MATRICIAL STRUCTURE OF THE HAAGERUP L^p-SPACES (WORK IN PROGRESS)

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Here, we review the construction of the Haagerup L^p-spaces for the special algebra M_n of all $n \times n$ complex matrices (n is fixed for the rest of the section) and describe the matricial structure of these spaces. Note that we consider the algebra M_n as the algebra of all bounded linear operators on ℓ^2_n.

Fix a n.s.f. weight ϕ on M_n. Suppose that the weight ϕ is given by

$$\phi(x) = \mathrm{Tr}\left(e^{\Phi} x\right), \quad \Phi = \mathrm{diag}\,\{\phi_\alpha\}_{\alpha=1}^n \in M_n,\ \phi_\alpha \in \mathbb{R},\ 1 \leqslant \alpha \leqslant n. \tag{1}$$

Let σ_t be the modular group for the weight ϕ, i.e., the group of $*$-automorphisms on M_n such that

(i) $\phi(\sigma_t(x)) = \phi(x)$, $x \in M_n$;

(ii) for every $x, y \in M_n$, there is a function $f_{x,y}(z)$ holomorphic in the strip $0 < \Im z < 1$ and continuous in the closure of that strip such that

$$\phi\left(\sigma_t(x) y\right) = f_{x,y}(t) \ \text{ and } \ \phi\left(y \sigma_t(x)\right) = f_{x,y}(i + t), \ \ t \in \mathbb{R}. \tag{2}$$

It is not very difficult to see that for the weight ϕ introduced in (1) the corresponding modular group is given by

$$\sigma_t(x) = e^{it\Phi} x e^{-it\Phi}.$$

Indeed, it is clear that the group σ introduced above is ϕ-invariant, furthermore, if $x, y \in M_n$, then, the identities (2) hold with the following holomorphic function

$$f_{x,y}(z) = \mathrm{Tr}\left(e^{(1+iz)\Phi} x e^{-iz\Phi} y\right).$$

The Haagerup L^p spaces construction starts with the crossed product R of the algebra M_n with the action $\sigma = \{\sigma_t\}_{t\in\mathbb{R}}$. The crossed product algebra R is the subalgebra of $B(L^2(\mathbb{R}, \ell^2_n))$ which is the minimal algebra containing the collections

$$\{\pi(x),\ x \in M_n\} \ \text{ and } \ \{\lambda_t\}_{t\in\mathbb{R}},$$

where

$$\pi(x)\xi(t) = \sigma_{-t}(x)\xi(t), \ \ x \in M_n$$

and

$$\lambda_t(\xi)(s) = \xi(s-t), \ \ t, s \in \mathbb{R}, \ \xi \in L^2(\mathbb{R}, \ell^2_n).$$

The construction of continuous crossed product is very abstract in its nature. The first step in our discussion is to present a constructive matricial description of the algebra R. Let us consider the linear space of matrix-valued functions

$$\hat{R} = \left\{ x(t) = [x_{\alpha\beta}(t)]^n_{\alpha,\beta=1}, \ \ x_{\alpha\beta} \in L^\infty(\mathbb{R}) \right\}$$

equipped with the following product

$$[xy]_{\alpha\beta}(t - \phi_\beta) = \sum_{\gamma=1}^n x_{\alpha\gamma}(t-\phi_\gamma) y_{\gamma\beta}(t-\phi_\beta)$$

and involution

$$[x^*]_{\alpha\beta}(t-\phi_\beta) = \bar{x}_{\beta\alpha}(t-\phi_\alpha), \ \ t \in \mathbb{R}, \ x, y \in \hat{R}.$$

Together with introduced product and involution, the space $\hat{R}$ becomes a $*$-algebra. We shall embed this algebra into $B(L^2(R, \ell^2_n))$ as follows: if $\eta = x(\xi)$ and $\eta = (\eta_\alpha)^n_{\alpha=1}$, $\xi = (\xi_\beta)^n_{\beta=1}$, then

$$\eta_\alpha(t-\phi_\alpha) = \sum_{\beta=1}^n x_{\alpha\beta}(t-\phi_\beta)\xi_\beta(t-\phi_\beta). \tag{3}$$

Observe that the algebra $\hat{R}$ has a subalgebra of constant matrix functions which is isomorphic to M_n. Furthermore, observe that identity (3) gives a (twisted) representation for the algebra M_n in $L^2(\mathbb{R}, \ell^2_n)$.

Observe also that when the matrix $\Phi = \mathrm{diag}\,\{\phi_\alpha\}^n_{\alpha=1}$ is null, i.e., the weight ϕ coincides with the standard trace Tr, then the algebra $\hat{R}$ clearly coincides with the tensor product von Neumann algebra $L^\infty(\mathbb{R})\bar{\otimes}M_n$.

Proposition 1. *The crossed product* R *is isomorphic to the algebra* $\hat{R}$.

Proof. Let $\mathcal{F}$ be the Fourier transform on $L^2(\mathbb{R}, \ell^2_n)$ and $\mathcal{F}^{-1}$ is the inverse Fourier transform, i.e.,

$$\hat{\xi}(t) = \mathcal{F}\xi(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \xi(s)\, e^{-its}\, ds, \ \ \xi \in L^2(\mathbb{R}, \ell^2_n)$$

and

$$\xi(t) = \mathcal{F}^{-1}\hat{\xi}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\xi}(s)\, e^{its}\, ds, \quad \hat{\xi} \in L^2(\mathbb{R}, \ell^2_n).$$

The mappings $\mathcal{F}$ and $\mathcal{F}^{-1}$ are unitary transformations of $L^2(\mathbb{R}, \ell^2)$. We shall show that

$$\hat{R} = \mathcal{F} R \mathcal{F}^{-1}.$$

Let $t \mapsto x(t) = [x_{\alpha\beta}]^n_{\alpha,\beta=1} \in M_n$ be a Schwartz matrix function, i.e., a matrix function with every entry being a Schwartz scalar function. The function $x(t)$ defines an operator $T \in R$ by the following formula

$$T = \int_{\mathbb{R}} \pi(x(t))\, \lambda_t\, dt.$$

The collection of all such operators T is weakly dense in R (see [Ta, Ch. X, Lemma 1.8]).

In order to describe the algebra $\mathcal{F} R \mathcal{F}^{-1}$, let us compute the operator $\hat{T} = \mathcal{F} T \mathcal{F}^{-1}$. To this end let us fix a vector function $\xi \in L^2(\mathbb{R}, \ell^2_n)$ and represent the functions $x(t)$ and $\xi(t)$ in terms of their Fourier transforms $\hat{x}(t)$ and $\hat{\xi}(t)$, i.e.,

$$\xi(t) = \int_{\mathbb{R}} \hat{\xi}(s)\, e^{its}\, \frac{ds}{\sqrt{2\pi}} \quad \text{and} \quad x(t) = \int_{\mathbb{R}} \hat{x}(s)\, e^{its}\, ds.$$

Furthermore, assuming that $\hat{x} = [\hat{x}_{\alpha\beta}]_{\alpha,\beta=1}^{n}$ and $\hat{\xi} = \left(\hat{\xi}_{\alpha}\right)_{\alpha=1}^{n}$,

$$
\begin{aligned}
T\xi(t) &= \int_{\mathbb{R}} \pi(x(k))\,\lambda_k(t)\xi(t)\,dk \\
&= \int_{\mathbb{R}} \sigma_{-t}(x(k))\,\xi(t-k)\,dk \\
&= \int_{\mathbb{R}} e^{-it\Phi}x(k)e^{it\Phi}\,\xi(t-k)\,dk \\
&= \int_{\mathbb{R}} e^{-it\Phi}\left[\int_{\mathbb{R}} \hat{x}(s)\,e^{iks}\,ds\right]\,e^{it\Phi}\left[\int_{\mathbb{R}} \hat{\xi}(m)\,e^{im(t-k)}\,\frac{dm}{\sqrt{2\pi}}\right]\,dk \\
&= \int_{\mathbb{R}^3} e^{-it\Phi}\hat{x}(s)e^{it\Phi}\hat{\xi}(m)\,e^{iks+im(t-k)}\,ds\,dk\,\frac{dm}{\sqrt{2\pi}} \\
&= \sum_{\beta=1}^{n}\int_{\mathbb{R}^3} e^{-it\phi_\alpha}\hat{x}_{\alpha\beta}(s)e^{it\phi_\beta}\hat{\xi}_\beta(m)\,e^{iks+im(t-k)}\,ds\,dk\,\frac{dm}{\sqrt{2\pi}} \\
&= \sum_{\beta=1}^{n}\int_{\mathbb{R}^2} \hat{x}_{\alpha\beta}(s)\,\hat{\xi}_\beta(m)\,e^{it(m-\phi_\alpha+\phi_\beta)}\left[\int_{\mathbb{R}} e^{ik(s-m)}\,dk\right]\,ds\,\frac{dm}{\sqrt{2\pi}} \\
&= \sum_{\beta=1}^{n}\int_{\mathbb{R}^2} \hat{x}_{\alpha\beta}(s)\,\hat{\xi}_\beta(m)\,e^{it(m-\phi_\alpha+\phi_\beta)}\delta(s-m)\,ds\,\frac{dm}{\sqrt{2\pi}} \\
&= \sum_{\beta=1}^{n}\int_{\mathbb{R}} \hat{x}_{\alpha\beta}(s)\hat{\xi}_\beta(s)e^{it(s-\phi_\alpha+\phi_\beta)}\,\frac{ds}{\sqrt{2\pi}} \\
&= \sum_{\beta=1}^{n}\int_{\mathbb{R}} \hat{x}_{\alpha\beta}(s-\phi_\beta)\,\hat{\xi}_\beta(s-\phi_\beta)\,e^{it(s-\phi_\alpha)}\,\frac{ds}{\sqrt{2\pi}}.
\end{aligned}
$$

Consequently, if $\eta = T\xi$ and $\hat{\eta} = (\hat{\eta}_\alpha)_{\alpha=1}^{n}$, then

$$
\hat{\eta}_\alpha(s-\phi_\alpha) = \sum_{\beta=1}^{n} \hat{x}_{\alpha\beta}(s-\phi_\beta)\,\hat{\xi}_\beta(s-\phi_\beta).
$$

Thus, we showed that the operator $\mathcal{F}T\mathcal{F}^{-1}$ belongs to $\hat{R}$. The proof of the proposition is finished. □

From now on we shall identify the algebras R and $\hat{R}$.

The crossed product $\hat{R}$ possesses a distinguished trace τ. Indeed, for an element $x \in \hat{R}$, introduce the functional τ by

$$
\tau(x) = \int_{\mathbb{R}} \phi(x(t))\,e^t dt = \sum_{\alpha=1}^{n}\int_{\mathbb{R}} x_{\alpha\alpha}(t-\phi_\alpha)\,e^t dt,\quad x = [x_{\alpha\beta}]_{\alpha,\beta=1}^{n} \in \hat{R}.
$$

Proposition 2. *The functional τ is a normal semi-finite faithful trace.*

Proof. Clearly, the functional τ is semi-finite. Let us show that it trace, i.e.,

$$
\tau(xy) = \tau(yx).
$$

Due to the polarization identity, it is sufficient to check that

$$\tau(x^*x) = \tau(xx^*).$$

Let us fix $x = [x_{\alpha\beta}]_{\alpha,\beta=1}^n$ and compute $\tau(x^*x)$. We have

$$\begin{aligned}[x^*x]_{\alpha\beta}(t-\phi_\beta) &= \sum_{\gamma=1}^n [x^*]_{\alpha\gamma}(t-\phi_\gamma)x_{\gamma\beta}(t-\phi_\beta)\\ &= \sum_{\gamma=1}^n \bar{x}_{\gamma\alpha}(t-\phi_\alpha)x_{\gamma\beta}(t-\phi_\beta).\end{aligned}$$

Consequently,

$$\begin{aligned}\tau(x^*x) &= \sum_{\alpha=1}^n \int_{\mathbb{R}} [x^*x]_{\alpha\alpha}(t-\phi_\alpha)\, e^t dt\\ &= \sum_{\alpha,\gamma=1}^n \int_{\mathbb{R}} \bar{x}_{\gamma\alpha}(t-\phi_\alpha)x_{\gamma\alpha}(t-\phi_\alpha)\, e^t dt\\ &= \sum_{\alpha,\gamma=1}^n \int_{\mathbb{R}} |x_{\gamma\alpha}(t-\phi_\alpha)|^2\, e^t ds.\end{aligned}$$

Clearly, if we replace x with x^*, i.e., $x_{\gamma\alpha}(t-\phi_\alpha)$ with $\bar{x}_{\alpha\gamma}(t-\phi_\gamma)$ the latter quantity does not change. In other words, $\tau(x^*x) = \tau(xx^*)$ and therefore τ is a trace.

The expression for $\tau(x^*x)$ above also clearly implies that the trace τ is faithful. Thus, the proposition is proved. □

Consider the group of translations on $\hat{R}$, i.e., the group $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ such that

$$\theta_t(x)(s) = x(s+t), \quad t,s \in \mathbb{R},\ x \in \hat{R}.$$

The action $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ is called *the dual action* on $\hat{R}$. It is not very difficult to see that

$$\tau(\theta_t(x)) = e^{-t}\,\tau(x), \quad x \in \hat{R}.$$

Let us denote by $\tilde{R}$ the algebra of all τ-measurable operators with respect to the couple (R,τ). Clearly, the $*$-algebra $\tilde{R}$ is alternatively described as follows

$$\tilde{R} = \left\{x(t) = [x_{\alpha\beta}]_{\alpha,\beta=1}^\infty,\ \ x_{\alpha\beta} \in S(e^t dt)\right\},$$

where $S(e^t dt)$ is the algebra of all measurable (with respect to the trace $e^t dt$) functions on $\mathbb{R}$.

The Haagerup L^p*-space* is the subspace of $\tilde{R}$ of all elements $x \in \hat{R}$ such that $\theta_t(x) = e^{-t/p}x$, $t \in \mathbb{R}$, i.e.,

$$L^p(M_n) = \left\{ x \in \tilde{R} : \ \theta_t(x) = e^{-t/p}x \right\}.$$

Theorem 3. (i) *The space* $L^p(M_n)$ *admits the following description*

$$L^p(M_n) = \left\{ x \in \tilde{R} : \ x = \left[e^{-(t+\phi_\beta)/p} \psi_{\alpha\beta} \right]_{\alpha,\beta=1}^n, \ [\psi_{\alpha\beta}] \in M_n \right\}.$$

(ii) *If* $\mu(x)$ *is the decreasing rearrangement of a element* $x \in \tilde{R}$ *(with respect to the trace* τ*), then*

$$\mu_t(x) = \frac{\text{const}}{t^{1/p}}, \ \ t \in \mathbb{R}, \ x \in L^p(M_n).$$

(iii) *If an element* $x \in L^p(M_n)$ *has the representation* $x = \left[e^{-(t+\phi_\beta)/p} \psi_{\alpha\beta} \right]_{\alpha,\beta=1}^n$ *for some matrix* $\psi = [\psi_{\alpha\beta}]_{\alpha,\beta=1}^n$*, then the constant from the statement above is equal to* $\|\psi\|_{\mathcal{S}^p}$*, where* $\mathcal{S}^p$ *is the Schatten-von Neumann class.*

Proof. Clearly every matrix function $x = \left[e^{-(t+\phi_\beta)/p} \psi_{\alpha\beta} \right]_{\alpha,\beta=1}^n$ satisfies the equation

$$\theta_t(x) = e^{-t/p}x. \tag{4}$$

Conversely, if $x = [x_{\alpha\beta}]_{\alpha,\beta=1}^n$ satisfies (4), then

$$x(t) = \theta_t(x)(0) = e^{-t/p}x(0).$$

Setting $\psi = x(0)\, e^{\Phi/p} \in M_n$ yields that

$$x(t) = \left[e^{-(t+\phi_\beta)/p} \psi_{\alpha\beta} \right]_{\alpha,\beta=1}^n.$$

Thus, (i) follows.

Let us show (ii) and (iii). Fix the element $x = \left[e^{-(t+\phi_\beta)} \psi_{\alpha\beta} \right]_{\alpha,\beta=1}^n \in L^p(M_n)$. Without loss of generality, we may assume that the matrix ψ is positive and diagonal. In this case x is also positive operator. Let $\psi_{\alpha\alpha} = \delta_\alpha > 0$ and $\psi_{\alpha\beta} = 0$ if $\alpha \neq \beta$.

Recall that

$$\mu_t(x) = \inf \left\{ s > 0 : \ \tau\left(\chi_{(s,+\infty)}(x) \right) \leqslant t \right\},$$

where $\chi(x)$ is the spectral measure of x. Observe that

$$e^{-(t+\phi_\alpha)/p}\delta_\alpha > s \Longleftrightarrow -\frac{t+\phi_\alpha}{p} > \log\frac{s}{\delta_\alpha} \Longleftrightarrow t < -p\log\frac{s}{\delta_\alpha} - \phi_\alpha.$$

Consequently,

$$\chi_{(s,+\infty)}(x) = \operatorname{diag}\left\{\chi_{\left(-\infty, p\log\frac{s}{\delta_\alpha} - \phi_\alpha\right)}\right\}_{\alpha=1}^{n} \in \hat{R}.$$

Furthermore,

$$\begin{aligned}
&\tau(\chi_{(s,+\infty)}(x)) \\
&= \sum_{\alpha=1}^{n}\int_{\mathbb{R}} \left[\chi_{(s,+\infty)}(x)\right]_{\alpha\alpha}(t-\phi_\alpha)\, e^t dt \\
&= \sum_{\alpha=1}^{n}\int_{\mathbb{R}} \chi_{\left(-\infty, p\log\frac{s}{\delta_\alpha}\right)}(t)\, e^t dt \\
&= \sum_{\alpha=1}^{n}\int_{-\infty}^{p\log\frac{s}{\delta_\alpha}} e^t\, dt \\
&= \frac{1}{s^p}\sum_{\alpha=1}^{n}\delta_\alpha^p.
\end{aligned}$$

The instantly implies that

$$\mu_t(x) = \frac{\text{const}}{t^{1/p}},$$

where

$$\text{const} = \left(\sum_{\alpha=1}^{n}\delta_\alpha^p\right)^{\frac{1}{p}} = \|\psi\|_{S^p}.$$

Thus, (ii) and (iii) are proved. The proof of the theorem is finished. □

At the end of this section, let us note that a similar argument may be given in order to describe Haagerup L^p-spaces with respect to the algebra $L^\infty(\mathbb{R})$. We shall state the result without the proof. We leave details to the audience.

Theorem 4. *Let* $M = L^\infty(\mathbb{R})$, $R = L^\infty(\mathbb{R}\times\mathbb{R})$ *and let* $\tilde{R} = S(dtds)$ *(=the space of all measurable functions on the plane* $\mathbb{R}\times\mathbb{R}$*).*

(i) *The space* $L^p(M)$ *admits the following description*

$$L^p(M) = \left\{x \in \tilde{R} : \ \ x(t,s) = e^{-t/p}\psi(s),\ \psi \in L^p(ds)\right\}.$$

(ii) *If* $\mu(x)$ *is the decreasing rearrangement of an element* $x \in \tilde{R}$ *(with respect to the trace* $e^t\,\mathrm{d}t\mathrm{d}s$*), then*

$$\mu_t(x) = \frac{\mathrm{const}}{t^{1/p}},\ \ t \in \mathbb{R},\ x \in L^p(M).$$

(iii) *If an element* $x \in L^p(M)$ *has the representation* $x(t,s) = e^{-t/p}\psi(s)$ *for some function* $\psi \in L^p(ds)$*, then the constant from the statement above is equal to* $\|\psi\|_{L^p(ds)}$.

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