Matricial Haagerup L^p-spaces

D. Potapov, F. Sukochev

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Appendix

Matricial structure of the Haagerup L^p-spaces (work in progress)

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Matrix algebra M_n

Let n be a fixed integer. Let M_n is the algebra of all $n \times n$ complex matrices

$$M_n = \left\{ x = \left[x_{\alpha\beta} \right]_{\alpha,\beta=1}^{n}, \quad x \in \mathbb{C} \right\}.$$

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Fix a n.s.f. weight ϕ on M_n. Suppose that ϕ is given by

$$\phi(x) = Tr\left(e^{\Phi} x\right), \quad \Phi = \mathrm{diag}\, \{\phi_\alpha\}_{\alpha=1}^{n} \in M_n, \qquad \phi_\alpha \in \mathbb{R},\ 1 \leqslant \alpha \leqslant n. \quad (1)$$

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The talk shall present the Haagerup construction of L^p-spaces associated with the couple (M_n, ϕ).

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Every n.s.f. weight ϕ on a von Neumann algebra possesses the unique modular group σ, i.e., a group of $*$-automorphisms such that

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Modular group

Every n.s.f. weight ϕ on a von Neumann algebra possesses the unique modular group σ, i.e., a group of $*$-automorphisms such that

1. $\phi(\sigma_t(x)) = \phi(x)$, $x \in M_n$;
2. $\forall x, y \in M_n$, $\exists f_{x,y}(z)$, holomorphic in $0 < \operatorname{Im} z < 1$ and continuous in $0 \leqslant \operatorname{Im} z \leqslant 1$ such that

$$\phi\left(\sigma_t(x)y\right) = f_{x,y}(t) \quad \text{and} \quad \phi\left(y\sigma_t(x)\right) = f_{x,y}(i+t), \quad t \in \mathbb{R}. \tag{2}$$

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Remark

If the weight ϕ is tracial, then the modular group is trivial and the function $f_{x,y}(z)$ is constantly $\phi(xy)$.

Modular group (cont.)

Lemma

The modular group σ for the weight ϕ defined in (1) is given by

$$\sigma_t(x) = e^{it\Phi} x e^{-it\Phi}, \quad t \in \mathbb{R}.$$

Proof.

□

Modular group (cont.)

Lemma

The modular group σ for the weight ϕ defined in (1) is given by

$$\sigma_t(x) = e^{it\Phi} x e^{-it\Phi}, \quad t \in \mathbb{R}.$$

Proof.

The group σ_t is ϕ-invariant:

$$\begin{aligned}\phi(\sigma_t(x)) = Tr\left(e^{\Phi}\sigma_t(x)\right) \\ = Tr\left(e^{\Phi}e^{it\Phi}xe^{-it\Phi}\right) \\ = Tr(e^{\Phi}x) = \phi(x).\end{aligned}$$

□

Modular group (cont.)

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The modular group σ for the weight ϕ defined in (1) is given by

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Proof.

If $x, y \in M_n$ and if $f_{x,y}(z) = Tr\left(e^{(1+iz)\Phi} x e^{-iz\Phi} y\right)$, then

1. $f_{x,y}$ is holomorphic in $0 < \operatorname{Im} z < 1$ and continuous in $0 \leqslant \operatorname{Im} z \leqslant 1$;

□

Modular group (cont.)

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If $x, y \in M_n$ and if $f_{x,y}(z) = Tr\left(e^{(1+iz)\Phi} x e^{-iz\Phi} y\right)$, then

1. $f_{x,y}$ is holomorphic in $0 < \operatorname{Im} z < 1$ and continuous in $0 \leqslant \operatorname{Im} z \leqslant 1$;
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□

Modular group (cont.)

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If $x, y \in M_n$ and if $f_{x,y}(z) = Tr\left(e^{(1+iz)\Phi} x e^{-iz\Phi} y\right)$, then

1. $f_{x,y}$ is holomorphic in $0 < \operatorname{Im} z < 1$ and continuous in $0 \leqslant \operatorname{Im} z \leqslant 1$;
2. $f_{x,y}(t) = Tr\left(e^{(1+it)\,\Phi} x e^{-it\,\Phi} y\right) = \phi\left(\sigma_t(x)\, y\right)$
3. $f_{x,y}(i+t) = Tr\left(e^{it\,\Phi} x e^{(1-it)\,\Phi} y\right) = \phi\left(y \sigma_t(x)\right)$.

□

Continuous crossed product (abstract approach)

Let R is the continuous crossed product of M_n and $\sigma = \{\sigma_t\}_{t \in \mathbb{R}}$.

Continuous crossed product (abstract approach)

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Let R is the continuous crossed product of M_n and $\sigma = \{\sigma_t\}_{t\in\mathbb{R}}$.
Algebra $R \subseteq B(L^2(\mathbb{R}, \ell^2_n))$ is the minimal von Neumann algebra induced by

$$\{\pi(x),\ x \in M_n\} \quad \text{and} \quad \{\lambda_t\}_{t\in\mathbb{R}}\,,$$

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where

$$\pi(x)\xi(t) = \sigma_{-t}(x)\xi(t), \quad x \in M_n$$

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$$\pi(x)\xi(t) = \sigma_{-t}(x)\xi(t), \quad x \in M_n$$

and

$$\lambda_t(\xi)(s) = \xi(s-t), \quad t, s \in \mathbb{R},\ \xi \in L^2(\mathbb{R}, \ell_n^2).$$

Continuous crossed product (abstract approach)

Let R is the continuous crossed product of M_n and $\sigma = \{\sigma_t\}_{t \in \mathbb{R}}$.
Algebra $R \subseteq B(L^2(\mathbb{R}, \ell_n^2))$ is the minimal von Neumann algebra induced by

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and

$$\lambda_t(\xi)(s) = \xi(s - t), \quad t, s \in \mathbb{R},\ \xi \in L^2(\mathbb{R}, \ell_n^2).$$

Remark

The mapping $\pi : M_n \mapsto R$ is a $*$-representation of the algebra M on $L^2(\mathbb{R}, \ell_n^2)$.

Continuous crossed product (constructive approach)

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$$\hat{R} = \left\{ x(t) = \left[x_{\alpha\beta}(t) \right]_{\alpha,\beta=1}^{n}, \quad x_{\alpha\beta} \in L^\infty(\mathbb{R}) \right\}$$

Continuous crossed product (constructive approach)

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$$[xy]_{\alpha\beta}(t - \phi_\beta) = \sum_{\gamma=1}^{n} x_{\alpha\gamma}(t - \phi_\gamma) y_{\gamma\beta}(t - \phi_\beta)$$

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$$[x^*]_{\alpha\beta}(t - \phi_\beta) = \bar{x}_{\beta\alpha}(t - \phi_\alpha), \quad t \in \mathbb{R}, \; x, y \in \hat{R}$$

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Remark

If $\Phi = \operatorname{diag}\{\phi_\alpha\}_{\alpha=1}^{n}$ is null, i.e., ϕ coincides with Tr, then $\hat{R} = L^\infty(\mathbb{R}) \bar{\otimes} M_n$.

Continuous crossed product (constructive approach)

$$\hat{R} = \left\{ x(t) = \left[x_{\alpha\beta}(t) \right]_{\alpha,\beta=1}^{n}, \quad x_{\alpha\beta} \in L^{\infty}(\mathbb{R}) \right\}$$

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Remark

The algebra $\hat{R}$ has a subalgebra of constant matrix functions isomorphic to M_n.

Continuous crossed product (constructive approach, cont.)

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Proposition

The crossed product R is isomorphic to $\hat{R}$. The isomprphism is implemented via the Fourier transform, i.e., the mapping $T \in R \mapsto \hat{T} \in \hat{R} = \mathfrak{F} T \mathfrak{F}^{-1}$, where $\mathfrak{F}$ is the Fourier transform on $L^2(\mathbb{R}, \ell_n^2)$.

Continuous crossed product (constructive approach, cont.)

Proposition

The crossed product R is isomorphic to $\hat{R}$. The isomprphism is implemented via the Fourier transform, i.e., the mapping $T \in R \mapsto \hat{T} \in \hat{R} = \mathfrak{F} T \mathfrak{F}^{-1}$, where $\mathfrak{F}$ is the Fourier transform on $L^2(\mathbb{R}, \ell_n^2)$.

The von Neumann algebra $\hat{R}$ is represented on $L^2(\mathbb{R}, \ell_n^2)$ as follows: if $\eta = x(\xi)$ and $\eta = (\eta_\alpha)_{\alpha=1}^n$, $\xi = \left(\xi_\beta\right)_{\beta=1}^n$, then

$$\eta_\alpha(t - \phi_\alpha) = \sum_{\beta=1}^n x_{\alpha\beta}(t - \phi_\beta)\xi_\beta(t - \phi_\beta). \tag{3}$$

The distinguished trace on $\hat{R}$

For $x = \left[x_{\alpha\beta}\right]_{\alpha,\beta=1}^{n} \in \hat{R}$, introduce τ by

$$\tau(x) = \int_{\mathbb{R}} \phi(x(t))\, e^{t} dt = \sum_{\alpha=1}^{n} \int_{\mathbb{R}} x_{\alpha\alpha}(t - \phi_{\alpha})\, e^{t} dt.$$

The distinguished trace on $\hat{R}$

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Proposition

The functional τ is a normal semi-finite faithful trace on $\hat{R}$.

The trace (cont.)

Proof.

τ is semi-finite. □

The trace (cont.)

Proof.

τ is tracial, i.e., $\tau(x^*x) = \tau(xx^*)$.

□

The trace (cont.)

Proof.

τ is tracial, i.e., $\tau(x^*x) = \tau(xx^*)$.

$$\tau\left(x^*x\right) = \sum_{\alpha=1}^{n} \int_{\mathbb{R}} [x^*x]_{\alpha\alpha}\left(t - \phi_\alpha\right) e^t dt$$

$$= \sum_{\alpha,\gamma=1}^{n} \int_{\mathbb{R}} \bar{x}_{\gamma\alpha}(t - \phi_\alpha) x_{\gamma\alpha}(t - \phi_\alpha)\, e^t dt$$

$$= \sum_{\alpha,\gamma=1}^{n} \int_{\mathbb{R}} |x_{\gamma\alpha}(t - \phi_\alpha)|^2 \; e^t ds$$

□

The trace (cont.)

Proof.

τ is tracial, i.e., $\tau(x^*x) = \tau(xx^*)$.

$$\tau(xx^*) = \sum_{\alpha=1}^{n} \int_{\mathbb{R}} [xx^*]_{\alpha\alpha}(t - \phi_\alpha)\, e^t dt$$

$$= \sum_{\alpha,\gamma=1}^{n} \int_{\mathbb{R}} x_{\alpha\gamma}(t - \phi_\gamma)\, \bar{x}_{\alpha\gamma}(t - \phi_\gamma)\, e^t dt$$

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$$= \sum_{\alpha,\gamma=1}^{n} \int_{\mathbb{R}} x_{\alpha\gamma}(t - \phi_\gamma)\, \bar{x}_{\alpha\gamma}(t - \phi_\gamma)\, e^t dt$$

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τ is faithful

□

The dual action

$\theta = \{\theta_t\}_{t\in\mathbb{R}}$ is the group of translations on $\hat{R}$:

$$\theta_t(x)(s) = x(s+t), \quad t, s \in \mathbb{R}, \ x \in \hat{R}.$$

The group θ is called *the dual group.*

The dual action

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The group θ is called *the dual group.*

Proposition

The group $\theta = \{\theta_t\}_{t \in \mathbb{R}}$ is dual (via the Fourier transform) to the group $\lambda = \{\lambda_t\}_{t \in \mathbb{R}}$.

Haagerup L^p-spaces, $1 \leqslant p \leqslant \infty$ (definition)

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$\tilde{R}$ the $*$-algebra of all τ-measurable operators for $(\hat{R}, \tau)$. $\tilde{R}$ is described as follows

$$\tilde{R} = \left\{ x(t) = \left[x_{\alpha\beta} \right]_{\alpha,\beta=1}^{\infty}, \quad x_{\alpha\beta} \in S(e^t dt) \right\},$$

where $S(e^t dt)$ is the algebra of all measurable (with respect to the trace $e^t dt$) functions on $\mathbb{R}$.

Haagerup L^p-spaces, $1 \leqslant p \leqslant \infty$ (definition)

$\tilde{R}$ the $*$-algebra of all τ-measurable operators for $(\hat{R}, \tau)$. $\tilde{R}$ is described as follows

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where $S(e^t dt)$ is the algebra of all measurable (with respect to the trace $e^t dt$) functions on $\mathbb{R}$.

Definition

The Haagerup L^p-space is the subspace of $\tilde{R}$ of all elements $x \in \hat{R}$ such that $\theta_t(x) = e^{-t/p} x$, $t \in \mathbb{R}$, i.e.,

$$L^p(M_n) = \left\{ x \in \tilde{R} : \quad \theta_t(x) = e^{-t/p} x \right\}.$$

Haagerup L^p-spaces, $1 \leqslant p \leqslant \infty$ (description)

Theorem

1. *The space* $L^p(M_n)$ *admits the following description*

$$L^p(M_n) = \left\{ \left[e^{-(t+\phi_\beta)/p} \psi_{\alpha\beta} \right]_{\alpha,\beta=1}^n, \ \left[\psi_{\alpha\beta}\right] \in M_n \right\}.$$

2. *If* μ *is the decreasing rearrangement (wrt* τ*), then*

$$\mu_t(x) = \frac{k(x)}{t^{1/p}}, \quad t \in \mathbb{R}, \ x \in L^p(M_n).$$

3. *If* $x = \left[e^{-(t+\phi_\beta)/p} \psi_{\alpha\beta} \right]_{\alpha,\beta=1}^n \in L^p(M_n)$ *for some* $\psi = \left[\psi_{\alpha\beta}\right]_{\alpha,\beta=1}^n$*, then* $k(x) = \|\psi\|_{\mathcal{S}^p}$.

Proof of the main theorem

Clearly every matrix function $x = \left[e^{-(t+\phi_\beta)/p}\psi_{\alpha\beta}\right]_{\alpha,\beta=1}^n$ satisfies the equation

$$\theta_t(x) = e^{-t/p}x. \tag{4}$$

Proof of the main theorem

Clearly every matrix function $x = \left[e^{-(t+\phi_\beta)/p}\psi_{\alpha\beta}\right]_{\alpha,\beta=1}^n$ satisfies the equation

$$\theta_t(x) = e^{-t/p}x. \tag{4}$$

Conversely, if $x = \left[x_{\alpha\beta}\right]_{\alpha,\beta=1}^n$ satisfies (4), then

$$x(t) = \theta_t(x)(0) = e^{-t/p}x(0).$$

Setting $\psi = x(0)\, e^{\Phi/p} \in M_n$ yields that

$$x(t) = \left[e^{-(t+\phi_\beta)/p}\psi_{\alpha\beta}\right]_{\alpha,\beta=1}^n.$$

Thus, (1) follows.

Proof of the main theorem

Fix $x = \left[e^{-(t+\phi_\beta)}\psi_{\alpha\beta}\right]_{\alpha,\beta=1}^{n} \in L^p(M_n)$.

Proof of the main theorem

Fix $x = \left[e^{-(t+\phi_\beta)}\psi_{\alpha\beta}\right]_{\alpha,\beta=1}^{n} \in L^p(M_n)$. We may assume that ψ is positive and diagonal, i.e., let $\psi_{\alpha\alpha} = \delta_\alpha > 0$ and $\psi_{\alpha\beta} = 0$ if $\alpha \neq \beta$.

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$$\mu_t(x) = \inf\left\{s > 0 : \ \tau\left(\chi_{(s,+\infty)}(x)\right) \leqslant t\right\},$$

where $\chi(x)$ is the spectral measure of x.

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$$\mu_t(x) = \inf\left\{s > 0 : \quad \tau\left(\chi_{(s,+\infty)}(x)\right) \leqslant t\right\},$$

where $\chi(x)$ is the spectral measure of x.

$$e^{-(t+\phi_\alpha)/p}\delta_\alpha > s \Longleftrightarrow -\frac{t+\phi_\alpha}{p} > \log\frac{s}{\delta_\alpha}$$

$$\Longleftrightarrow t < -p\log\frac{s}{\delta_\alpha} - \phi_\alpha.$$

Proof of the main theorem

Fix $x = \left[e^{-(t+\phi_\beta)}\psi_{\alpha\beta}\right]_{\alpha,\beta=1}^{n} \in L^p(M_n)$. We may assume that ψ is positive and diagonal, i.e., let $\psi_{\alpha\alpha} = \delta_\alpha > 0$ and $\psi_{\alpha\beta} = 0$ if $\alpha \neq \beta$.

$$\mu_t(x) = \inf\left\{s > 0 : \quad \tau\left(\chi_{(s,+\infty)}(x)\right) \leqslant t\right\},$$

where $\chi(x)$ is the spectral measure of x.

$$e^{-(t+\phi_\alpha)/p}\delta_\alpha > s \iff -\frac{t+\phi_\alpha}{p} > \log\frac{s}{\delta_\alpha}$$
$$\iff t < -p\log\frac{s}{\delta_\alpha} - \phi_\alpha.$$

$$\chi_{(s,+\infty)}(x) = \mathrm{diag}\left\{\chi_{\left(-\infty,-p\log\frac{s}{\delta_\alpha}-\phi_\alpha\right)}\right\}_{\alpha=1}^{n} \in \hat{R}.$$

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where $\chi(x)$ is the spectral measure of x.

$$\begin{aligned}
\tau(\chi_{(s,+\infty)}(x)) &= \sum_{\alpha=1}^{n} \int_{\mathbb{R}} \left[\chi_{(s,+\infty)}(x)\right]_{\alpha\alpha} (t - \phi_\alpha)\, e^t dt \\
&= \sum_{\alpha=1}^{n} \int_{\mathbb{R}} \chi_{\left(-\infty, -p\log\frac{s}{\delta_\alpha}\right)}(t)\, e^t dt \\
&= \sum_{\alpha=1}^{n} \int_{-\infty}^{-p\log\frac{s}{\delta_\alpha}} e^t\, dt = \frac{1}{s^p} \sum_{\alpha=1}^{n} \delta_\alpha^p.
\end{aligned}$$

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$$\mu_t(x) = \frac{k(x)}{t^{1/p}}, \quad \text{where } k(x) = \left(\sum_{\alpha=1}^{n}\delta_\alpha^p\right)^{\frac{1}{p}} = \|\psi\|_{\mathcal{S}^p}.$$

Commutative version

A similar argument describes L^p-spaces for $M = L^\infty(\mathbb{R})$.

Theorem

1. *The space $L^p(M)$ admits the following description*

$$L^p(M) = \left\{ x \in \tilde{R} : \quad x(t,s) = e^{-t/p}\psi(s),\ \psi \in L^p(ds) \right\}.$$

2. *If μ is the decreasing rearrangement (wrt $e^t dtds$), then*

$$\mu_t(x) = \frac{\|\psi\|_{L^p(ds)}}{t^{1/p}}, \quad t \in \mathbb{R},$$

$$x(t,s) = e^{-t/p}\psi(s) \in L^p(M).$$

Appendix (construction of the isomorphism between R and $\hat{R}$)

Let $\mathcal{F}$ be the Fourier transform on $L^2(\mathbb{R}, \ell_n^2)$ and $\mathcal{F}^{-1}$ is the inverse Fourier transform, i.e.,

$$\hat{\xi}(t) = \mathcal{F}\xi(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \xi(s)\, e^{-its}\, ds, \quad \xi \in L^2(\mathbb{R}, \ell_n^2)$$

and

$$\xi(t) = \mathcal{F}^{-1}\hat{\xi}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\xi}(s)\, e^{its}\, ds, \quad \hat{\xi} \in L^2(\mathbb{R}, \ell_n^2).$$

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The mappings $\mathfrak{F}$ and $\mathfrak{F}^{-1}$ are unitary transformations of $L^2(\mathbb{R}, \ell^2)$.

Appendix (construction of the isomorphism between R and $\hat{R}$)

We shall show that

$$\hat{R} = \mathfrak{F} R \mathfrak{F}^{-1}.$$

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$$T = \int_{\mathbb{R}} \pi(x(t))\, \lambda_t \, dt.$$

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The collection of all such operators T is weakly dense in R (see [Ta, Ch. X, Lemma 1.8]).

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let also $\hat{x} = [\hat{x}_{\alpha\beta}]_{\alpha,\beta=1}^n$ and $\hat{\xi} = \left(\hat{\xi}_\alpha\right)_{\alpha=1}^n$

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$$T\xi(t) = \int_{\mathbb{R}} \pi(x(k))\, \lambda_k(t)\xi(t)\, dk$$

$$= \int_{\mathbb{R}} \sigma_{-t}(x(k))\, \xi(t-k)\, dk$$

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let also $\hat{x} = [\hat{x}_{\alpha\beta}]_{\alpha,\beta=1}^n$ and $\hat{\xi} = \left(\hat{\xi}_\alpha\right)_{\alpha=1}^n$

$$\begin{aligned} T\xi(t) &= \int_{\mathbb{R}} \pi(x(k))\, \lambda_k(t)\xi(t)\, dk \\ &= \int_{\mathbb{R}} \sigma_{-t}(x(k))\, \xi(t-k)\, dk \\ &= \int_{\mathbb{R}} e^{-it\Phi} x(k) e^{it\Phi}\, \xi(t-k)\, dk \end{aligned}$$

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$$\begin{aligned} T\xi(t) &= \int_{\mathbb{R}} \pi(x(k))\, \lambda_k(t) \xi(t)\, dk \\ &= \int_{\mathbb{R}} e^{-it\Phi} x(k) e^{it\Phi}\, \xi(t-k)\, dk \\ &= \int_{\mathbb{R}} e^{-it\Phi} \left[\int_{\mathbb{R}} \hat{x}(s)\, e^{iks}\, ds \right] e^{it\Phi} \left[\int_{\mathbb{R}} \hat{\xi}(m)\, e^{im(t-k)} \frac{dm}{\sqrt{2\pi}} \right] dk \end{aligned}$$

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$$\begin{aligned} T\xi(t) &= \int_{\mathbb{R}} \pi(x(k))\, \lambda_k(t) \xi(t)\, dk \\ &= \int_{\mathbb{R}} e^{-it\Phi} \left[\int_{\mathbb{R}} \hat{x}(s)\, e^{iks}\, ds\right] e^{it\Phi} \left[\int_{\mathbb{R}} \hat{\xi}(m)\, e^{im(t-k)} \frac{dm}{\sqrt{2\pi}}\right] dk \\ &= \int_{\mathbb{R}^3} e^{-it\Phi}\, \hat{x}(s) e^{it\Phi}\, \hat{\xi}(m)\, e^{iks+im(t-k)}\, ds\, dk \frac{dm}{\sqrt{2\pi}} \end{aligned}$$

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$$T\xi(t) = \int_{\mathbb{R}} \pi(x(k))\, \lambda_k(t) \xi(t)\, dk$$

$$= \sum_{\beta=1}^n \int_{\mathbb{R}} \hat{x}_{\alpha\beta}(s - \phi_\beta)\, \hat{\xi}_\beta(s - \phi_\beta)\, e^{it(s-\phi_\alpha)}\, \frac{ds}{\sqrt{2\pi}}$$

Consequently, if $\eta = T\xi$ and $\hat{\eta} = (\hat{\eta}_\alpha)_{\alpha=1}^n$, then

$$\hat{\eta}_\alpha(s - \phi_\alpha) = \sum_{\beta=1}^n \hat{x}_{\alpha\beta}(s - \phi_\beta)\, \hat{\xi}_\beta(s - \phi_\beta).$$

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Thus, we showed that the operator $\mathcal{F} T \mathcal{F}^{-1}$ belongs to $\hat{R}$. The proof of the proposition is finished.