Noncommutative L_p-spaces and their generalizations: symmetric operator spaces

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1 Introduction

Brief review of preceding material.

We denote by $\mathcal{N}$ (or $\mathcal{M}$) a semifinite von Neumann algebra on the Hilbert space $\mathcal{H}$, with a fixed faithful and normal semifinite trace τ. We shall be mainly concerned with $\tau(1) = \infty$, where 1 is the identity in $\mathcal{N}$. A linear operator x:dom$(x) \to \mathcal{H}$, with domain dom$(x) \subseteq \mathcal{H}$, is called **affiliated** with $\mathcal{N}$ if $ux = xu$ for all unitary u in the commutant $\mathcal{N}'$ of $\mathcal{N}$. The closed and densely defined operator x, affiliated with $\mathcal{N}$, is called **τ-measurable** if for every $\epsilon > 0$ there exists an orthogonal projection $p \in \mathcal{N}$ such the $p(\mathcal{H}) \subseteq$dom(x) and $\tau(1-p) < \epsilon$. The set of all τ-measurable operators is denoted $\widetilde{\mathcal{N}}$, or $S(\widetilde{\mathcal{N}}, \tau)$, or $S(\tau)$.

We next recall the notion of generalized singular value function. Given a self-adjoint operator x in $\mathcal{H}$, we denote by $e^x(\cdot)$ the spectral measure of x. Now assume that x is τ- measurable. Then $e^{|x|}(B) \in \mathcal{N}$ for all Borel sets $B \subseteq \mathbb{R}$, and there exists $s > 0$ such that $\tau(e^{|x|}(s, \infty)) < \infty$. For $t \geqslant 0$, we define

$$\mu_t(x) = \inf\{s \geqslant 0 : \tau(e^{|x|}(s, \infty)) \leqslant t\}.$$

The function $\mu(x) : [0, \infty) \to [0, \infty]$ is called the **generalized singular value function** (or decreasing rearrangement) of x; note that $\mu_t(x) < \infty$ for all $t > 0$.

For each $x \in S(\tau)$, the support projection of x is denoted by $s(x)$, that is, $s(x) = \mathbf{1} - n(x)$. For $t > 0$, define

$$\mathcal{R}_t(\tau) = \{x \in S(\tau) : \tau(s(x)) \leqslant t\}.$$

the singular value function in terms of what might be called generalized approximation numbers.

Proposition 1.1. If $x \in S(\tau)$, then

$$\mu(t; x) = \inf\left\{\|x - y\|_{B(H)} : y \in \mathcal{R}_t(\tau), x - y \in \mathcal{M}\right\}$$

for all $t > 0$.

1.1 Basic examples

(i) Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra and suppose that $a = \sum_{j=1}^{m} \alpha_j p_j$, $m \leqslant \infty$ and where $p_1, \ldots, p_m \in P(\mathcal{M})$ with $p_j p_k = 0$ whenever $j \neq k$ and $\tau(p_j) = 1$ for all $1 \leqslant j \leqslant m$, and $0 < \alpha_j \in \mathbb{R}$ $(j = 1, \ldots, m)$ are such that $\alpha_j \neq \alpha_k$ whenever $j \neq k$. For the computation of $\mu(a)$, it may be assumed that $\alpha_1 > \alpha_2 > \cdots > \alpha_m > 0$. Setting $p_{m+1} = \mathbf{1} - \sum_{j=1}^{m} p_j$ and $\alpha_{m+1} = 0$, the spectral measure of a is then given by

$$e^{a} = \sum_{j=1}^{m+1} p_j \delta_{\alpha_j},$$

where δ_{α_j} denotes the Dirac measure at the point α_j. Since

$$e^{a}(\lambda, \infty) = \sum_{\alpha_j > \lambda} p_j, \qquad \lambda \geqslant 0.$$

It is now easily verified that $\mu(a)$ is given by

$$\mu(a) = \sum_{j=1}^{m-1} \alpha_j \chi_{[\chi_{j-1}, \chi_j)} + 0\chi_{[\chi_m, \infty)} \tag{1}$$

Note in particular, if $a = \mathbf{1}$ and $\tau(\mathbf{1}) = \infty$, then $\mu(\mathbf{1}) = \chi_{[0,\infty)}$.

(ii) If $\mathcal{M} = \ell_\infty(\mathbb{N})$ or $\mathcal{M} = \ell_\infty^m$ (with $m < \infty$) and τ is the counting measure on $\mathbb{N}$, then the set of all τ-measurable operators coincides with $\mathcal{M}$ and the above computation shows that $\mu(a)$ of an arbitrary element $a = \{a(n)\}_{n=1}^{\infty} \in \mathcal{M}$ may be also viewed as the decreasing rearrangement of the sequence $|a| = \{|a(n)|\}_{n=1}^{\infty}$.

(iii) Consider the special case that $H = \mathbb{C}^n$ and $\mathcal{M} = B(H) \cong M_n(\mathbb{C})$ equipped with the standard trace τ_n. If $a \in M_n(\mathbb{C})$ is positive self-adjoint, then a may be written as $a = \sum_{j=1}^{m} \alpha_j p_j$, where $\alpha_1 > \cdots > \alpha_m > 0$ are the distinct non-zero eigenvalues of a and p_j is the orthogonal projection onto the eigenspace corresponding to α_j. Consequently,

$$\mu(a) = \sum_{j=1}^{n} \lambda_j \chi_{[j-1, j)},$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$ is the sequence of eigenvalues of a in which each eigenvalue is repeated according to its multiplicity.

If $x \in M_n(\mathbb{C})$ is arbitrary, then $\mu(x) = \mu(|x|)$ and the eigenvalues of $|x|$ are usually called the singular values of x. Applying the above observations to

$a = |x|$, it follows that

$$\mu(x) = \sum_{j=1}^{n} \mu_j \chi_{[j-1,j)},$$

where $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n \geqslant 0$ is the sequence of singular values of x, repeated according to multiplicity.

(iv) Now suppose that $\mathcal{M} = B(H)$ (H is any Hilbert space), equipped with the standard trace τ. Suppose that $a \in B(H)$ is a positive self-adjoint compact operator. From the spectral theorem it follows that a can be written as

$$a = \sum_j \alpha_j p_j,$$

(norm convergent series in $B(H)$), where $\alpha_1 > \alpha_2 > \cdots > 0$ is the (finite or infinite) sequence of distinct non-zero eigenvalues of a and each p_j is the orthogonal projection onto the eigenspace corresponding to α_j.

So, as above

$$\mu(a) = \sum_j \lambda_j \chi_{[j-1,j)},$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots > 0$ is the sequence of non-zero eigenvalues of a, repeated according to multiplicity. If $x \in B(H)$ is an arbitrary compact operator, then $|x|$ is also compact and the eigenvalues of $|x|$ are called the singular values of x. Accordingly,

$$\mu(x) = \sum_j \mu_j \chi_{[j-1,j)},$$

where $\mu_1 \geqslant \mu_2 \geqslant \cdots > 0$ is the sequence of non-zero singular values of x, repeated according to multiplicity.

(v) If we consider $\mathcal{N} = L_\infty([0,\infty), m)$, where m denotes Lebesgue measure on $[0,\infty)$, as an abelian von Neumann algebra acting via multiplication on the Hilbert space $\mathcal{H} = L^2([0,\infty), m)$, with the trace given by integration with respect to m, it is easy to see that the set of all τ-measurable operators affiliated with $\mathcal{N}$ consists of all measurable functions on $[0,\infty)$ which are bounded except on a set of finite measure, and that the generalized singular value function $\mu(f)$ is precisely the decreasing rearrangement f^*. Here, x^* denote the non-increasing, right-continuous rearrangement of $|x|$ given by

$$x^*(t) = \inf\{s \geqslant 0 \,|\, m(\{|x| > s\}) \leqslant t\},\ t > 0.$$

In future, I shall avoid the notation x^* do denote the rearrangement and use $\mu(x)$, however the audience should keep in mind that this is a classical notation which is widely used in the (commutative) literature.

2 Noncommutative L_p-spaces

Let $(\mathcal{M},\tau)$ be a semi-finite von Neumann algebra. As is well known, the trace of a positive $n\times n$ matrix is the sum of its eigenvalues. In the general case, the trace of a positive element of $\mathcal{M}$ is obtained by integrating the singular value function, as the following formula shows

$$\tau(x) := \int_0^\infty \mu(t;x)dt, \quad x\in S(\tau)^+.$$

That is the trace τ extends to a positive tracial functional on the positive part of $S(\tau)$, still denoted by τ. For $0<p<\infty$, we define

$$L_p(\mathcal{M},\tau) = L_p(\tau) =: \{x\in S(\tau):\ \tau(|x|^p)<\infty\},\ \|x\|_p := \tau(|x|^p)^{1/p}.$$

The question whether $\|\cdot\|_p$ is actually a norm is important for understanding. We shall address this question below. Let us just list a number of (expected and easy to verify) properties.

(i) $x\in L_p(\tau)$ iff $\mu(x)\in L_p(0,\infty)$ and $\|x\|_p = \|\mu(x)\|_p$;

(ii) Since $\mu(x)=\mu(x^*)=\mu(|x|)$, we see that $x\in L_p(\tau)$ iff $x^*\in L_p(\mathcal{M},\tau)$ iff $|x|\in L_p(\tau)$ and we have $\|x\|_p = \|x^*\|_p = \||x|\|_p$.

(iii) The classical Hölder inequality extends to the noncommutative setting. For $0<r,p,q\leqslant\infty$ such that $1/r = 1/p+1/q$, we have

$$x\in L_p(\tau),\ y\in L_q(\tau) \Longrightarrow xy\in L_r(\tau)$$

and

$$\|xy\|_r \leqslant \|x\|_p\|y\|_q.$$

(iv) In particular, when $r=1$,

$$|\tau(xy)| \leqslant \|xy\|_1 \leqslant \|x\|_p\|y\|_q,\ x\in L_p(\tau),\ y\in L_q(\tau).$$

(v) The inequality above defines a natural pairing between $L_p(\tau)$ and $L_q(\tau)$: $\langle x,y\rangle = \tau(xy)$ and for any $1\leqslant p<\infty$ we have (isometrically)

$$L_p(\mathcal{M},\tau)^* = L_q(\mathcal{M}).$$

In particular, $L_1(\mathcal{M},\tau)$ is the predual of $\mathcal{M}$, and $L_p(\mathcal{M},\tau)$ is reflexive for $1<p<\infty$.

Actually, the proof of the fact that $\|\cdot\|_1$ is a norm on $L_1(\tau)$, which we left opened earlier may be easily seen from the alternative description of $\|\cdot\|_1$ presented in the next proposition.

Proposition 2.1. If $x \in L_1(\tau)$, then

$$\|x\|_1 = \sup\left\{|\tau(xy)| : y \in \mathcal{M}, \|y\|_{B(H)} \leqslant 1\right\}. \tag{2}$$

Similarly,

Proposition 2.2. If $x \in L_p(\tau)$, $1 < p < \infty$ and $1/p + 1/q = 1$, then

$$\|x\|_p = \sup\left\{|\tau(xy)| : y \in L_q(\tau), \|y\|_{L_q(\tau)} \leqslant 1\right\}. \tag{3}$$

Finally, let us state the result which is easily guessed (but whose proof requires some effort).

Proposition 2.3. The normed linear space $L_p(\tau)$, $1 \leqslant p \leqslant \infty$ is a Banach space.

An excellent source on the modern theory of and up-to-date developments in the theory of noncommutative L_p-spaces is [Pisier-Xu; survey].

2.1 Basic examples

(i) Let $\mathcal{M}$ be an Abelian von Neumann algebra. Then $\mathcal{M}$ coincides with $L_\infty(\Omega, \nu)$ and the trace τ is given by integration against the measure ν. In this case $L_p(\mathcal{M}, \tau)$ is just the commutative L_p-space $L_p(\Omega, \nu)$. In particular, if $\mathcal{M} = \ell_\infty(\mathbb{N})$ or $\mathcal{M} = \ell_\infty^m$ (with $m < \infty$) and τ is the counting measure on $\mathbb{N}$, then $L_p(\mathcal{M}, \tau)$ coincides with the space $\ell_p(\mathbb{N})$ or ℓ_p^m. These are of course classical (commutative) L_p-spaces.

(ii) If $\mathcal{M} = B(H_n) \cong M_n(\mathbb{C})$ or $\mathcal{M} = B(H)$ (H any Hilbert space), equipped with the standard trace Tr, then $L_p(\mathcal{M}, \tau)$ coincides with (Schatten-von Neumann class) C_p^n or C_p. In the literature, these classes are also frequently denoted by S_p^n and S_p or also $\mathfrak{S}_p^n$ and $\mathfrak{S}_p$. Historically, the spaces C_p^n (and more generally, symmetric operator spaces on algebras M_n) were introduced by John von Neumann in 1937. Having constructed the n^2-dimensional space matrix space C_p^n, von Neumann remarks that another natural way to metrize the n^2-dimensional linear space M_n is to identify standard matrix units e_{ij}, $1 \leqslant i, j \leqslant n$ with the first n^2 coordinate vectors of the space l_p, in other words to convert M_n into the n^2-dimensional (commutative) space $l_p^{n^2}$. This leads to an immediate problem: whether these two n^2-dimensional spaces, C_p^n (non-commutative) and $l_p^{n^2}$ (commutative), coincide. In the same paper, von Neumann shows that the spaces C_p^n and $l_p^{n^2}$ are non-isometric.

Another natural question would be whether the Banach-Mazur distance between C_p^n and $l_p^{n^2}$ is uniformly bounded. Recall that the Banach-Mazur distance

$d(X, Y)$ between Banach spaces X and Y is

$$d(X, Y) := \inf\{\|T\| \cdot \|T^{-1}\| \mid T : X \longrightarrow Y \text{ isomorphism}\},$$

where we adopt the convention $\inf \varnothing = +\infty$. This question was answered only 30 years later, in the negative, by McCarthy. **Theorem, McCarthy, (1967)**

There exists a constant $C > 0$ such that for any $n \in \mathbb{N}$ and n^2-dimensional subspace X of L_p we have

$$d(C_p^n, X) > Cn^{\frac{1}{3}\left|\frac{1}{p}-\frac{1}{2}\right|}.$$

(iii) Let $\nu \in \mathbb{N}$ and $0 < \alpha \leqslant \frac{1}{2}$ be fixed throughout the text. Let $\mathcal{N}_\nu$ be the class of all complex $2^\nu \times 2^\nu$-matrices with the unit matrix 1_ν. Tr is the standard trace on matrices. The state ρ_ν on $\mathcal{N}_\nu$ is given by

$$\rho_\nu(x) = \mathrm{Tr}(xA_\nu), \quad x \in \mathcal{N}_\nu, \quad A_\nu = \bigotimes_{k=1}^{\nu} \begin{bmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{bmatrix}. \tag{4}$$

The definition of the state ρ_ν immediately implies that

$$\rho_{\nu+\mu}(x \otimes y) = \rho_\nu(x)\, \rho_\mu(y), \quad x \in \mathcal{N}_\nu, y \in \mathcal{N}_\mu. \tag{5}$$

We consider the ultra-weak continuous $*$-isomorphic embedding $i_\nu : \mathcal{N}_\nu \to \mathcal{N}_{\nu+1}$ given by

$$i_\nu(x) = x \otimes 1_1, \quad x \in \mathcal{N}_\nu. \tag{6}$$

Due to (5), we have that $\rho_{\nu+1}(i_\nu(x)) = \rho_\nu(x)$, $x \in \mathcal{N}_\nu$, i.e. the restriction of the state $\rho_{\nu+1}$ onto the subalgebra $i_\nu(\mathcal{N}_\nu)$ is equal to the state ρ_ν. The collection of the algebras $\{(\mathcal{N}_\nu, \rho_\nu)\}_{\nu\in\mathbb{N}}$ together with the embedding (6) forms *a directed system of C^*-algebras*, [Section 11.4, Kadison-Ringrose, II]. *The inductive limit* of this system possesses a state ρ_α, induced by ρ_ν, $\nu \geqslant 1$. We denote the GNS representation of this inductive limit with respect to the state ρ_α as $\mathcal{R}_\alpha$.

$\mathcal{R}_\alpha$ is a factor of type III_λ if $0 < \alpha < \frac{1}{2}$, with $\lambda = \frac{\alpha}{1-\alpha}$ and a factor of type II_1 if $\alpha = \frac{1}{2}$. The construction of noncommutative L_p-spaces suggested above is not suitable to factors III_λ; to define L_p-spaces on them we need to resort to so-called Haagerup construction of L_p-spaces (to be discussed tomorrow). However, in the case of the hyperfinite factor $\mathcal{R}_{\frac{1}{2}}$ of type II_1, we obtain a very interesting L_p-space $L_p\left(\mathcal{R}_{\frac{1}{2}}, \rho_{\frac{1}{2}}\right)$ (observe that $\rho_{\frac{1}{2}}$ is a tracial state on $\mathcal{R}_{\frac{1}{2}}$).

If in the construction above, we choose only diagonal subalgebras of full matrix algebras $\mathcal{N}_\nu$, then at the outcome we would have received the classical

$L_p(0,1)$. Thus, $L_p\left(\mathcal{R}_{\frac{1}{2}}, \rho_{\frac{1}{2}}\right)$ contains an isometric copy of $L_p(0,1)$. However, the space $L_p\left(\mathcal{R}_{\frac{1}{2}}, \rho_{\frac{1}{2}}\right)$ is "highly noncommutative". To see this, one needs to resort to another classical description of $\left(\mathcal{R}_{\frac{1}{2}}, \rho_{\frac{1}{2}}\right)$ through the canonical **anticommutation relation**.

Let $(\epsilon_n)_{n\geqslant 1}$ be a sequence of self-adjoint unitaries on a (infinite-dimensional) Hilbert space H, satisfying

$$\epsilon_i\epsilon_j + \epsilon_j\epsilon_i = 2\delta_{ij}, \quad i,j \in \mathbb{N}.$$

Let R_0 be the C^*-algebra generated by $(\epsilon_n)_{n\geqslant 1}$. The R_0 admits a unique faithful tracial state τ, defined as follows. For any finite subset $A = \{i_1, i_2, \ldots, i_n\} \subset \mathbb{N}$ with $i_1 < i_2 < \cdots < i_n$ we set $w_A := \epsilon_{i_1}\epsilon_{i_2}\ldots\epsilon_{i_n}$ and $w_\varnothing = 1$. The trace τ is uniquely determined by its action on the w_A's: $\tau(w_A)+1$ (resp. 0) if $A = \varnothing$ (resp. $\neq \varnothing$). Consider R_0 as a C^*-algebra acting on $L^2(\tau)$ by left multiplication. Then the von Neumann algebra generated by R_0 in $B(L_2(\tau))$ is $*$-isomorphic with $\mathcal{R}_{\frac{1}{2}}$. Note that the family of all linear combinations of w_A's are dense in $L_p\left(\mathcal{R}_{\frac{1}{2}}, \rho_{\frac{1}{2}}\right)$ for all $1 \leqslant p < \infty$.

A good analogy here is with the classical Rademacher sequence $\{r_n\}_{n\geqslant 0}$, where r_n denotes the n^{th} Rademacher function, i.e.

$$r_n(t) = \text{sign } \sin 2^n \pi t, \quad \text{for} \quad 0 \leqslant t \leqslant 1 \quad (n = 1, 2, ...).$$

The fact that the products r_A's (properly enumerated) generate the so-called Walsh-Paley basis in $L_p(0,1)$, $1 < p < \infty$ is one of the basic facts of the classical Harmonic analysis (equivalent to the fact that the dyadic decomposition is unconditional and also to the fact that the trigonometric system is a Schauder basis in all $L_p(T)$, $1 < p < \infty$).

The fact that w_A's (properly enumerated) generate the so-called noncommutative Walsh-Paley basis in $L_p\left(\mathcal{R}_{\frac{1}{2}}, \rho_{\frac{1}{2}}\right)$, $1 < p < \infty$ was obtained in 1994 by S.V. Ferleger and F.S.

It is of interest to mention also another fact from comparative study of the classical Schatten-von Neumann classes C_p and $L_p\left(\mathcal{R}_{\frac{1}{2}}, \rho_{\frac{1}{2}}\right)$, $1 < p < \infty$:

Theorem 2.4. The Banach space C_p does not isomorphically embeds into the Banach space $L_p\left(\mathcal{R}_{\frac{1}{2}}, \rho_{\frac{1}{2}}\right)$, $1 \leqslant p \neq 2 < \infty$.

The case $p \in [1,2)$ was obtained by U. Haagerup, H. Rosenthal and F.S. in 2001, the case $p > 2$ by F.S. in 1996. In [Haagerup-Rosenthal-F.S.] the *complete isomorphic classification* of non-commutative L_p-spaces, $1 \leqslant p \neq 2 <$

∞ associated with *hyperfinite* von Neumann algebras is obtained. In fact, there exists precisely 13 pairwise non-isomorphic such spaces.

(iv) Consider a discrete group Γ. Let $\nu N(\Gamma) \subset B(\ell_2(\Gamma))$ be the associated von Neumann algebra and τ_Γ be the canonical trace on $\nu N(\Gamma)$ (see exercises). The Banach space geometry of the spaces $L_p(\nu N(\Gamma), \tau_\Gamma)$ is poorly understood. Especially interesting case is when $\Gamma = F_n$, the free group with n generators.

3 Symmetric operator spaces.

Let us again review the definition of L_p-spaces: firstly we extend the trace

$$\tau(x) := \int_0^\infty \mu(t; x) dt, \quad x \in S(\tau)^+.$$

on the positive part of $S(\tau)$ and then we define

$$L_p(\mathcal{M}, \tau) := \{x \in S(\tau) : \ \tau(|x|^p) < \infty\}, \quad \text{and} \quad \|x\|_p := \tau(|x|^p)^{1/p}.$$

You may think about purely commutative setup: $L_p(0, \infty)$. Note that it follows immediately from the definition of L_p-space that if $x \in L_p$ and $\mu(y) \leqslant \mu(x)$ then $y \in L_p$ and $\|y\|_p \leqslant \|x\|_p$. In particular, if $\mu(x) = \mu(y)$ and $x \in L_p$, then $y \in L_p$ and $\|y\|_p = \|x\|_p$. This feature of L_p-spaces is taken as the basis for the definition of the class of symmetric (function and operator) spaces.

Consider a Banach space $(E, \|\cdot\|_E)$ of real valued Lebesgue measurable functions (with identification m a.e.) on the interval $J = [0, \infty)$ or else on $J = \mathbb{N}$. Then E is called **symmetric** if $x \in E$ and $y \in S(J)$ with $\mu(y) \leqslant \mu(x)$, then $y \in E$ and $\|y\|_E \leqslant \|x\|_E$. A detailed exposition of the theory of symmetric function spaces may be found in a number of books e.g. [Krein-Petunin-Semenov], [Lindenstruss-Tzafiri, I,II] and many others.

Similarly, let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space H, equipped with a fixed semi-finite faithful normal trace τ.

Definition 3.1. A linear subspace E of $S(\tau)$, equipped with a norm $\|\cdot\|_E$, is called *symmetrically normed* if $y \in S(\tau)$, $x \in E$ and $\mu(y) \leqslant \mu(x)$ imply that $y \in E$ and $\|y\|_E \leqslant \|x\|_E$. If, in addition, E is a Banach space, then E is termed a *symmetric space (of τ-measurable operators).*

Given a semifinite von Neumann algebra $(\mathcal{M}, \tau)$ and a symmetric Banach function space $(E, \|\cdot\|_E)$ on $([0, \infty), m)$, we (**would like to**) define the corresponding non-commutative symmetric space $E(\mathcal{N}, \tau)$ by setting

$$E(\mathcal{M}, \tau) = E(\tau) = \{x \in S(\mathcal{M}) : \mu(x) \in E\}, \ \|x\|_{E(\tau)} := \|\mu(x)\|_E.$$

Theorem 3.2. (**Nigel Kalton and F.S., 2008**) The space $(E(\mathcal{M},\tau), \|\cdot\|_{E(\tau)})$ is a Banach space.

The space $(E(\tau), \|\cdot\|_{E(\tau)})$ is called the (non-commutative) symmetric operator space associated with $(\mathcal{M},\tau)$ and $(E, \|\cdot\|_E)$. If $\mathcal{M} = \ell_\infty(\mathbb{N})$, then the space $E(\tau)$ is simply the symmetric sequence space ℓ_E, which may be viewed as the linear span in E of the vectors $e_n = \chi_{[n-1,n)}$, $n \geqslant 1$ (see e.g. [Lindenstrauss-Tzafriri]). If $\mathcal{M} = (B(H), Tr)$, then the space $(E(\tau), \|\cdot\|_{E(\tau)})$ is the familiar symmetrically-normed ideal of compact operators on H associated with symmetric sequence space ℓ_E (see e.g [Gohberg-Krein]).

The exists a very substantial literature treating various geometrical, topological and order properties of symmetric operator spaces. The detailed exposition of their theory will be given in the forthcoming book [Dodds-de Pagter-Sukochev]. It should however be pointed out that the proof of the theorem above is rather complicated. It is much easier to deal with the class of **fully symmetric spaces** which is a proper subclass of symmetric operator spaces. For "applications" (that is concrete questions of noncommutative analysis and noncommutative geometry) the class of fully symmetric spaces suffices in 99 cases out of 100. We shall explain this class in the section below since it based on the properties of singular values of interest in its own right.

Also, we shall only briefly discuss (tomorrow) the special case of Marcinkiewicz operator spaces and their connection with Dixmier traces.

4 Fully symmetric operator spaces.

A great deal of operator inequalities are all centered around the important concept of submajorization in the sense of Hardy, Littlewood and Polya. We shall first discuss this notion and then demonstrate its usefulness and relevance to the theory of fully symmetric spaces.

Throughout this section it will be assumed that $\mathcal{M}$ is a von Neumann algebra on a Hilbert space H, equipped with a fixed semi-finite faithful normal trace τ.

4.1 Submajorization.

Definition 4.1. Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are von Neumann algebras on Hilbert spaces H_1 and H_2, respectively, equipped with semi-finite normal faithful traces τ_1 and τ_2, respectively. If $x \in S(\tau_1)$ and $y \in S(\tau_2)$, then x is said to be *submajorized* by y, denoted by $x \prec\prec y$, if

$$\int_0^t \mu(s;x)\,ds \leqslant \int_0^t \mu(s;y)\,ds \tag{7}$$

for all $t \geqslant 0$.

Remark 4.2. (i). The notion of submajorization may, of course, in particular be considered for functions on $[0, \infty)$, that is, with respect to the commutative von Neumann algebra $L_\infty(0, \infty)$ with Lebesgue integral as trace. If $x \in S(\tau_1)$ and $y \in S(\tau_2)$, then it is clear that $x \prec\prec y$ if and only if $\mu(x) \prec\prec \mu(y)$.

(ii). Evidently, if $x \in S(\tau_1)$ and $y \in S(\tau_2)$ are such that $\mu(x) \leqslant \mu(y)$, in particular if $|x| \leqslant |y|$, then $x \prec\prec y$ (but the converse does not hold). It is also clear that $x \prec\prec y$ implies that $\alpha x \prec\prec \alpha y$ for all $0 \leqslant \alpha \in \mathbb{R}$.

To get a better feel of the quantity $\int_0^t \mu(s; a)\, ds$, I state below the following result and suggest to the audience to prove it in the special case of commutative non-atomic von Neumann algebra $L_\infty(0, \infty)$. Recall that a von Neumann algebra is called *non-atomic* if it does not contain any minimal projections.

Theorem 4.3. If $\mathcal{M}$ is non-atomic, if $a \in S(\tau)^+$ and if $0 \leqslant t \leqslant \tau(\mathbf{1})$, then

$$\int_0^t \mu(s; a)\, ds = \sup\{\tau(eae) : e \in P(\mathcal{M}), \tau(e) = t\}. \tag{8}$$

Remark 4.4. Suppose that the von Neumann algebra $\mathcal{M}$ is atomic and that all minimal projections have trace equal to one. The result of Theorem above is also valid in this case, if the values of t are restricted to $\mathbb{N}$, that is,

$$\sum_{k=0}^{n-1} \mu(k) = \int_0^n \mu(s; a)\, ds = \sup\{\tau(eae) : e \in P(\mathcal{M}), \tau(e) = n\}$$

for all $n \in \mathbb{N}$ satisfying $n \leqslant \tau(\mathbf{1})$.

An absolutely indispensable (in the study of fully symmetric spaces and great many other topic of operator theory) is the following property of submajorization.

Theorem 4.5. Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space H, equipped with the semi-finite faithful normal trace τ. If $x, y \in S(\tau)$, then $\mu(x+y) \prec\prec \mu(x) + \mu(y)$, that is,

$$\int_0^t \mu(s; x+y)\, ds \leqslant \int_0^t \mu(s; x)\, ds + \int_0^t \mu(s; y)\, ds, \quad t \geqslant 0. \tag{9}$$

Proof. It is sufficient to prove this theorem under the additional assumption without that $\mathcal{M}$ is non-atomic.

Given $t \geqslant 0$, suppose that $e \in P(\mathcal{M})$ satisfies $\tau(e) = t$ and let $x+y = v|x+y|$ be the polar decomposition of $x+y$. Using simplest properties of the polar decomposition, it follows that

$$\begin{aligned}\tau(e|x+y|e) &= \tau(ev^*(x+y)e) = \tau(|ev^*xe+ev^*ye|)\\ &\leqslant \tau(|ev^*xe|)+\tau(|ev^*ye|)\\ &\leqslant \int_0^t \mu(s;v^*x)\,ds + \int_0^t \mu(s;v^*y)\,ds\\ &\leqslant \int_0^t \mu(s;x)\,ds + \int_0^t \mu(s;y)\,ds.\end{aligned}$$

Since this holds for all $e \in P(\mathcal{M})$ with $\tau(e) = t$, inequality (9) is an immediate consequence of Theorem 4.3.□

4.2 Submajorization and fully symmetrically normed spaces

Definition 4.6. A linear subspace E of $S(\tau)$, equipped with a norm $\|\cdot\|_E$, is called *fully symmetrically normed* if $x \in S(\tau)$, $y \in E$ and $x \prec\prec y$ imply that $x \in E$ and $\|x\|_E \leqslant \|y\|_E$. If, in addition, E is a Banach space, then E is called a *fully symmetric space (of τ-measurable operators).*

Now fix a fully symmetric function space $E = E(0,\infty)$ and recast the question/construction from the section above:

Given a semifinite von Neumann algebra $(\mathcal{M},\tau)$ and a fully symmetric Banach function space $(E, \|\cdot\|_E)$ on $([0,\infty), m)$, we (**would like to**) define the corresponding non-commutative fully symmetric space $E(\mathcal{N},\tau)$ by setting

$$E(\tau) = \{x \in S(\mathcal{M}) : \mu(x) \in E\}, \quad \|x\|_{E(\tau)} := \|\mu(x)\|_E.$$

Let us now demonstrate how easy the question of whether $\|\cdot\|_{E(\tau)}$ is a norm has become thanks to submajorization inequalities.

Fix $x, y \in E(\tau)$. We have to prove that

$$\|x+y\|_{E(\tau)} \leqslant \|x\|_{E(\tau)} + \|y\|_{E(\tau)},$$

or equivalently (see the definition of $\|\cdot\|_{E(\tau)}$ above) that

$$\|\mu(x+y)\|_{E(0,\infty)} \leqslant \|\mu(x)\|_{E(0,\infty)} + \|\mu(y)\|_{E(0,\infty)}. \tag{10}$$

However, by the preceding Theorem, we have $\mu(x+y) \prec\prec \mu(x) + \mu(y)$ and since the space E is assumed to be fully symmetric, this fact guarantees

$$\|\mu(x+y)\|_{E(0,\infty)} \leqslant \|\mu(x)+\mu(y)\|_{E(0,\infty)}.$$

Now, triangle inequality in the space $E(0,\infty)$ obviously guarantees that

$$\|\mu(x)+\mu(y)\|_{E(0,\infty)} \leqslant \|\mu(x)\|_{E(0,\infty)} + \|\mu(y)\|_{E(0,\infty)},$$

and a combination of the two inequalities above yields immediately the required claim (10).

The question of completeness of the space $(E(\tau), \|\cdot\|_{E(\tau)})$ requires addition arguments (also based on submajorization inequalities) and won't be discussed here.

4.3 Which spaces are fully symmetric?

We have seen above how easy it is to deal with fully symmetric spaces (which is all based on the fundamental fact that the Hardy, Littlewood and Polya submajorization is excellently suited for the operator theory). Now, the natural question would be to have natural criteria to recognize such spaces. Basically, there are two such criteria:

(i). Every space E with order continuous norm (that is $x_\alpha \downarrow 0$ implies $\|x_\alpha\|_E \to 0$ is fully symmetric. In particular, any *separable* symmetric function space $E(0,\infty)$ has an order continuous norm.

(ii). Every maximal space (that is $E = E^{\times\times}$ is a fully symmetric space. Here, the space $E^\times$ (=the Köthe dual to E) is defined by

$$\|y\|_{E^\times} = \sup\left\{\int_0^\infty \mu(t;z)\,\mu(t;y)\,dt : z \in E, \|z\|_E \leqslant 1\right\} < \infty.$$

In particular, the Köthe dual $E^\times$ to any symmetrically normed space E is fully symmetric.

Observe that the criterion implies immediately that ANY symmetric space on the algebra ℓ_∞^n or on matrix algebra M_n is automatically fully symmetric. In particular, any L_p-space $L_p(\tau)$, $1 \leqslant p < \infty$ has an order continuous norm and is also maximal. By the way, Proposition 2.2 above shows that the Köthe dual to $L_p(\mathcal{M},\tau)$, $1 \leqslant p < \infty$ is precisely $L_q(\mathcal{M},\tau)$, $1/p + 1/q = 1$, which immediately shows that $L_p(\mathcal{M},\tau)$, $1 \leqslant p < \infty$ is maximal. Ultimately, this is exactly the fact underlying the proof of Theorem 2.3. However, a demanding listener may ask for a direct verification of the fact that $f \prec\prec g$, $g \in L_p$ implies that $\|f\|_p \leqslant \|g\|_p$. Indeed, such a fact should be directly verifiable at least for *finite-dimensional* L_p spaces (that is for ℓ_p^n, $1 \leqslant n < \infty$).

I know one such a proof, which I could recommend as an excellent (and in fact rather important exercise). Such a proof is an *immediate* corollary of the following beautiful geometrical fact. Fix $1 \leqslant n < \infty$ and a positive element $0 \leqslant f = \mu(f) \in \ell_\infty^n$. Consider the sets

$$\Omega_+(f) := \{0 \leqslant g \prec\prec f\}, \quad \Omega'_+(f) := \{0 \leqslant g \prec\prec f, \quad \tau(g) = \tau(f)\}.$$

Theorem 4.7. The set of extreme points of $\Omega_+(f)$ (resp. of $\Omega'_+(f)$) is

$$\mathrm{extr}(\Omega_+(f)) = \{g \in \Omega_+(f) : \mu(g) = \mu(f)\chi_{[0,m]}, m \leqslant n\},$$

(respectively,

$$\mathrm{extr}(\Omega'_+(f)) = \{g \in \Omega_+(f) : \mu(g) = \mu(f)\}).$$

It follows immediately from Theorem above that ANY finite dimensional symmetric space is *automatically* fully symmetric.