, the *negative part* of a. From the properties of the functional calculus it is clear that $|a|, a^+, a^- \in \mathcal{A}^+$, $a = a^+ - a^-$ and $|a| = a^+ + a^-$. This shows in particular that the positive cone $\mathcal{A}^+$ is generating in $\mathcal{A}_h$, that is, $\mathcal{A}_h = \mathcal{A}^+ - \mathcal{A}^+$. Note that the absolute value of $a \in \mathcal{A}_h$ is also given by $|a| = (a^2)^{1/2}$.

It is worth noting, that for every $a \in \mathcal{A}_h$ with $\|a\|_{\mathcal{A}} \leq 1$, the element $\mathbf{1} - a^2$ is positive and the elements $u_1 = a + i(\mathbf{1} - a^2)^{1/2}$ and $u_2 = a - i(\mathbf{1} - a^2)^{1/2}$ are unitary. Moreover, $a = \frac{1}{2}(u_1 + u_2)$. This implies, in particular, that $\mathcal{A}$ is the linear span of its unitary elements.

A linear functional φ on a C^*-algebra $\mathcal{A}$ is said to be *positive* if $\varphi(a) \geq 0$ for all $a \in \mathcal{A}^+$. If φ is a positive functional on $\mathcal{A}$, then φ is necessarily continuous, $\varphi(x^*) = \overline{\varphi(x)}$ and $|\varphi(x)|^2 \leq \|\varphi\|_{\mathcal{A}^*} \varphi(x^*x)$ for all $x \in \mathcal{A}$. Moreover, if $\mathcal{A}$ is unital, then $\|\varphi\|_{\mathcal{A}^*} = \varphi(\mathbf{1})$. The set of all positive functionals in $\mathcal{A}^*$ is denoted by $(\mathcal{A}^*)^+$.

Let $\mathcal{A}^*$ denote the Banach space dual of the C^*-algebra $\mathcal{A}$. For any $\varphi \in \mathcal{A}^*$, the functional $\bar{\varphi} \in \mathcal{A}^*$ is defined by setting

$$\bar{\varphi}(x) = \overline{\varphi(x^*)}, \quad x \in \mathcal{A}.$$

A functional $\varphi \in \mathcal{A}^*$ is called *hermitian* if $\bar{\varphi} = \varphi$. It should be observed that a functional $\varphi \in \mathcal{A}^*$ is hermitian if and only if $\varphi(a) \in \mathbb{R}$ for all $a \in \mathcal{A}_h$. In particular, any positive functional is hermitian. The collection of all hermitian functionals in $\mathcal{A}$ is denoted by $\mathcal{A}_h^*$, which is a real linear subspace of $\mathcal{A}^*$. For any $\varphi \in \mathcal{A}^*$, the *real and imaginary* part are defined by setting

$$\operatorname{Re}\varphi = \frac{1}{2}(\varphi + \bar{\varphi}), \quad \operatorname{Im}\varphi = \frac{1}{2i}(\varphi - \bar{\varphi}),$$

respectively. It is clear that $\operatorname{Re}\varphi, \operatorname{Im}\varphi \in \mathcal{A}_h^*$ and that $\varphi = \operatorname{Re}\varphi + i\operatorname{Im}\varphi$. Consequently, $\mathcal{A}^* = \mathcal{A}_h^* \oplus i\mathcal{A}_h^*$. A partial ordering in $\mathcal{A}_h^*$ is defined by setting $\varphi \leq \psi$ whenever $\varphi, \psi \in \mathcal{A}_h^*$ satisfy $\psi - \varphi \in (\mathcal{A}^*)^+$. With respect to this partial ordering the space $\mathcal{A}_h^*$ is a partially ordered vector space with $(\mathcal{A}^*)^+$ as its positive cone.

Let $\{\mathcal{A}_j\}_{j\in J}$ be a family of C^*-algebras, where J is an arbitrary set of indices. Denote by $\mathcal{A}$ the set of all families $(x_j) = (x_j)_{j\in J}$, with $x_j \in \mathcal{A}_j$ for all $j \in J$ and $\sup_j \|x_j\|_{\mathcal{A}_j} < \infty$. Defining the algebraic operations, involution and norm in $\mathcal{A}$ by

$$\begin{aligned} (x_j) + (y_j) &= (x_j + y_j), \quad (x_j)(y_j) = (x_j y_j), \quad \lambda (x_j) = (\lambda x_j), \\ (x_j)^* &= \left(x_j^*\right), \quad \|(x_j)\|_{\mathcal{A}} = \sup_j \|x_j\|_{\mathcal{A}_j}, \end{aligned}$$

respectively, it follows that $\mathcal{A}$ is a C^*-algebra, which is called the *C^*-algebra product* of the family $\{\mathcal{A}_j\}_{j\in J}$. This product C^*-algebra is denoted by $C^* - \prod_j \mathcal{A}_j$, or simply by $\prod_j \mathcal{A}_j$, if no confusion may arise.

11 Von Neumann algebras

Given a non-empty subset $\mathcal{S}$ of $B(H)$, the *commutant* $\mathcal{S}'$ of $\mathcal{S}$ is defined by

$$\mathcal{S}' = \{x \in B(H) : xy = yx \ \ \forall y \in \mathcal{S}\},$$

which is a wo-closed unital subalgebra of $B(H)$. If $\mathcal{S}$ is self-adjoint, then $\mathcal{S}'$ is a wo-closed unital C^*-subalgebra of $B(H)$. Defining the *bi-commutant* $\mathcal{S}''$ of $\mathcal{S}$ by $\mathcal{S}'' = (\mathcal{S}')'$, it is clear that $\mathcal{S} \subseteq \mathcal{S}''$ and $\mathcal{S}' = \mathcal{S}'''$.

Definition 11.1 *A $*$-subalgebra $\mathcal{M}$ of $B(H)$ is said to be a* von Neumann algebra *if $\mathcal{M} = \mathcal{M}''$.*

In order to indicate that $\mathcal{M}$ is contained in the algebra $B(H)$, it is said that $\mathcal{M}$ acts on the Hilbert space H (briefly, *$\mathcal{M}$ is a von Neumann algebra on H*). If $\mathcal{M}$ is a von Neumann algebra, then $\mathcal{M}$ is a wo-closed unital C^*-subalgebra of $B(H)$. The simplest examples of von Neumann algebras are given by the algebra $B(H)$ itself and the subalgebra $\mathbb{C}\mathbf{1} = \mathbb{C}_H = \{\lambda\mathbf{1} : \lambda \in \mathbb{C}\}$. For any non-empty subset $\mathcal{S}$ of $B(H)$, the commutant $\mathcal{S}'$ and the bi-commutant $\mathcal{S}''$ are von Neumann algebras. Actually, $\mathcal{S}''$ is the von Neumann algebra generated by $\mathcal{S}$, that is, the smallest von Neumann algebra on H containing $\mathcal{S}$.

Suppose that $\mathcal{M}$ is a von Neumann algebra. By the observations at the end of Section 10, every element of $\mathcal{M}'$ is a linear combination of unitary elements of $\mathcal{M}'$. Consequently, if $x \in B(H)$, then $x \in \mathcal{M}$ if and only if $xu = ux$ (equivalently, $x = uxu^*$) for all $u \in U(\mathcal{M}')$. Furthermore, if $x \in B(H)$ is normal, it follows from Theorem 5.12 that the following statements are equivalent:

1. $x \in \mathcal{M}$;
2. $e^x(\delta) \in \mathcal{M}$ for all Borel sets $\delta \subseteq \mathbb{C}$;
3. $f(x) \in \mathcal{M}$ for all bounded Borel functions $f : \sigma(x) \to \mathbb{C}$.

In particular, if $a \in \mathcal{M}^+$, then $a^{1/2} \in \mathcal{M}^+$. Consequently, if $x \in \mathcal{M}$, then $x^*x \in \mathcal{M}^+$ and so, $|x| = (x^*x)^{1/2} \in \mathcal{M}^+$. Furthermore, if $x \in \mathcal{M}$, with polar decomposition $x = v|x|$, then it follows from Proposition 7.4 that $v = uvu^*$ for all $u \in U(\mathcal{M}')$ and hence, $v \in \mathcal{M}$. This also implies that the support projection $s(x) = v^*v$ and the range projection $r(x) = vv^*$ both belong to $\mathcal{M}$.

The following result is an immediate consequence of Lemma 3.4 and Theorem 3.3.

Theorem 11.2 *Suppose that $\{a_\beta\}$ is an increasing net in $\mathcal{M}_h$.*

(i). If $a \in B_h(H)$ is such that $a_\beta \uparrow_\beta a$ in $B_h(H)$, then $a \in \mathcal{M}_h$ (and hence, $a_\beta \uparrow_\beta a$ in $\mathcal{M}_h$).

(ii). If $\{a_\beta\}$ is bounded from above in $B_h(H)$, then there exists $a \in \mathcal{M}_h$ such that $a_\beta \uparrow_\beta a$ in $\mathcal{M}_h$ (and $a_\beta \uparrow_\beta a$ in $B_h(H)$).

The following observation is easily checked.

Lemma 11.3 *If $\{a_\beta\}$ is an increasing net in $\mathcal{M}_h$ and $a \in \mathcal{M}_h$, then $a_\beta \uparrow_\beta a$ in $\mathcal{M}_h$ if and only if $a_\beta \overset{uwo}{\to} a$.*

Suppose that $\{\mathcal{M}_i : i \in I\}$ is a collection of von Neumann algebras acting on a Hilbert space H and define $\mathcal{M} = \bigcap_{i \in I} \mathcal{M}_i$. Since $x \in \mathcal{M}$ if and only if $x \in \mathcal{M}_i''$ for all $i \in I$, it is clear that

$$\mathcal{M} = \left(\bigcup_{i \in I} \mathcal{M}_i' \right)',$$

which shows, in particular, that $\mathcal{M}$ is a von Neumann algebra. Moreover,

$$\mathcal{M}' = \left(\bigcup_{i \in I} \mathcal{M}_i' \right)''$$

and so, $\mathcal{M}'$ is the von Neumann algebra generated by $\{\mathcal{M}_i' : i \in I\}$.

The *center* $Z(\mathcal{M})$ of a von Neumann algebra $\mathcal{M}$ is defined by

$$Z(\mathcal{M}) = \{x \in \mathcal{M} : xy = yx \;\; \forall y \in \mathcal{M}\}.$$

Since $Z(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$, it follows from the above observations that $Z(\mathcal{M})$ is a von Neumann algebra, which is clearly commutative. Note that $Z(\mathcal{M}') = Z(\mathcal{M})$. If $Z(\mathcal{M}) = \mathbb{C}_H$, then the von Neumann algebra $\mathcal{M}$ is said to be a *factor*. Since $B(H)' = \mathbb{C}_H$, it is evident that $B(H)$ is a factor.

As has been observed before, if $\mathcal{M}$ is a von Neumann algebra, then $\mathcal{M}$ is a wo-closed subalgebra of $B(H)$ (and hence, $\mathcal{M}$ is also so-, uwo-, and uso-closed, as these topologies are stronger than the wo-topology). Moreover, by Theorem 2.2, the norm closed unit ball $B(H)_1$ is wo-compact and so, $\mathcal{M}_1$ is wo-closed (and hence, $\mathcal{M}_1$ is also so-, uwo-, and uso-closed). These topological properties actually characterize von Neumann algebras. This is the famous *Double Commutant Theorem* of J. von Neumann.

Theorem 11.4 (Double Commutant Theorem) *Let $\mathcal{M}$ be a unital $*$-subalgebra of $B(H)$ and let $\mathcal{M}_1$ be its closed unit ball (with respect to the operator norm). The following statements are equivalent.*

(i). $\mathcal{M}$ is a von Neumann algebra, that is, $\mathcal{M} = \mathcal{M}''$.

(ii). $\mathcal{M}$ is wo-closed (or, equivalently, so-, uwo-, uso-closed).

(iii). $\mathcal{M}_1$ is wo-closed (or, equivalently, so-, uwo-, uso-closed).

Note that the only non-trivial implication in this theorem is that $\mathcal{M}$ is a von Neumann algebra whenever $\mathcal{M}_1$ is uso-closed. If $\mathcal{M}$ is a von Neumann algebra, then $\mathcal{M}_1 = \mathcal{M} \cap B(H)_1$ is wo-compact and hence, $\mathcal{M}_1$ is uwo-compact, as the wo- and uwo-topology coincide on norm bounded subsets of $B(H)$.

The dual space of $\mathcal{M}$ with respect to the norm topology is denoted by $\mathcal{M}^*$. With slight abuse of notation, for $\xi, \eta \in H$, the linear functional $\omega_{\xi,\eta}$ on $\mathcal{M}$ is defined by setting $\omega_{\xi,\eta}(x) = \langle x\xi, \eta \rangle$, $x \in \mathcal{M}$ (see Section 2). Let $\mathcal{M}_\sim$ be the linear subspace of $\mathcal{M}^*$ generated by $\{\omega_{\xi,\eta} : \xi, \eta \in H\}$. Evidently, the wo-topology is equal to $\sigma(\mathcal{M}, \mathcal{M}_\sim)$. The norm closure of $\mathcal{M}_\sim$ is denoted by $\mathcal{M}_*$. Furthermore, $\mathcal{M}_{wo}'$, $\mathcal{M}_{so}'$, $\mathcal{M}_{uwo}'$ and $\mathcal{M}_{uso}'$ denote the dual spaces of $\mathcal{M}$ with respect to the wo-, so-, uwo- and uso-topology, respectively. The following theorem collects together the main features of the duality theory of von Neumann algebras.

Theorem 11.5 *Let $\mathcal{M}$ be a von Neumann algebra on the Hilbert space H.*

(i). $\mathcal{M}'_{so} = \mathcal{M}'_{wo}$ *and* $\varphi \in \mathcal{M}'_{wo}$ *if and only if* φ *is given by* $\varphi = \sum_{j=1}^{n} \omega_{\xi_j,\eta_j}$ *with* $\xi_1, \ldots, \xi_n \in H$ *and* $\eta_1, \ldots, \eta_n \in H$.

(ii). $\mathcal{M}'_{uso} = \mathcal{M}'_{uwo} = \mathcal{M}_*$ *and* $\varphi \in \mathcal{M}'_{uwo}$ *if and only if* φ *is given by* $\varphi = \sum_{j=1}^{\infty} \omega_{\xi_j,\eta_j}$, *as a norm convergent series in* $\mathcal{M}^*$, *with* $\{\xi_j\}_{j=1}^{\infty}$ *and* $\{\eta_j\}_{j=1}^{\infty}$ *in* H *satisfying* $\sum_{j=1}^{\infty} \|\xi_j\|_H^2 < \infty$ *and* $\sum_{j=1}^{\infty} \|\eta_j\|_H^2 < \infty$.

(iii). If φ *is a linear functional on* $\mathcal{M}$, *then* φ *belongs to* $\mathcal{M}_*$ *if and only if the restriction of* φ *to the unit ball* $\mathcal{M}_1$ *is wo-continuous (equivalently, is so-, uso- or uwo-continuous).*

(iv). The uwo-topology in $\mathcal{M}$ *is equal to* $\sigma(\mathcal{M}, \mathcal{M}_*)$.

(v). Every $x \in \mathcal{M}$ *defines a bounded linear functional* $\hat{x} \in (\mathcal{M}_*)^*$ *by setting* $\hat{x}(\varphi) = \varphi(x)$ *for all* $\varphi \in \mathcal{M}_*$. *The map* $x \longmapsto \hat{x}$ *is a isometrically isomorphism from* $\mathcal{M}$ *onto* $(\mathcal{M}_*)^*$.

By (v) of the above theorem, $\mathcal{M}$ may be identified with the Banach dual space of $\mathcal{M}_*$ via the mapping $x \longmapsto \hat{x}$. Therefore, $\mathcal{M}_*$ is called the *pre-dual* of $\mathcal{M}$.

For *positive linear functionals* on $\mathcal{M}$ there is another important characterization of uwo-continuity. Recall that a linear functional φ on $\mathcal{M}$ is called positive if $\varphi(a) \geq 0$ whenever $a \in \mathcal{M}^+$. This is denoted by $\varphi \geq 0$. Such positive functionals belong to $\mathcal{M}^*$ and it may be shown that every element of $\mathcal{M}^*$ can be written as a linear combination of at most four positive linear functionals. Examples of positive functional on $\mathcal{M}$ are given by functionals of the form $\omega_\xi = \omega_{\xi,\xi}$, where $\xi \in H$. Note that it follows immediately from (ii) of Theorem 11.5, via polarization, that every $\varphi \in \mathcal{M}_*$ may be written as a linear combination of positive functionals in $\mathcal{M}_*$.

Definition 11.6 *Let* φ *be a positive linear functional on the von Neumann algebra* $\mathcal{M}$.

(i). φ *is said to be* normal *if* $a_\beta \uparrow_\beta a$ *in* $\mathcal{M}^+$ *implies that* $\varphi(a_\beta) \uparrow_\beta \varphi(a)$.

(ii). φ *is called* completely additive *if* $\varphi(\sum_\alpha p_\alpha) = \sum_\alpha \varphi(p_\alpha)$ *for every system* $\{p_\alpha\}$ *of pairwise orthogonal projections in* $\mathcal{M}$.

Theorem 11.7 *For a positive functional* φ *on* $\mathcal{M}$ *the following statements are equivalent:*

(i). φ *is normal;*

(ii). φ *is completely additive;*

(iii). φ *is uwo-continuous (equivalently,* $\varphi \in \mathcal{M}_*$*);*

(iv). φ *is given by* $\varphi = \sum_{j=1}^{\infty} \omega_{\xi_j}$*, as a norm convergent series in* $\mathcal{M}^*$*, with* $\{\xi_j\}_{j=1}^{\infty}$ *in* H *satisfying* $\sum_{j=1}^{\infty} \|\xi_j\|_H^2 < \infty$*.*

For normal positive functionals the following non-commutative version of the Radon-Nikodym theorem holds, which is due to S. Sakai.

Theorem 11.8 *If* $\varphi \in \mathcal{M}_*$ *and* $\psi \in \mathcal{M}^*$ *are such that* $0 \leq \psi \leq \varphi$*, then* $\psi \in \mathcal{M}_*$ *and there exists* $a \in \mathcal{M}_h$ *such that* $0 \leq a \leq \mathbf{1}$ *and* $\psi(x) = \varphi(axa)$ *for all* $x \in \mathcal{M}$*.*

The notions of positivity and normality can also be introduced for linear mappings between von Neumann algebras.

Definition 11.9 *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be von Neumann algebras, acting on the Hilbert spaces* H *and* K *respectively, and* π *be a linear mapping from* $\mathcal{M}$ *into* $\mathcal{N}$*.*

(i). π *is called* positive *if* $\pi(a) \geq 0$ *in* $\mathcal{N}$ *whenever* $a \geq 0$ *in* $\mathcal{M}$*.*

(ii). If π *is positive, then* π *is said to be* normal *if* $\pi(a_\beta) \uparrow_\beta \pi(a)$ *in* $\mathcal{N}$ *whenever* $a_\beta \uparrow_\beta a$ *in* $\mathcal{M}$*.*

Note that any $*$-homomorphism $\pi : \mathcal{M} \to \mathcal{N}$ is positive. Furthermore, a positive linear mapping $\pi : \mathcal{M} \to \mathcal{N}$ is normal if and only if it is continuous with respect to the ultra-weak operator topologies.

Theorem 11.10 *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be von Neumann algebras. If* $\pi : \mathcal{M} \to \mathcal{N}$ *is a unital* $*$*-homomorphism, then* π *is normal if and only if* $\pi(\mathcal{M})$ *is a von Neumann algebra.*

Some important constructions concerning von Neumann algebras are discussed next. First, the necessary terminology is introduced. Let $e \in P(B(H))$ be a projection onto $K = e(H)$. For $x \in B(H)$, define $x_e \in B(K)$ by setting $x_e(\xi) = ex\xi$ for all $\xi \in K$, that is, $x_e = (ex)|_K$. Note that $x_e = (exe)_e$ for all $x \in B(H)$. For any non-empty subset $\mathcal{D} \subseteq B(H)$, denote $\mathcal{D}_e = \{x_e : x \in \mathcal{D}\}$. In particular, the following two situations will be considered:

1. Let $\mathcal{M} \subseteq B(H)$ be a von Neumann algebra and $e \in P(\mathcal{M})$. Define

$$e\mathcal{M}e = \{exe : x \in \mathcal{M}\},$$

which is a $*$-subalgebra of $\mathcal{M}$ with unit element e. Note that an element $x \in \mathcal{M}$ belongs to $e\mathcal{M}e$ if and only if $x(K) \subseteq K$ and $x\left(K^{\perp}\right) = \{0\}$. If $x \in e\mathcal{M}e$, then $x_e = x_{|K}$. In this situation, $\mathcal{M}_e$ is a unital $*$-subalgebra of $B(K)$ and the mapping $\phi_e : e\mathcal{M}e \to \mathcal{M}_e$, defined by $\phi_e(x) = x_e$ for all $x \in e\mathcal{M}e$, is a surjective unital $*$-isomorphism.

2. Let $\mathcal{M} \subseteq B(H)$ be a von Neumann algebra and $e \in P(\mathcal{M}')$. If $x \in \mathcal{M}$, then $xe = ex$, that is, $x(K) \subseteq K$ and $x\left(K^{\perp}\right) \subseteq K^{\perp}$. Moreover, $x_e = x_{|K}$ for all $x \in \mathcal{M}$. The set $\mathcal{M}_e$ is a unital $*$-subalgebra of $B(K)$ and the mapping $\psi_e : \mathcal{M} \to \mathcal{M}_e$, defined by $\psi_e(x) = x_e$, is a surjective $*$-homomorphism. The kernel of ψ_e is given by

$$\operatorname{Ker}(\psi_e) = \{x(\mathbf{1} - z(e)) : x \in \mathcal{M}\},$$

where $z(e)$ denotes the *central support* of e (see Definition 13.2). In particular, ψ_e is an isomorphism if and only if $z(e) = \mathbf{1}$.

Theorem 11.11 *Let $\mathcal{M}$ be a von Neumann algebra on the Hilbert space H. Suppose that $e \in P(\mathcal{M})$ and put $K = e(H)$.*

(i). $\mathcal{M}_e$ and $(\mathcal{M}')_e$ are von Neumann algebras on the Hilbert space K and $(\mathcal{M}_e)' = (\mathcal{M}')_e$.

(ii). The center of $\mathcal{M}_e$ is equal to $(Z(\mathcal{M}))_e$.

(iii). If $\mathcal{M}$ is a factor, then $\mathcal{M}_e$ and $(\mathcal{M}')_e$ are also factors.

By (i) of the above theorem, there is no danger of confusion to denote $(\mathcal{M}')_e$ simply by $\mathcal{M}'_e$ whenever $e \in P(\mathcal{M})$. The von Neumann algebra $\mathcal{M}_e$ is called the *reduced von Neumann algebra* of $\mathcal{M}$ with respect to $e \in P(\mathcal{M})$. The von Neumann algebra $\mathcal{M}'_e$ is called the *induced von Neumann algebra* by $\mathcal{M}'$ on $K = e(H)$. Frequently, the $*$-algebra $e\mathcal{M}e$ is identified with the von Neumann algebra $\mathcal{M}_e$ on K.

12 Direct products of von Neumann algebras

Suppose that $\{H_j : j \in J\}$ is a collection of Hilbert spaces. The inner product on each H_j will be denoted simply by $\langle \cdot, \cdot \rangle$. The set of all families $\left(\xi_j\right)_{j \in J} =$

(ξ_j), with $\xi_j \in H_j$ for all $j \in J$, satisfying $\sum_{j\in J} \left\|\xi_j\right\|_{H_j}^2 < \infty$ is denoted by $\sum_{j\in J} H_j$.With addition and scalar multiplication given by

$$(\xi_j) + (\eta_j) = (\xi_j + \eta_j), \quad \lambda(\xi_j) = (\lambda\xi_j)$$

for all $(\xi_j), (\eta_j) \in \sum_{j\in J} H_j$ and all $\lambda \in \mathbb{C}$, the set $\sum_{j\in J} H_j$ is a vector space. Defining the inner product of two vectors $(\xi_j), (\eta_j) \in \sum_{j\in J} H_j$ by

$$\langle (\xi_j), (\eta_j) \rangle = \sum_j \langle \xi_j, \eta_j \rangle,$$

it follows that $\sum_{j\in J} H_j$ is a Hilbert space. Note that the norm of $(\xi_j) \in \sum_{j\in J} H_j$ is given by $\left(\sum_j \left\|\xi_j\right\|_{H_j}^2\right)^{1/2}$. The Hilbert space $\sum_{j\in J} H_j$, also denoted by $\sum_j H_j$, is called the *direct sum of the Hilbert spaces* $\{H_j\}_{j\in J}$. If $J = \{1, \ldots, n\}$, then $\sum_j H_j$ is also written as $H_1 \oplus \cdots \oplus H_n$. Note that the algebraic direct sum $\bigoplus_j H_j$ is dense in $\sum_j H_j$. For any $k \in J$, the mapping U_k, which assigns to every $\xi \in H_k$ the vector $(\xi_j) \in \sum_j H_j$ given by $\xi_k = \xi$ and $\xi_j = 0$ whenever $j \neq k$, is an isometric isomorphism from H_k onto a closed subspace $\tilde{H}_k$ of $\sum_j H_j$. Usually, H_k with $\tilde{H}_k$ are identified and so, H_k is considered as a closed subspace of the direct sum $\sum_j H_j$. Note that the subspaces $\{H_j\}_{j\in J}$ are pairwise orthogonal and that $\bigvee_j H_j = \sum_j H_j$. For $k \in J$, the projection in $\sum_j H_j$ onto the subspace H_k is denoted by p_k.

For the sake of convenience, the direct sum Hilbert space $\sum_j H_j$ is denoted, for the moment, simply by H. Suppose that operators $x_j \in B(H_j)$, $j \in J$, are given such that $\sup_j \|x_j\|_{B(H_j)} < \infty$. Then, the linear operator $\oplus_j x_j$ on H may be defined by setting

$$(\oplus_j x_j)(\xi_j) = (x_j \xi_j), \quad (\xi_j) \in H.$$

The operator $\oplus_j x_j$ is bounded and $\|\oplus_j x_j\|_{B(H)} = \sup_j \|x_j\|_{B(H_j)}$. The map $(x_j) \longmapsto \oplus_j x_j$ is a $*$-isomorphism from the C^*-algebra product $\prod_j B(H_j)$ into $B(H)$, the range of which is denoted by $\sum_j B(H_j)$. It is easily verified that an operator $x \in B(H)$ belongs to the C^*-subalgebra $\sum_j B(H_j)$ if and only if $xp_j = p_j x$ for all $j \in J$. Hence, $\sum_j B(H_j)$ is the commutant of the set $\{p_j : j \in J\}$. Therefore, $\sum_j B(H_j)$ is a von Neumann algebra on the Hilbert space H.

Suppose now that $\mathcal{M}_j$ is a von Neumann algebra on each of the Hilbert spaces H_j, $j \in J$. Define

$$\sum_j \mathcal{M}_j = \left\{ \oplus_j x_j : x_j \in \mathcal{M}_j \ \forall j \in J, \sup_j \|x_j\|_{B(H_j)} < \infty \right\},$$

which is a C^*-subalgebra of $\sum_j B(H_j)$ and hence, of $B(H)$, where $H = \sum_j H_j$. Note that $p_k \in \sum_j \mathcal{M}_j$ for all $k \in J$. The commutant of $\sum_j \mathcal{M}_j$ is given by $\left(\sum_j \mathcal{M}_j\right)' = \sum_j \mathcal{M}_j'$ and so, $\left(\sum_j \mathcal{M}_j\right)'' = \sum_j \mathcal{M}_j$, which shows that $\sum_j \mathcal{M}_j$ is a von Neumann algebra on H. The algebra $\sum_j \mathcal{M}_j$ is called the *product von Neumann algebra* of the family $\{\mathcal{M}_j\}_{j \in J}$. The projections p_k, $k \in J$, belong to the center of $\sum_j \mathcal{M}_j$, satisfy $\sum_k p_k = \mathbf{1}$ and $\left(\sum_j \mathcal{M}_j\right)_{p_k} = \mathcal{M}_k$. If the index set J is finite, say $J = \{1, \ldots, n\}$, then $\sum_j \mathcal{M}_j$ is also denoted by $\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n$.

As a converse to the above construction, the following should be observed. Suppose that $\mathcal{M}$ is a von Neumann algebra on a Hilbert space H and assume that $\{p_\alpha\}$ is a collection of pairwise orthogonal projections in the center $Z(\mathcal{M})$ of $\mathcal{M}$ such that $\sum_\alpha p_\alpha = \mathbf{1}$. If $\mathcal{M}_\alpha = \mathcal{M}_{p_\alpha}$, the reduced von Neumann algebra on the Hilbert space $H_\alpha = p_\alpha(H)$, then $H = \sum_\alpha H_\alpha$ and $\mathcal{M} = \sum_\alpha \mathcal{M}_\alpha$.

13 Comparison of projections

In this section, $\mathcal{M}$ is a von Neumann algebra on the Hilbert space H. Let $P(\mathcal{M})$ be the collection of all (orthogonal) projections belonging to $\mathcal{M}$, that is,

$$P(\mathcal{M}) = \left\{p \in \mathcal{M} : p^2 = p, p^* = p\right\}.$$

Evidently, $P(\mathcal{M}) \subseteq P(B(H))$. As has been noted in Section 3, $P(B(H))$ is a complete lattice. The following fundamental observation follows immediately from Lemma 3.4.

Proposition 13.1 *For any non-empty subset $D \subseteq P(\mathcal{M})$ the supremum $\bigvee D$ of D in $P(B(H))$ belongs to $P(\mathcal{M})$. Consequently, $P(\mathcal{M})$ is a complete sublattice of $P(B(H))$ (that is, $P(\mathcal{M})$ is a complete lattice and the lattice operations, finite and infinite, in $P(\mathcal{M})$ coincide with the lattice operations in $P(B(H))$).*

If the von Neumann algebra $\mathcal{M}$ is commutative, then $p \wedge q = pq$ and $p \vee q = p + q - pq$ for all $p, q \in P(\mathcal{M})$. In this case $P(\mathcal{M})$ is a complete Boolean algebra, where the complement of each $p \in P(\mathcal{M})$ is given by $p^\perp = \mathbf{1} - p$. In particular, $P(Z(\mathcal{M}))$ is a complete Boolean algebra and a complete sublattice of $P(\mathcal{M})$, for any von Neumann algebra $\mathcal{M}$.

For any $x \in \mathcal{M}$, the support projection $s(x)$ and the range projection $r(x)$ both belong to $P(\mathcal{M})$ (see the observations following Definition 11.1).

Therefore, it follows from Theorem 4.5 that $s(x)$ (respectively, $r(x)$) is the smallest of all projections $p \in P(\mathcal{M})$ satisfying $x = xp$ (respectively, $x = px$).

Projections belonging to the center $Z(\mathcal{M})$ of $\mathcal{M}$ are called *central projections.*

Definition 13.2 *For $x \in \mathcal{M}$ the* central support $z(x) \in P(Z(\mathcal{M}))$ *is defined by setting*

$$z(x) = \inf\{p \in P(Z(\mathcal{M})) : x = xp\}.$$

Note that $x = xz(x)$, so the above infimum is actually a minimum. For $p \in P(Z(\mathcal{M}))$, the conditions $x = xp$ and $x^* = x^*p$ are equivalent. Hence, $z(x) = z(x^*)$ for all $x \in \mathcal{M}$. From the discussion preceding Definition 13.2, it is clear that $s(x), r(x) \le z(x)$ and that $z(x)$ is also given by

$$z(x) = \inf\{p \in P(Z(\mathcal{M})) : s(x) \le p\} = \inf\{p \in P(Z(\mathcal{M})) : r(x) \le p\}.$$

Note, furthermore, that

$$z(q) = \inf\{p \in P(Z(\mathcal{M})) : q \le p\}$$

whenever $q \in P(\mathcal{M})$. In particular, $z(x) = z(s(x)) = z(r(x))$ for all $x \in \mathcal{M}$. An alternative description of the central support is given in the next proposition.

Proposition 13.3 *The central support $z(x)$ of an element $x \in \mathcal{M}$ is the projection onto the closed subspace of H given by*

$$\overline{\text{span}}\left\{\sum_{j=1}^{n} y_j x \xi_j : y_j \in \mathcal{M}, \xi_j \in H, j = 1, \dots, n; n \in \mathbb{N}\right\}.$$

It is easily verified that

$$z\left(\bigvee_{\alpha} p_\alpha\right) = \bigvee_{\alpha} z(p_\alpha) \tag{7}$$

for any non-empty collection $\{p_\alpha\}$ in $P(\mathcal{M})$.

Definition 13.4 *Let $e, f \in P(\mathcal{M})$ be given.*

(i). The projections e and f are said to be equivalent *(relative to the von Neumann algebra $\mathcal{M}$) if there exists a partial isometry $v \in \mathcal{M}$ with initial projection e and final projection f (that is, $e = v^*v$ and $f = vv^*$). This is denoted by $e \sim f$ (or by $e \overset{\mathcal{M}}{\sim} f$, if it is necessary to emphasize the von Neumann algebra relative to which the projections are equivalent).*

(ii). The projection e is said to be majorized *by f (relative to $\mathcal{M}$) if there exists a projection $f_1 \in P(\mathcal{M})$ such that $f_1 \leq f$ and $e \sim f_1$. This is denoted by $e \precsim f$ (or, $e \precsim_{\mathcal{M}} f$).*

If $x \in \mathcal{M}$, with polar decomposition $x = v|x|$, then $v \in \mathcal{M}$ and $v^*v = s(x)$ and $vv^* = r(x)$. Therefore, $s(x) \sim r(x)$. Evidently, if $\mathcal{M}$ is an abelian von Neumann algebra and $e, f \in P(\mathcal{M})$, then $e \sim f$ if and only if $e = f$ and $e \precsim f$ if and only if $e \leq f$. In the next proposition, some of the properties of the relation $\sim$ are listed.

Proposition 13.5 *(i). The relation $\sim$ is an equivalence relation on the set $P(\mathcal{M})$.*

(ii). If $x \in \mathcal{M}$, then $s(x) \sim r(x)$.

(iii). If $e, f \in P(\mathcal{M})$, then

$$e \vee f - f \sim e - e \wedge f.$$

In particular, if $e \wedge f = 0$, then $e \precsim f^{\perp}$.

(iv). If $e, f \in P(\mathcal{M})$ and $e \sim f$, then $z(e) = z(f)$.

(v). Given $e, f \in P(\mathcal{M})$, there exist $e_1, f_1 \in P(\mathcal{M})$ such that $e_1 \leq e$, $f_1 \leq f$ and $e_1 \sim f_1$ if and only if $z(e)z(f) \neq 0$ (equivalently, there exists $x \in \mathcal{M}$ such that $exf \neq 0$).

(vi). If $e, f \in P(\mathcal{M})$ such that $e \sim f$, then $ep \sim fp$ for all $p \in P(Z(\mathcal{M}))$.

(vii). Suppose that $\{f_i\}_{i \in I}$ and $\{g_i\}_{i \in I}$ are two families of pairwise orthogonal projections in $P(\mathcal{M})$. If $e_i \sim f_i$ for all $i \in I$, then $\sum_{i \in I} e_i \sim \sum_{i \in I} f_i$.

Some properties of the relation $\precsim$ are collected together in the following proposition.

Proposition 13.6 *(i). If $e, f, g \in P(\mathcal{M})$ are such that $e \precsim f$ and $f \precsim g$, then $e \precsim g$.*

(ii). If $e, f \in P(\mathcal{M})$ are such that $e \precsim f$ and $f \precsim e$, then $e \sim f$.

(iii). Suppose that $\{f_i\}_{i \in I}$ and $\{g_i\}_{i \in I}$ are two families of pairwise orthogonal projections in $P(\mathcal{M})$. If $e_i \preccurlyeq f_i$ for all $i \in I$, then $\sum_{i \in I} e_i \precsim \sum_{i \in I} f_i$.

(iv). If $e, f \in P(\mathcal{M})$, then there exists a central projection $p \in P(Z(\mathcal{M}))$ such that $pe \precsim pf$ and $p^{\perp}f \precsim p^{\perp}e$.

(v). If $\mathcal{M}$ is a factor and $e, f \in P(\mathcal{M})$, then either $e \precsim f$ or $f \precsim e$.

Definition 13.7 *Let $\mathcal{M}$ be a von Neumann algebra on the Hilbert space H.*

(i). A projection $e \in P(\mathcal{M})$ is said to be finite *(relative to $\mathcal{M}$) if it follows from $f \in P(\mathcal{M})$, $e \sim f$ and $f \leq e$ that $f = e$. If e is not finite, then e is termed* infinite.

(ii). A projection $e \in P(\mathcal{M})$ is said to be properly infinite *(relative to $\mathcal{M}$) if $e \neq 0$ and for every $p \in P(Z(\mathcal{M}))$, either $pe = 0$ or pe is infinite.*

Before collecting some properties of finite and infinite projections in the next proposition, it should be recalled that a projection $p \in P(\mathcal{M})$ is said to be *countably decomposable* (also called *σ-finite* or, *of countable type*) if every system of $\{e_\alpha\}$ of non-zero pairwise orthogonal projections in $P(\mathcal{M})$, satisfying $e_\alpha \leq p$ for all α, is at most countable. On a separable Hilbert space, every $p \in P(\mathcal{M})$ is clearly countably decomposable.

Proposition 13.8 *(i). If $e \in P(\mathcal{M})$ is finite and if $f \in P(\mathcal{M})$ satisfies $f \precsim e$, then f is finite.*

(ii). If $e, f \in P(\mathcal{M})$ are finite, then $e \vee f$ is a finite projections.

(iii). If $e \in P(\mathcal{M})$ is properly infinite, then there exists $f \in P(\mathcal{M})$ such that $f \leq e$, $f \sim e$ and $f \sim e - f$.

(iv). Let $\{p_\alpha\}$ be a collection of projections in $P(Z(\mathcal{M}))$ and let $p = \bigvee_\alpha p_\alpha$. If $e \in P(\mathcal{M})$ is such that $p_\alpha e$ is finite for all α, then pe is finite.

(v). If $e \in P(\mathcal{M})$ is infinite, then there exists a unique $p \in P(Z(\mathcal{M}))$, satisfying $p \leq z(e)$, such that $(\mathbf{1} - p)e$ is finite and pe is properly infinite.

(vi). If $e \in P(\mathcal{M})$ is properly infinite and if $f \in P(\mathcal{M})$ such that $f \sim e$, then f is properly infinite.

(vii). If $e \in P(\mathcal{M})$ is properly infinite and if $f \in P(\mathcal{M})$ is countably decomposable and $z(f) \leq z(e)$, then $f \precsim e$. Consequently, if $e, f \in P(\mathcal{M})$ are properly infinite and countably decomposable, then $e \sim f$ if and only if $z(e) = z(f)$. In particular, if $\mathcal{M}$ is a factor on a separable Hilbert space, then any two properly infinite projections in $P(\mathcal{M})$ are equivalent.

Definition 13.9 *A projection $e \in P(\mathcal{M})$ is said to be* abelian *if the reduced von Neumann algebra $\mathcal{M}_e$ is abelian (equivalently, $e\mathcal{M}e$ is a commutative $*$-subalgebra of $\mathcal{M}$).*

The following proposition lists some of the properties of abelian projections.

Proposition 13.10 *(i). If $e \in P(\mathcal{M})$ is abelian, then e is finite.*

(ii). A projection $e \in P(\mathcal{M})$ is abelian if and only if e is a minimal element of the set $\{p \in P(\mathcal{M}) : z(p) = z(e)\}$. In particular, if $\mathcal{M}$ is a factor, a non-zero projection $e \in P(\mathcal{M})$ is abelian if and only if e is a minimal projection of $\mathcal{M}$.

(iii). If $\{e_\alpha\}$ is a collection of abelian projections in $P(\mathcal{M})$ such that the central supports $\{z(e_\alpha)\}$ are pairwise orthogonal, then $\sum_\alpha e_\alpha$ is an abelian projection.

(iv). If $e \in P(\mathcal{M})$ is abelian and if $f \in P(\mathcal{M})$ is such that $f \precsim e$, then f is also abelian.

(v). If $e \in P(\mathcal{M})$ is abelian and if $f \in P(\mathcal{M})$ is such that $z(e) \leq z(f)$, then $e \precsim f$.

(vi). If $e, f \in P(\mathcal{M})$ are abelian and $z(e) = z(f)$, then $e \sim f$.

(vii). Suppose that $e \in P(\mathcal{M})$ is abelian. If $f \in P(\mathcal{M})$ satisfies $f \leq z(e)$, then f is a sum of pairwise orthogonal abelian projections.

The *type decomposition* of von Neumann algebras is discussed next.

Definition 13.11 *Let $\mathcal{M}$ be a von Neumann algebra on the Hilbert space H.*

(i). $\mathcal{M}$ is of type I *if there exists an abelian projection $e \in P(\mathcal{M})$ with $z(e) = \mathbf{1}$.*

(ii). $\mathcal{M}$ is of type II *if $\mathcal{M}$ does not contain any non-zero abelian projections and there exists a finite projection $e \in P(\mathcal{M})$ such that $z(e) = \mathbf{1}$.*

(iii). $\mathcal{M}$ is of type III *if $\mathcal{M}$ does not contain any non-zero finite projection.*

(iv). $\mathcal{M}$ is of type I_n*, where n is a cardinal number satisfying $1 \leq n \leq \dim H$, if $\mathbf{1}$ is the sum of n mutually equivalent abelian projections in $P(\mathcal{M})$.*

(v). If $\mathcal{M}$ is of type II, then $\mathcal{M}$ is said to be of type II_1 *(respectively,* type II_∞*), if* $\mathbf{1}$ *is a finite projection (respectively,* $\mathbf{1}$ *is a properly infinite projection).*

Type I von Neumann algebras are also called *discrete* and type III von Neumann algebras are also known as *purely infinite* von Neumann algebras. Observe that any von Neumann algebra $\mathcal{M}$ of type I_n, for some n, is also of type I. Indeed, suppose that $\{e_\alpha\}_{\alpha\in\mathbb{A}}$ is a collection of pairwise disjoint, mutually equivalent abelian projections in $P(\mathcal{M})$ such that $\sum_\alpha e_\alpha = \mathbf{1}$, where the cardinality of $\mathbb{A}$ satisfies $|\mathbb{A}| = n$. It follows from Proposition 13.5 (iv) that $z(e_{\alpha_1}) = z(e_{\alpha_2})$ for all $\alpha_1, \alpha_2 \in \mathbb{A}$ and so, (7) implies that $z(e_\alpha) = \mathbf{1}$ for all $\alpha \in A$. Hence, $\mathcal{M}$ is of type I.

Theorem 13.12 (Type decomposition) *Suppose that $\mathcal{M}$ is a von Neumann algebra on H.*

(i). There exist unique, pairwise orthogonal, central projections p_I, p_{II}, $p_{III} \in P(Z(\mathcal{M}))$, satisfying $p_I + p_{II} + p_{III} = \mathbf{1}$, such that $\mathcal{M}_{p_I}$ is of type I or $p_I = 0$, $\mathcal{M}_{p_{II}}$ is of type II or $p_{II} = 0$, and $\mathcal{M}_{p_{III}}$ is of type III or $p_{III} = 0$.

(ii). Suppose that $\mathcal{M}$ is of type I. There exists a unique system $\{p_n : 1 \leq n \leq \dim H\}$ of pairwise orthogonal projections in $P(Z(\mathcal{M}))$, satisfying $\sum_n p_n = \mathbf{1}$, such that $\mathcal{M}_{p_n}$ is of type I_n or $p_n = 0$, for each n.

(iii). Suppose that $\mathcal{M}$ is of type II. There exist unique, mutually orthogonal projections q_1, $q_\infty \in P(Z(\mathcal{M}))$, with $q_1 + q_\infty = \mathbf{1}$, such that $\mathcal{M}_{q_1}$ is of type I_1 or $q_1 = 0$, and $\mathcal{M}_{q_\infty}$ is of type II_∞ or $q_\infty = 0$.

Clearly, it is possible to apply (ii) and (iii) in the above theorem to $\mathcal{M}_{p_I}$ and $\mathcal{M}_{p_{II}}$ of (i) to obtain a further decomposition of an arbitrary von Neumann algebra $\mathcal{M}$.

Corollary 13.13 *A factor is either of type I_n (for a unique cardinal n satisfying $1 \leq n \leq \dim H$), or type II_1, or type II_∞, or type III.*

Some further terminology is introduced next.

Definition 13.14 *The notation introduced in Theorem 13.12 is used.*

(i). If $p_{III} = 0$, then $\mathcal{M}$ is said to be a semi-finite.

(ii). If $p_I = 0$*, then* $\mathcal{M}$ *is called a* continuous von Neumann algebra*.*

(iii). If $\mathbf{1}$ *is a finite projection, then* $\mathcal{M}$ *is called a* finite von Neumann algebra*.*

(iv). If $\mathbf{1}$ *is a properly infinite projection, then* $\mathcal{M}$ *is called a* properly infinite von Neumann algebra*.*

(v). $\mathcal{M}$ *is said to be of* type I_{fin} *if* $\mathcal{M}$ *is of type I and* $\mathcal{M}$ *is finite.*

(vi). $\mathcal{M}$ *is said to be of* type I_{∞} *if* $\mathcal{M}$ *is of type I and* $\mathcal{M}$ *is properly infinite.*

(vii). If $\mathbf{1}$ *is a countably decomposable, then* $\mathcal{M}$ *is said to be* countably decomposable *(or,* σ-finite*).*

In the next proposition, some alternative characterizations of the types of von Neumann algebras introduced above, are presented.

Proposition 13.15 *(i).* $\mathcal{M}$ *is of type I if and only if for every non-zero central projection* $p \in P(Z(\mathcal{M}))$ *there exists a non-zero abelian projection* $e \in P(\mathcal{M})$ *such that* $e \leq p$*.*

(ii). $\mathcal{M}$ *is semi-finite if and only if for every non-zero central projection* $p \in P(Z(\mathcal{M}))$ *there exists a non-zero finite projection* $e \in P(\mathcal{M})$ *such that* $e \leq p$*.*

(iii). $\mathcal{M}$ *is of type* I_{fin} *(respectively, type* I_{∞}*) if and only if* $\mathcal{M}$ *is of type I and* $p_n = 0$ *for all infinite (respectively, finite) cardinals* n *(here, the central projections* p_n *are as introduced in (ii) of Theorem 13.12).*

(iv). If $\mathcal{M}$ *is of type I, then there exist unique projections* e_{fin}*,* $e_{\infty} \in P(Z(\mathcal{M}))$*, with* $e_{fin} + e_{\infty} = \mathbf{1}$*, such that* $\mathcal{M}_{e_{fin}}$ *is of type* I_{fin} *and* $\mathcal{M}_{e_{\infty}}$ *is of type* I_{∞} *or* $e_{\infty} = 0$*. Actually,* $e_{fin} = \sum_{\{n:n \text{ finite}\}} p_n$ *and* $e_{\infty} = \sum_{\{n:n \text{ infinite}\}} p_n$*.*

14 Traces

First, the notion of a *center-valued trace* on a von Neumann algebra is discussed. Let $\mathcal{M}$ be a von Neumann algebra, with center $Z(\mathcal{M})$, on the Hilbert space H.

Definition 14.1 *A* center-valued trace *on* $\mathcal{M}$ *is a linear map* $T : \mathcal{M} \rightarrow Z(\mathcal{M})$ *satisfying:*

(i). $T(a) > 0$ *whenever* $0 < a \in \mathcal{M}^+$*;*

(ii). $T(xy) = T(yx)$ *for all* $x, y \in \mathcal{M}$*;*

(iii). $T(z) = z$ *whenever* $z \in Z(\mathcal{M})$*.*

Suppose that $T : \mathcal{M} \to Z(\mathcal{M})$ is a center-valued trace on the von Neumann algebra $\mathcal{M}$. If $e, f \in P(\mathcal{M})$ are equivalent projections, then there exists a partial isometry $v \in \mathcal{M}$ such that $e = v^*v$ and $f = vv^*$. Hence, it follows from (ii) above that $T(e) = T(v^*v) = T(vv^*) = T(f)$. In particular, if $e \in P(\mathcal{M})$ is such that $e \sim \mathbf{1}$, then $T(e) = T(\mathbf{1})$, that is, $T(\mathbf{1} - e) = 0$. Hence, by (i) in the above definition, it follows that $e = \mathbf{1}$. This shows that any von Neumann algebra on which a center-valued trace exists, is necessarily finite.

Theorem 14.2 *If* $\mathcal{M}$ *is a finite von Neumann algebra, then there exists a unique center-valued trace* $T : \mathcal{M} \to Z(\mathcal{M})$*. Moreover,* T *has the following properties:*

(i). $T(zx) = zT(x)$ *whenever* $x \in \mathcal{M}$ *and* $z \in Z(\mathcal{M})$*;*

(ii). $\|T(x)\|_{B(H)} \leq \|x\|_{B(H)}$ *for all* $x \in \mathcal{M}$*;*

(iii). T *is uwo-continuous.*

Observe that it follows from (iii) of the above theorem, in combination with Lemma 11.3, that the center-valued trace T is *normal*, that is, $Ta_\beta \downarrow_\beta 0$ in $Z(\mathcal{M})$ whenever $a_\beta \downarrow_\beta 0$ in $\mathcal{M}_h$.

Next, consider *numerical traces* on von Neumann algebras are considered. Let $\mathcal{M}$ be a von Neumann algebra with positive cone $\mathcal{M}^+$.

Definition 14.3 *A function* $\tau : \mathcal{M}^+ \to [0, \infty]$ *is said to be a* weight *on* $\mathcal{M}^+$ *if:*

(i). $\tau(a + b) = \tau(a) + \tau(b)$ *for all* $a, b \in \mathcal{M}^+$*;*

(ii). $\tau(\lambda a) = \lambda\tau(a)$ *for all* $a \in \mathcal{M}^+$ *and* $0 \leq \lambda \in \mathbb{R}$ *(with the convention that* $0 \cdot \infty = 0$*).*

If τ *has the additional property that*

(iii). $\tau(u^*au) = \tau(a)$ *whenever* $a \in \mathcal{M}^+$ *and* $u \in U(\mathcal{M})$*,*

then τ *is called a* trace *(or,* tracial weight*) on* $\mathcal{M}^+$*.*

If $\tau : \mathcal{M}^+ \to [0, \infty]$ is a weight, then it follows immediately from (i) in the above definition that $\tau(a) \leq \tau(b)$ whenever $a \leq b$ in $\mathcal{M}^+$. Furthermore, observe that a weight τ is a trace if and only if $\tau(x^*x) = \tau(xx^*)$ for all $x \in \mathcal{M}$.

Definition 14.4 *A trace $\tau : \mathcal{M}^+ \to [0, \infty]$ is called:*

(i). finite *if $\tau(\mathbf{1}) < \infty$;*

(ii). semi-finite *if*

$$\tau(a) = \sup\left\{\tau(b) : b \in \mathcal{M}^+, b \leq a, \tau(b) < \infty\right\}$$

for all $a \in \mathcal{M}^+$;

(iii). faithful *if $a \in \mathcal{M}^+$ and $\tau(a) = 0$ imply that $a = 0$;*

(iv). normal *if $a_\beta \uparrow_\beta a$ in $\mathcal{M}^+$ implies that $\tau(a_\beta) \uparrow_\beta \tau(a)$.*

Remark 14.5 *If $\tau : \mathcal{M}^+ \to [0, \infty]$ is a normal trace, then τ is semi-finite if and only if for every $0 < a \in \mathcal{M}^+$, there exists $b \in \mathcal{M}^+$ such that $0 < b \leq a$ and $\tau(b) < \infty$.*

The following theorem characterizes finite and semi-finite von Neumann algebras in terms of traces.

Theorem 14.6 *Let $\mathcal{M}$ be a von Neumann algebra.*

(i). $\mathcal{M}$ is finite if and only if for every non-zero $a \in \mathcal{M}^+$ there exists a finite trace τ on $\mathcal{M}^+$ such that $\tau(a) > 0$.

(ii). $\mathcal{M}$ is semi-finite if and only if there exists a faithful normal semi-finite trace τ on $\mathcal{M}^+$.

Suppose that $\mathcal{M}$ is a semi-finite von Neumann algebra and that $\tau : \mathcal{M}^+ \to [0, \infty]$ is a trace. Define

$$\mathfrak{N}_\tau = \{x \in \mathcal{M} : \tau(x^*x) < \infty\}.$$

It can be shown that $\mathfrak{N}_\tau$ is an (two-sided) ideal in $\mathcal{M}$. The ideal $\mathfrak{N}_\tau^2$ is denoted by $\mathfrak{M}_\tau$, that is,

$$\mathfrak{M}_\tau = \left\{\sum_{j=1}^{n} x_j y_j : x_j, y_j \in \mathfrak{N}_\tau, j = 1, \ldots, n; n \in \mathbb{N}\right\}.$$

Evidently, $\mathfrak{M}_\tau \subseteq \mathfrak{N}_\tau$. Since $\mathfrak{M}_\tau$ is an ideal in $\mathcal{M}$, it follows, in particular, that $\mathfrak{M}_\tau$ is self-adjoint and that $\mathfrak{M}_\tau$ is the linear span of its positive cone $\mathfrak{M}_\tau^+ = \mathfrak{M}_\tau \cap \mathcal{M}^+$. Moreover, if $x \in \mathcal{M}$, then $x \in \mathfrak{M}_\tau$ if and only if $|x| \in \mathfrak{M}_\tau^+$.

Theorem 14.7 *Suppose that $\tau : \mathcal{M}^+ \to [0, \infty]$ is a trace.*

(i). $\mathfrak{M}_\tau^+ = \{a \in \mathcal{M}^+ : \tau(a) < \infty\}$ *and* $\mathfrak{M}_\tau = \{x \in \mathcal{M} : \tau(|x|) < \infty\}$.

(ii). The trace τ extends uniquely from $\mathfrak{M}_\tau^+$ to a positive linear functional $\dot{\tau} : \mathfrak{M}_\tau \to \mathbb{C}$, satisfying the following conditions:

- *(a)* $\dot{\tau}(x^*) = \overline{\dot{\tau}(x)}$ *for all $x \in \mathfrak{M}_\tau$;*
- *(b)* $\dot{\tau}(xy) = \dot{\tau}(yx)$ *for all $x \in \mathfrak{M}_\tau$ and all $y \in \mathcal{M}$;*
- *(c)* $\dot{\tau}(xy) = \dot{\tau}(yx)$ *for all $x, y \in \mathfrak{N}_\tau$.*

(iii). Given $y \in \mathfrak{M}_\tau$, the linear functional $\psi_y : \mathcal{M} \to \mathbb{C}$ is defined by $\psi_y(x) = \dot{\tau}(xy)$ for all $x \in \mathcal{M}$. If $y \in \mathfrak{M}_\tau^+$, then ψ_y is a positive functional on $\mathcal{M}$. If τ is a normal trace, then ψ_y is uwo-continuous for every $y \in \mathfrak{M}_\tau$ (equivalently, ψ_y is a normal positive functional for every $y \in \mathfrak{M}_\tau^+$).

The ideal $\mathfrak{M}_\tau$ is called the *ideal of definition* of the trace τ.