DIXMIER TRACES: Analytical approach and applications.

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1 Introduction

The Dixmier trace τ_ω arose from the problem of whether the algebra $B(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$ had a unique non-trivial trace. Dixmier resolved this question in the negative in 1966 in a note in Comptes Rendus [Dix66CR]. To construct the trace he used an invariant mean ω on the solvable 'ax+b' group. His trace vanishes on the ideal of trace class operators and hence is completely disjoint from the usual trace. It is also non-normal.

Applications of the Dixmier trace to classical geometry are facilitated by a remarkable result [Co3] relating the trace to the Wodzicki [Wo84KT] residue for pseudo-differential operators on a closed manifold. Both the Dixmier trace and the Wodzicki residue play important roles in noncommutative geometry and its applications [Co1]. The usefulness of the Dixmier trace is extended by the results of Connes [Co1] which relates it to residues of zeta functions.

Our aim is to give a unified and coherent account of some recent functional analytic advances in the theory of Dixmier traces. In addition to surveying these new results we also offer in some cases new proofs. Part of our motivation is to extend and clarify questions raised by [Co1] Chapter IV. Specifically we characterise the class of positive measurable operators defined in [Co1], explain the role of the Cesaro mean in Connes' version of Dixmier traces (called Connes-Dixmier traces here) and give a complete analysis of zeta function formulae for the Dixmier trace. Most importantly, however, our whole treatment is within the framework of 'semifinite spectral triples'. This notion arises when one extends the theory in [Co1], which deals with subalgebras of the bounded operators on a separable Hilbert space equipped with the standard trace, to the situation where the von Neumann algebra of bounded operators is replaced by a general semifinite von Neumann algebra equipped with some faithful normal semifinite trace (cf [BCPRSW, BeF]). As a result of the extra effort needed to handle this level of generality it is possible to find improvements even in the standard type I theory of [Co1].

Our exposition is based on the approach articulated in [DPSS, CPS2], which considers such traces as a special class of continuous linear functionals on the corresponding operator ideals. Many features of the theory may be well understood, even in the most trivial situation, when the von Neumann algebras in question are commutative. In this situation, the theory of Dixmier traces roughly corresponds to the theory of symmetric functionals on symmetric and fully symmetric function spaces [DPSSS1, DPSSS2, DPSS] and allows an alternative treatment based on the methods drawn from real analysis.

The key observation, made in Section 4, that enables us to prove a considerable generalisation of Proposition IV.2.4 of [Co1] and also the measurability theorem of Section 6, is the existence of two kinds of Dixmier trace. One is associated with the multiplicative group of the positive reals and its invariant mean and the second, which is naturally associated with the zeta function, arises from the invariant mean of the additive group of the reals. The relationship between the two captures the formula for the Dixmier trace in terms of the zeta function.

Finally, following [AzSu], we show how Lidskii's formula may be extended to Dixmier traces in the von Neumann setting in Section 11. In a series of corollaries we explain its relevance to the question of measurability. In [Co1], except for the case of pseudodifferential operators, measurability results are proved for positive operators. The approach of [AzSu] allows one to address the problem of removing the positivity assumption.

We present below a short list of symbols and terminology used in this paper with the indication of the place where these symbols and notations are introduced:

- Symmetric and fully symmetric functionals and spaces $E(J)$ (Section 2, Definition 2.1);
- Marcinkiewicz spaces, $M(\psi)$, $\mathcal{L}^{(1,\infty)}$, $\mathcal{L}^{(p,\infty)}$ (Section 2.1);
- Generalized singular value function $\mu_{(\cdot)}(x)$ (Section 4);
- Semifinite von Neumann algebra $\mathcal{N}$, faithful normal semifinite trace on $\mathcal{N}$, τ. τ-measurable operators $\widetilde{\mathcal{N}}$, the fully symmetric operator space associated to $(\mathcal{N},\tau)$ and Banach function space E is denoted $E(\mathcal{N},\tau)$ (Section 2.3);
- Operator Marcinkiewicz spaces $M(\psi)(\mathcal{N},\tau)$ (Section 2.3);
- Symmetric functionals on a fully symmetric space $E = E(0,\infty)$: E^*_{sym} (Section 3.1);
- Sets of states $BL(\mathbb{R})$, $D(\mathbb{R}^*_+)$, $BL(\mathbb{R}_+)$ (Section 4);
- Dixmier traces τ_ω, traces $F_{\mathbb{L}}, F_{\mathbb{L}}$ (Section 5.1);
- Connes-Dixmier traces τ_ω with $\omega \in CD(\mathbb{R}^*_+)$ (Section 5.2);
- Measurable operators (Definitions 6.1 and 6.2);

Acknowledgement This exposition is partly contained in the recent survey "Dixmier traces and some applications in noncommutative geometry" [CaS]. Contents of Sections 8-10 (resp., 11) is taken from [CRSS] (resp., [AzSu]).

1.1 Illustration of an important special case: Dixmier ideal $\mathcal{L}^{(1,\infty)}$.

The natural domain of a Dixmier trace is the Banach (symmetrically normed) ideal (of compact operators) $\mathcal{L}^{(1,\infty)}(\mathcal{H})$, which we now introduce.

For our purposes the following definition of s-numbers is most convenient. $\mathcal{K}(\mathcal{H})$ stands for compact operators on $\mathcal{H}$.

Definition 1.1. Let $\mathcal{R}_n = \{A \in \mathcal{B}(\mathcal{H}) : \mathrm{rank}(A) \leq n\}$. Then n-th characteristic number of an operator $T \in \mathcal{K}(\mathcal{H})$ is by definition $\mu_n(T) := \mathrm{dist}(\mathcal{R}_{n-1}, T) = \inf\limits_{A \in \mathcal{R}_{n-1}} \|T - A\|$.

Let $\sigma_N(T) := \sum\limits_{n=1}^{N} \mu_n(T)$. We use the following properties of s-numbers, all of which, with a possible exclusion of the last equality, follow easily from Definition 1.1.

Proposition 1.2. Let $T, S \in \mathcal{K}(\mathcal{N}, \tau)$. Then
(1) $\mu_1(T) = \|T\|$;
(2) $\mu_n(\alpha T) = |\alpha| \mu_n(T)$;
(3) $|\mu_n(S) - \mu_n(T)| \leq \|S - T\|$;
(4) $\mu_n(T)$ is norm-continuous function;
(5) $\mu_n(ATB) \leq \|A\| \mu_n(T) \|B\|$ for all $A, B \in \mathcal{B}(\mathcal{H})$;
(6) $\mu_n(UTU^*) = \mu_n(TU) = \mu_n(UT) = \mu_n(T)$ for any unitary $U \in \mathcal{B}(\mathcal{H})$;
(7) $\sigma_N(T) = \sup\{\|TE\|_1 : E \text{ is an orthogonal projection, } \dim E = N\}$.

For T, we define a non-negative number (maybe infinite)

$$||T||_{(1,\infty)} := \sup_N \frac{1}{\log(1+N)} \sum_{n=1}^{N} \mu_n(T).$$

Let

$$\mathcal{L}^{(1,\infty)}(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) \;\;:\;\; ||T||_{(1,\infty)} < +\infty\}.$$

Lemma 1.3. The pair $(\mathcal{L}^{(1,\infty)}(\mathcal{H}), ||.||_{(1,\infty)})$ is a Banach space.

Proof. We have to prove the following: 1) $||T||_{(1,\infty)} = 0 \Rightarrow T = 0$, 2) $||\alpha T||_{(1,\infty)} = |\alpha| \cdot ||T||_{(1,\infty)}$, $\alpha \in \mathbb{C}$, 3) $||S + T||_{(1,\infty)} \leq ||S||_{(1,\infty)} + ||T||_{(1,\infty)}$, 4) $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ is complete in the norm $||.||_{(1,\infty)}$. 1) $||T||_{(1,\infty)} = 0$ implies that $\mu_1(T) = 0$ which implies that $T = 0$. 2) This follows from the following property of s-numbers: $\mu_n(\alpha T) = |\alpha| \mu_n(T)$. 3) The statement follows from (7) of proposition 1.2; 4) Note that $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ is a linear space, indeed, it follows from 3) above. We have now to prove that if $T_j \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, $j = 1, 2, ...$ is a Cauchy sequence (in norm $||.||_{(1,\infty)}$) then there exists $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ such that $T_j \to T$ in norm $||.||_{(1,\infty)}$. We have

$$\|T_i - T_j\| = \mu_1(T_i - T_j) \leq \log 2 \cdot ||T_i - T_j||_{(1,\infty)}.$$

This means that $\{T_j\}_{j=1}^{\infty}$ is a Cauchy sequence in the uniform operator norm, so it converges in that norm to some compact operator T (since all T_j are compact). The inequality $|\mu_n(S) - \mu_n(T)| \leq \|S - T\|$ implies

$$\liminf_k |\mu_n(T_k) - \mu_n(T)| \leq \liminf_k \|T_k - T\| = 0, \; n = 1, 2, ...$$

and so,

$$\liminf_k \mu_n(T_k) = \mu_n(T),\ n = 1, 2, ...$$

We now must show that T belongs to $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ and that T_k converges to T in $||.||_{(1,\infty)}$ norm. Fix $M \geq 1$. We have

$$\sup_{N=1,\dots,M} \frac{1}{\log(1+N)} \sum_{n=1}^{N} \mu_n(T_k) \leq \sup_{p\geq 1} ||T_p||_{(1,\infty)} < +\infty,\ k = 1, 2, ...$$

and passing to the limit $k \to \infty$ we obtain

$$\sup_{N=1,\dots,M} \frac{1}{\log(1+N)} \sum_{n=1}^{N} \mu_n(T) \leq \sup_{p\geq 1} ||T_p||_{(1,\infty)}.$$

Since the inequality above holds for every $M \geq 1$, we obtain

$$\sup_N \frac{1}{\log(1+N)} \sum_{n=1}^{N} \mu_n(T) \leq \sup_{p\geq 1} ||T_p||_{(1,\infty)},$$

hence, $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$. Now we show that T_k converges to T in $||.||_{(1,\infty)}$ norm. Fix $\epsilon > 0$. Let $M \in \mathbb{N}$ be such that for all $k, m > M$

$$||T_k - T_m||_{(1,\infty)} < \epsilon.$$

Then

$$\sup_N \frac{1}{\log(1+N)} \sum_{n=1}^{N} \mu_n(T_k - T_m) < \epsilon,\ k, m > M. \tag{1}$$

Bearing in mind that

$$\liminf_m \mu_n(T_k - T_m) = \mu_n(T_k - T),\ n = 1, 2, ...$$

we find that for every fixed $N \in \mathbb{N}$,

$$\frac{1}{\log(1+N)} \sum_{n=1}^{N} \mu_n(T_k - T) = \liminf_m \frac{1}{\log(1+N)} \sum_{n=1}^{N} \mu_n(T_k - T_m). \tag{2}$$

Therefore, it follows from (1) and (2)

$$\sup_N \frac{1}{\log(1+N)} \sum_{n=1}^{N} \mu_n(T_k - T) \leq \epsilon,\ k > M,$$

i.e. $||T_k - T||_{(1,\infty)} \leq \epsilon$. $\square$

Let $\mathcal{L}_+^{(1,\infty)}(\mathcal{H}) := \{T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}) : T \geqslant 0\}$.

Proposition 1.4. $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ is a Banach ideal (for the norm $\|\cdot\|_{(1,\infty)}$) in the algebra $\mathcal{B}(\mathcal{H})$ (in the literature this ideal is also known as the dual to the Macaev's ideal).

Proof. We have to prove that if $A, B \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ then $ATB \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$.

The inequality $\mu_n(ATB) \leq \|A\|\|B\|\mu_n(T)$ (Proposition 1.2, (5)) implies

$$\frac{1}{\log(1+N)} \sum_{n=1}^{N} \mu_n(ATB)$$

$$\leq \frac{1}{\log(1+N)} \sum_{n=1}^{N} \|A\|\|B\|\mu_n(T) \leq \|A\|\|B\| ||T||_{(1,\infty)}.$$

Hence, $ATB \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$. $\square$

Lemma 1.5. For any $\epsilon > 0$

$$\mathcal{L}^1(\mathcal{H}) \subset \mathcal{L}^{(1,\infty)}(\mathcal{H}) \subset \mathcal{L}^{1+\epsilon}(\mathcal{H}).$$

Here, $\mathcal{L}^p(\mathcal{H})$ stands for Schatten-von Neumann p-class. Thus,

$$\mathcal{L}^p(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) \ : \ \sum_{n=1}^{\infty} \mu_n^p(T) < \infty\}.$$

1.2 Dixmier traces

Let ω be a state on l^∞ which is invariant under the dilation operator D_n (see Theorem 4.3 and also Theorem 4.7 and Corollary 4.8 below). It is sufficient to require that ω is D_2-invariant, that is $\omega \circ D_2 = \omega$.

Definition 1.6. *Dixmier trace* of $T \in \mathcal{L}_+^{(1,\infty)}(\mathcal{H})$ is a number

$$\mathrm{Tr}_\omega(T) := \omega\text{-}\lim_{N\to\infty} \frac{1}{\log(1+N)} \sum_{n=1}^{N} \mu_n(T).$$

Proposition 1.7. $\tau_\omega(S+T) = \tau_\omega(S) + \tau_\omega(T) \quad \forall S,\ T \in \mathcal{L}_+^{(1,\infty)}(\mathcal{H})$.

Proof. Set, for brevity

$$\sigma_N(X) := \sum_{n=1}^{N} \mu_n(X), \quad X \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$$

and note that for positive X,

$$\sigma_N(X) = \sup\{Tr(XP) \ : \ P = P(\mathcal{H}) \text{ is a projection and } \dim P(\mathcal{H}) = N\}.$$

(see Proposition 1.2 (7) and [FK]). For a given $\epsilon > 0$, let projections P_1 and P_2 satisfy the conditions $\dim P_1(\mathcal{H}) = \dim P_2(\mathcal{H}) = N$ and $Tr(SP_1) > \sigma_N(S) - \epsilon$, $Tr(TP_2) > \sigma_N(T) - \epsilon$. Setting $P := P_1 \vee P_2$, we have

$$\begin{aligned} Tr((S+T)P) &= Tr(SP) + Tr(TP) \geq Tr(SP_1) + Tr(TP_2) \\ &> \sigma_N(S) + \sigma_N(T) - 2\epsilon. \end{aligned}$$

Since $\dim P(\mathcal{H}) \leq 2N$ and ϵ is an arbitrary positive number, we have

$$\sigma_{2N}(S+T) \geq \sigma_N(S) + \sigma_N(T), \quad N \geq 1.$$

Setting, for brevity,

$$\alpha_N = \frac{1}{\log(1+N)}\sigma_N(S), \quad \beta_N = \frac{1}{\log(1+N)}\sigma_N(T),$$

$$\gamma_N = \frac{1}{\log(1+N)}\sigma_N(S+T).$$

we restate the above inequality as

$$\frac{\log(2N+1)}{\log(N+1)}\gamma_{2N} \geq \alpha_N + \beta_N, \quad N \geq 1.$$

Assume, for a moment, that we know

$$\omega\text{-}\lim_{N\to\infty} \gamma_{2N} = \omega\text{-}\lim_{N\to\infty} \gamma_N. \tag{3}$$

Noting that $\{\gamma_{2N}\}_{N\geq 1} \in \ell_\infty$ and so $\frac{\log(1+2N)}{\log(1+N)}\gamma_{2N} - \gamma_{2N} \to 0$, we infer from the above inequality that

$$\tau_\omega(S+T) = \omega\text{-}\lim_{N\to\infty} \gamma_N \geq \omega\text{-}\lim_{N\to\infty} \alpha_N + \omega\text{-}\lim_{N\to\infty} \beta_N = \tau_\omega(S) + \tau_\omega(T).$$

Since the converse inequality $\tau_\omega(S+T) \leq \tau_\omega(S) + \tau_\omega(T)$ follows immediately from the inequality $\sigma_N(S+T) \leq \sigma_N(S) + \sigma_N(T)$, which we already used in the proof of the triangle inequality for the norm $\|\cdot\|_{(1,\infty)}$ and which immediately follows from Proposition 1.2 (7), the proof is completed. It remains to explain equality (3).

Note that D_2-invariance of ω immediately implies that

$$\begin{aligned} \omega(\{\gamma_{2N}\}_{N=1}^\infty) &= \omega(D_2\{\gamma_{2N}\}_{N=1}^\infty) \\ &= \omega(\{\gamma_2, \gamma_2, \gamma_4, \gamma_4, \gamma_6, \gamma_6, \ldots\}) \end{aligned}$$

and therefore, in order to prove (3) it is sufficient to verify that

$$D_2\{\gamma_{2N}\}_{N=1}^{\infty} - \{\gamma_N\}_{N=1}^{\infty} = \{\gamma_2, \gamma_2, \gamma_4, \gamma_4, \gamma_6, \gamma_6, \dots\} \\ - \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \dots\} \in c_0,$$

or equivalently, that $\gamma_{2N} - \gamma_{2N-1} \to 0$, as $N \to \infty$. For $N \geq 2$, we have

$$\gamma_{2N} - \gamma_{2N-1} = \left(\frac{1}{\log 2N} - \frac{1}{\log(2N-1)}\right)\sigma_{2N-1}(T+S) \\ + \frac{1}{\log 2N} \cdot \mu_{2N}(T+S).$$

It is obvious that the second summand above tends to 0 as $N \to \infty$. Noting that the condition $T + S \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ guarantees $\sigma_{2N-1}(T+S) = O(\log(2N-1))$ and that $\frac{1}{\log 2N} - \frac{1}{\log(2N-1)} = o\left(\frac{1}{\log(2N-1)}\right)$, we see that the first summand also tends to 0 as $N \to \infty$. □

Definition 1.8. Dixmier trace of a s.-a. operator T is $\mathrm{Tr}_\omega(T) := \mathrm{Tr}_\omega(T_+) - \mathrm{Tr}_\omega(T_-)$ and Dixmier trace of an arbitrary operator is $\mathrm{Tr}_\omega(T) := \mathrm{Tr}_\omega(Re(T)) + i\,\mathrm{Tr}_\omega(Im(T))$.

Proposition 1.9. Dixmier trace has the following properties.
1) $\mathrm{Tr}_\omega(ST) = \mathrm{Tr}_\omega(TS)\ \forall T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}),\ S \in \mathcal{B}(\mathcal{H})$;
2) $\mathrm{Tr}_\omega(T) = 0\ \forall T \in \mathcal{L}^1$.
3) Dixmier trace is continuous in the norm $||\cdot||_{(1,\infty)}$.

Proof. 1) Since every operator $S \in \mathcal{B}(\mathcal{H})$ is a linear combination of four unitary operators it is sufficient to prove the equality $\mathrm{Tr}_\omega(UT) = \mathrm{Tr}_\omega(TU)$ for a unitary U. Since every operator from $\mathcal{B}(\mathcal{H})$ is a linear combination of positive operators, it is sufficient to prove the last equality for positive T's. In this case, the latter equality follows immediately from the fact $\mu_n(UTU^*) = \mu_n(UT) = \mu_n(TU) = \mu_n(T),\ \forall n \geq 1$ (see Proposition 1.2, (6)).
2) For $T \in \mathcal{L}^1$ we have $\sigma_N(T) \leq \mathrm{Tr}(T)$ for all N. So, $\mathrm{Tr}_\omega(T) = 0$.
3) The ideal $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ is a Banach space and the Dixmier trace Tr_ω is a linear functional on it, so it suffices to prove continuity at zero. So let $T_j \to 0$ in $||.||_{(1,\infty)}$ topology. This means that

$$||T_j||_{(1,\infty)} := \sup_N \frac{1}{\log(1+N)} \sum_{n=1}^{N} \mu_n(T_j) \to 0 \quad \text{as } j \to \infty.$$

We must prove that

$$\mathrm{Tr}_\omega(T_j) := \omega\text{-}\lim_{N\to\infty} \frac{1}{\log(1+N)} \sum_{n=1}^{N} \mu_n(T_j) \to 0 \quad \text{as } j \to \infty.$$

This follows from $|\,\mathrm{Tr}_\omega(T_j)| \leq ||T_j||_{(1,\infty)}$, which in turn follows immediately from the properties of ω (see Corollary 4.8). □

2 Extension to general Marcinkiewicz spaces.

Consider a Banach space $(E, \|\cdot\|_E)$ of real valued Lebesgue measurable functions (with identification λ a.e.) on the interval $J = [0, \infty)$ or else on $J = \mathbb{N}$. Let x^* denote the non-increasing, right-continuous rearrangement of $|x|$ given by

$$x^*(t) = \inf\{s \geqslant 0 \,|\, \lambda(\{|x| > s\}) \leqslant t\},\ t > 0,$$

where λ denotes Lebesgue measure. Then E will be called symmetric if

(i). E is an ideal lattice, that is if $y \in E$, and x is any measurable function on J with $0 \leqslant |x| \leqslant |y|$, then $x \in E$ and $\|x\|_E \leqslant \|y\|_E$;

(ii). if $y \in E$ and if x is any measurable function on J with $x^* = y^*$, then $x \in E$ and $\|x\|_E = \|y\|_E$.

In the case $J = \mathbb{N}$, it is convenient to identify x^* with the rearrangement of the sequence $|x| = \{|x_n|\}_{n=1}^\infty$ in the descending order. For basic properties of symmetric spaces we refer to the monographs [KPS], [LT], [LT2]. We note that for any symmetric space $E = E(J)$ the following continuous embeddings hold

$$L_1 \cap L_\infty(J) \subseteq E \subseteq L_1 + L_\infty(J).$$

The symmetric space E is said to be fully symmetric Banach space if it has the additional property that if $y \in E$ and $L_1 + L_\infty(J) \ni x \prec\prec y$, then $x \in E$ and $\|x\|_E \leqslant \|y\|_E$. Here, $x \prec\prec y$ denotes submajorization in the sense of Hardy-Littlewood-Pólya:

$$\int_0^t x^*(s) ds \leqslant \int_0^t y^*(s) ds, \qquad \forall\, t > 0.$$

A classical example of non-separable fully symmetric function and sequence spaces $E(J)$ is given by Marcinkiewicz spaces.

2.1 Marcinkiewicz function and sequence spaces

Let Ω denote the set of concave functions $\psi : [0, \infty) \to [0, \infty)$ such that $\lim_{t\to 0^+} \psi(t) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. Important functions belonging to Ω include t, $\log(1+t)$, t^α and $(\log(1+t))^\alpha$ for $0 < \alpha < 1$. Let $\psi \in \Omega$. Define the weighted mean function

$$a\,(x,t) = \frac{1}{\psi\,(t)} \int_0^t x^*\,(s)\,ds \quad t > 0$$

(function $a(x,t)$ plays the same role as $\sigma_n(T)$ intrudoced in the preceding Section) and denote by $M(\psi)$ the Marcinkiewicz space of measurable functions x on $[0,\infty)$ such that

$$\|x\|_{M(\psi)} := \sup_{t>0} a\,(x,t) = \|a\,(x,\cdot)\,\|_{\infty} < \infty. \tag{4}$$

The definition of the Marcinkiewicz sequence space $(m(\psi), \|x\|_{m(\psi)})$ is similar,

$$m(\psi) = \left\{ x = \{x_n\}_{n=1}^{\infty} \ : \ \|x\|_{m(\psi)} := \sup_{N\geqslant 1} \frac{1}{\psi(N)} \sum_{n=1}^{N} x_n^* < \infty \right\}.$$

Let $M_+(\psi)$ (respectively, $m_+(\psi)$) denote the set of positive functions of $M(\psi)$ (respectively, $m(\psi)$). For every $x \in M(\psi)$, we write $x = x_+ - x_-$, where $x_+ := x\chi_{\{t:x(t)>0\}}$ and $x_- := x - x_+$. The spaces

$$\mathcal{L}^{(1,\infty)} := M(\log(1+t)) \cap L_\infty$$

$$\text{and} \quad \mathcal{L}^{(p,\infty)} := M(t^{1-\frac{1}{p}}) \cap L_\infty, \quad 1 < p < \infty$$

play very important part in the sequel. Note that these spaces are still Marcinkiewicz spaces. Indeed, $\mathcal{L}^{(1,\infty)}$ (respectively, $\mathcal{L}^{(p,\infty)}$, $p>1$) may be identified with the space $M(\psi_1)$ (respectively $M(\psi_p)$, $p>1$), where

$$\psi_1(t) = \begin{cases} t\cdot\log 2, & 0 \leqslant t \leqslant 1 \\ \log(1+t), & 1 \leqslant t < \infty \end{cases},$$

respectively,

$$\psi_p(t) = \begin{cases} t, & 0 \leqslant t \leqslant 1 \\ t^{1-\frac{1}{p}}, & 1 \leqslant t < \infty \end{cases}.$$

The (Marcinkiewicz) norm given by formula (4) on the space $\mathcal{L}^{(p,\infty)}$ is denoted by $\|\cdot\|_{(p,\infty)}$, $1 \leqslant p < \infty$.

2.2 Singular symmetric functionals on Marcinkiewicz spaces.

Definition 2.1 (cf. [DPSS], Definition 2.1). A positive functional $f \in M(\psi)^*$ is said to be symmetric (respectively, fully symmetric) if $f(x) \leqslant f(y)$ for all $x,y \in M_+(\psi)$ such that $x^* = y^*$ (respectively, $x \prec\prec y$). Such a functional is said to be supported at infinity (or singular) if $f(|x|) = 0$ for all $x \in M_1(\psi)$ (equivalently, $f(x^*\chi_{[0,s]}) = 0$, for every $x \in M(\psi)$ and the indicator function $\chi_{[0,s]}$ of the interval $[0,s]$ for all $s > 0$).

Observe, that Dixmier trace Tr_ω introduced in Definition 1.6 is symmetric in the sense of Definition 2.1 above. Indeed, it is obvious that $\sigma_n(T) \leqslant \sigma_n(S)$, $n \geqslant 1$ for positive T and S imply $Tr_\omega(T) \leqslant Tr_\omega(S)$.

The following theorem completely characterizes Marcinkiewicz spaces admitting non-trivial fully symmetric functionals.

Theorem 2.2 ([DPSS, DPSSS1, DPSSS2]). If $\psi \in \Omega$, then

(i). a non-zero fully symmetric functional on $M(\psi)$ (respectively, $m(\psi)$) supported at infinity exists if and only if

$$\liminf_{t\to\infty} \frac{\psi(2t)}{\psi(t)} = 1, \tag{5}$$

(ii). a non-zero fully symmetric functional on $M(\psi)$ supported at zero exists if and only if

$$\liminf_{t\downarrow 0} \frac{\psi(2t)}{\psi(t)} = 1. \tag{6}$$

Thus, for example, $(\mathcal{L}^{(1,\infty)})^*_{sym,\infty} \neq \{0\}$, whereas $(\mathcal{L}^{(p,\infty)})^*_{sym,\infty} = \{0\}$, for all $1 < p < \infty$. The conditions (5) and (6) admit the following geometric interpretation. Let us denote by $N(\psi)$ the norm closure in $M(\psi)$ of the (order) ideal

$$N(\psi)^0 := \{f \in M(\psi) : f^*(\cdot) \leqslant k\psi'(\frac{\cdot}{k}) \text{ for some } k \in \mathbb{N}\}.$$

Clearly, $N(\psi)$ is a Banach function space (a subspace of $M(\psi)$) and is symmetric. Assuming (for simplicity) that ψ is linear in a neighbourhood of 0, the space $N(\psi)$ is fully symmetric (and thus coincides with $M(\psi)$) if and only if (5) fails. In other words, $M(\psi)^*_{sym,\infty} \neq \{0\}$ if and only if $N(\psi) \neq M(\psi)$.

Our focus on (fully) symmetric functionals supported at infinity is explained by the numerous applications of their non-commutative counterparts in non-commutative geometry. Non-commutative analogues of symmetric functionals supported at zero can be thought of as "Dixmier traces associated with von Neumann algebras of type II_1" and have not found any applications to date.

2.3 Symmetric operator spaces and functionals.

Here, we extend the ideas of the previous sections to the setting of (noncommutative) spaces of measurable operators. We denote by $\mathcal{N}$ a semifinite von Neumann algebra on the Hilbert space $\mathcal{H}$, with a fixed faithful and normal semifinite trace τ. We shall be mainly concerned with $\tau(1) = \infty$, where 1 is the identity in $\mathcal{N}$. A linear operator x:dom$(x) \to \mathcal{H}$, with domain dom$(x) \subseteq \mathcal{H}$, is called affiliated with $\mathcal{N}$ if $ux = xu$ for all unitary u in the commutant $\mathcal{N}'$ of

$\mathcal{N}$. The closed and densely defined operator x, affiliated with $\mathcal{N}$, is called τ-measurable if for every $\epsilon > 0$ there exists an orthogonal projection $p \in \mathcal{N}$ such the $p(\mathcal{H}) \subseteq \mathrm{dom}(x)$ and $\tau(1-p) < \epsilon$. The set of all τ-measurable operators is denoted $\widetilde{\mathcal{N}}$.

We next recall the notion of generalized singular value function [FK]. Given a self-adjoint operator x in $\mathcal{H}$, we denote by $e^x(\cdot)$ the spectral measure of x. Now assume that x is τ- measurable. Then $e^{|x|}(B) \in \mathcal{N}$ for all Borel sets $B \subseteq \mathbb{R}$, and there exists $s > 0$ such that $\tau(e^{|x|}(s,\infty)) < \infty$. For $t \geqslant 0$, we define

$$\mu_t(x) = \inf\{s \geqslant 0 : \tau(e^{|x|}(s,\infty)) \leqslant t\}.$$

The function $\mu(x) : [0,\infty) \to [0,\infty]$ is called the *generalized singular value function* (or decreasing rearrangement) of x; note that $\mu_t(x) < \infty$ for all $t > 0$. For the basic properties of this singular value function we refer the reader to [FK].

If we consider $\mathcal{N} = L_\infty([0,\infty), m)$, where m denotes Lebesgue measure on $[0,\infty)$, as an abelian von Neumann algebra acting via multiplication on the Hilbert space $\mathcal{H} = L^2([0,\infty), m)$, with the trace given by integration with respect to m, it is easy to see that the set of all τ-measurable operators affiliated with $\mathcal{N}$ consists of all measurable functions on $[0,\infty)$ which are bounded except on a set of finite measure, and that the generalized singular value function $\mu(f)$ is precisely the decreasing rearrangement f^*.

If $\mathcal{N} = \mathcal{L}(\mathcal{H})$ (respectively, $\ell_\infty(\mathbb{N})$) and τ is the standard trace Tr (respectively, the counting measure on $\mathbb{N}$), then it is not difficult to see that $\widetilde{\mathcal{N}} = \mathcal{N}$. In this case, $x \in \mathcal{N}$ is compact if and only if $\lim_{t\to\infty} \mu_t(x) = 0$; moreover,

$$\mu_n(x) = \mu_t(x), \quad t \in [n, n+1), \quad n = 0, 1, 2, \dots,$$

and the sequence $\{\mu_n(x)\}_{n=0}^{\infty}$ is just the sequence of eigenvalues of $|x|$ in non-increasing order and counted according to multiplicity.

Given a semifinite von Neumann algebra $(\mathcal{N}, \tau)$ and a (fully) symmetric Banach function space $(E, \|\cdot\|_E)$ on $([0,\infty), m)$, we define the corresponding non-commutative space $E(\mathcal{N}, \tau)$ by setting

$$E(\mathcal{N}, \tau) = \{x \in \widetilde{\mathcal{N}} : \mu(x) \in E\}.$$

Equipped with the norm $\|x\|_{E(\mathcal{N},\tau)} := \|\mu(x)\|_E$, the space $(E(\mathcal{N},\tau), \|\cdot\|_{E(\mathcal{N},\tau)})$ is a Banach space and is called the (non-commutative) (fully) symmetric operator space associated with $(\mathcal{N}, \tau)$ corresponding to $(E, \|\cdot\|_E)$. If $\mathcal{N} = \ell_\infty(\mathbb{N})$, then the space $E(\mathcal{N}, \tau)$ is simply the (fully) symmetric sequence space ℓ_E, which may be viewed as the linear span in E of the vectors $e_n = \chi_{[n-1,n)}$, $n \geqslant 1$ (see e.g. [LT]). In the case $(\mathcal{N}, \tau) = (\mathcal{L}(\mathcal{H}), Tr)$, we denote $E(\mathcal{N}, \tau)$ simply by

$E(\mathcal{H})$. Note, that the latter space coincides with the (symmetrically-normed) ideal of compact operators on $\mathcal{H}$ associated with (fully) symmetric sequence space ℓ_E (see e.g [GK]).

We shall be mostly concerned with fully symmetric operator spaces $E(\mathcal{N},\tau)$ and $E(\mathcal{H})$ when $E = M(\psi)$, in particular, when $E = \mathcal{L}^{(p,\infty)}$, $1 \leqslant p < \infty$. We refer to the spaces $M(\psi)(\mathcal{N},\tau)$ as to operator Marcinkiewicz spaces. Sometimes, for brevity, we shall omit the symbols $(\mathcal{N},\tau)$ and $\mathcal{H}$ from the notations and this should not cause a confusion.

Further references to the theory of fully symmetric operator spaces can be found in [DDP1, DDP2, CS1, CS2].

Definition 2.3. A linear functional $\varphi \in E(\mathcal{N},\tau)^*$ is called symmetric (respectively, fully symmetric) if φ is positive, (that is, $\varphi(x) \geqslant 0$ whenever $0 \leqslant x \in E(\mathcal{N},\tau)$) and $\varphi(x) = \varphi(y)$ (respectively, $\varphi(x) \leqslant \varphi(y)$) whenever $\mu(x) = \mu(y)$ (respectively, $\mu(x) \prec\prec \mu(y)$ and $x, y \geqslant 0$.

2.4 Connection with interpolation theory

For a given $x \in \widetilde{\mathcal{N}}$, the set $\Omega(x) = \{y \in \widetilde{\mathcal{N}} \ : \ y \prec\prec x\}$ is called the orbit of the operator x. If $x \in L_1(\mathcal{N},\tau) + \mathcal{N}$, then the set $\Omega(x)$ is conveniently described in terms of absolute contractions. Denote by Σ the set of all linear operators $T : L_1(\mathcal{N},\tau) + \mathcal{N} \to L_1(\mathcal{N},\tau) + \mathcal{N}$ such that $T(a) \in L_1(\mathcal{N},\tau)$ (respectively, $\mathcal{N}$) if $a \in L_1(\mathcal{N},\tau)$ (respectively, $\mathcal{N}$) and such that $\|T\|_{L_1(\mathcal{N},\tau)\to L_1(\mathcal{N},\tau)} \leqslant 1$, $\|T\|_{\mathcal{N}\to\mathcal{N}} \leqslant 1$. It follows from [DDP2] that $y \prec\prec x$, $x \in L_1(\mathcal{N},\tau) + \mathcal{N}$, $y \in \widetilde{\mathcal{N}}$ if and only if there exists $T \in \Sigma$ such that $T(x) = y$. Thus,

$$\Omega(x) = \{Tx \ : \ T \in \Sigma\}.$$

If $E(\mathcal{N},\tau)$ is a fully symmetric operator space, we have $\Omega(x) \subseteq E(\mathcal{N},\tau)$ for every $x \in E(\mathcal{N},\tau)$ and therefore a bounded positive linear functional φ on $E(\mathcal{N},\tau)$ is symmetric if and only if $\varphi(|Tx|) \leqslant \varphi(x)$ for every $T \in \Sigma$ and $0 \leqslant x \in E(\mathcal{N},\tau)$.

Now we assume that $\alpha : \mathcal{N} \to \mathcal{N}$ is a $*$-automorphism which is in addition trace preserving, that is, $\tau(\alpha(a)) = \tau(a)$ for all $0 \leqslant a \in \mathcal{N}$. It is easy to see that such an automorphism extends (uniquely) to a $*$-automorphism $\tilde{\alpha} : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}$, which is rearrangement preserving, that is, $\mu(\tilde{\alpha}(x)) = \mu(x)$ for all $x \in \widetilde{\mathcal{N}}$. Thus, we can view symmetric functionals on $E(\mathcal{N},\tau)$ as positive functionals which are invariant with respect to the action of the group of all trace preserving $*$-automorphisms of $\mathcal{N}$, which is a subgroup of Σ. If $\mathcal{N} = \mathcal{L}(\mathcal{H})$, then every symmetric functional on $E(\mathcal{H})$ is simply a trace (i.e. unitary invariant positive functional on $\mathcal{L}(\mathcal{H})$), which extends to a continuous linear functional on $E(\mathcal{H})$. Clearly, every fully symmetric functional is symmetric. However, *a priori*, it is not clear whether there exists (for example on the ideal $\mathcal{L}^{(1,\infty)}(\mathcal{H})$) a symmetric singular functional, which is not necessarily fully symmetric, or whether there exists a trace on $\mathcal{L}^{(1,\infty)}(\mathcal{H})$, which does not coincide with a Dixmier trace (see Section 5 below). Very recently the first example of such a trace has appeared in [KS]. In fact, if

$$\lim_{t\to\infty} \frac{\psi(2t)}{\psi(t)} = 1, \tag{7}$$

then there exists a non-zero trace (or symmetric functional) on every operator Marcinkiewicz space $M(\psi)(\mathcal{N},\tau)$ which vanishes on $N(\psi)(\mathcal{N},\tau)$. In particular, if $\psi(t) = \log(1+t)$, then such a functional vanishes on every operator $0 \leqslant x \in \mathcal{L}^{(1,\infty)}(\mathcal{N},\tau)$ with $\mu_t(x) = \frac{1}{t}$ (or, if $\mathcal{N} = B(\mathcal{H})$ then on every compact operator $0 \leqslant x \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, such that $\mu_n(x) = \frac{1}{n}$, $n \geqslant 1$). It is not clear yet whether symmetric functionals which are not fully symmetric exist on every $M(\psi)(\mathcal{N},\tau)$ with ψ satisfying condition (5).

3 General facts about symmetric functionals\-traces.

Let $E = E(0,\infty)$ be a fully symmetric space. By E_+ we denote the set of all nonnegative functions from E and by E^*_{sym} the set of all symmetric functionals of E.

A positive linear functional φ on E is called *normal* (or *order continuous*), if from $f_n \downarrow 0$ it follows that $\varphi(f_n) \downarrow 0$.

Proposition 3.1. [DPSS] If a functional $\varphi \in E^*_{sym}$ is order continuous, then $E \subset L^1[0,\infty)$, and φ is proportional to the integral against the Lebesgue measure.

We define the dilation operator D_s as in [KPS] by $D_s f(t) = f(t/s)$. Note that D_s is a bounded operator on E and $\|D_s\|_{E\to E} \leqslant \max\{1,s\}$, moreover $(D_s f)^* = D_s f^*$ for any function $f \in E$. The following result is established in [DPSS, Proposition 2.3] under the assumption that φ is fully symmetric, however the proof holds also for symmetric functionals.

Proposition 3.2. If $0 \leqslant \varphi \in E^*$ is symmetric, then $\varphi(D_s f) = s\varphi(f)$ for all $f \in E$ and $s > 0$.

A positive $\varphi \in E^*$ is said to be *singular*, if from $0 \leqslant \varphi' \leqslant \varphi$, $\varphi' \in E^*$, φ' is order-continuous, it follows that $\varphi' = 0$.

Proposition 3.3. [DPSS] (i) Every symmetric functional on E can be uniquely decomposed into the sum of a normal functional and a singular symmetric functional. Moreover, the normal functional is zero unless $E \subseteq L^1[0,\infty)$.

(ii) Any singular symmetric functional can be uniquely decomposed into the sum of singular symmetric functionals, supported at zero and at infinity.
(iii) The set of symmetric functionals forms a lattice.

The following result shows that every fully symmetric functional on E admits a "natural extension" up to a fully symmetric functional on $E(\mathcal{N},\tau)$ for every semifinite von Neumann algebra $(\mathcal{N},\tau)$. By $E(\mathcal{N},\tau)_+$ we denote the set of all positive operators from $E(\mathcal{N},\tau)$.

Theorem 3.4 ([DPSS]). Let $\varphi_0 \in E^*_{sym}$ be fully symmetric. If $\varphi(x) := \varphi_0(\mu(x))$, for all $x \in E(\mathcal{N},\tau)_+$, then φ extends to a fully symmetric functional $0 \leqslant \varphi \in E(\mathcal{N},\tau)^*$.

Proof. It clearly suffices to show that φ is additive on $E(\mathcal{N},\tau)_+$. Let $x, y \in E(\mathcal{N},\tau)_+$. Since $\mu(x+y) \prec\prec \mu(x)+\mu(y)$ ([FK, Theorem 4.4]), and since φ_0 is fully symmetric, it follows that

$$\varphi(x+y) = \varphi_0(\mu(x+y)) \leqslant \varphi_0(\mu(x)+\mu(y)) = \varphi(x)+\varphi(y).$$

To prove the converse inequality, we use the easily verified fact (see e.g. the beginning of the proof of Proposition 1.7) that

$$\int_0^t \mu_s(x)ds + \int_0^t \mu_s(y)ds \leqslant \int_0^{2t} \mu_s(x+y)ds, \quad \forall t > 0. \tag{8}$$

Observing that (8) is equivalent to the submajorization

$$\mu(x)+\mu(y) \prec\prec 2D_{\frac{1}{2}}\mu(x+y), \tag{9}$$

it follows from Proposition 3.2 that

$$\begin{aligned}\varphi(x)+\varphi(y) &= \varphi_0(\mu(x)+\mu(y)) \\ &\leqslant 2\varphi_0(D_{\frac{1}{2}}\mu(x+y)) = \varphi_0(\mu(x+y)) = \varphi(x+y).\end{aligned}$$

Thus φ is additive on $E(\mathcal{N},\tau)_+$ and this suffices to complete the proof of the Theorem. □

Theorem 3.5. Let $(\mathcal{N},\tau)$ be a semifinite von Neumann algebra without minimal projections, and let E be a fully symmetric Banach function space on $[0,\infty)$. If $0 \leqslant \varphi \in E(\mathcal{N},\tau)^*$ is a symmetric functional, then there exists a symmetric functional $0 \leqslant \varphi_0 \in E^*$ such that $\varphi(x) = \varphi_0(\mu(x))$ for all $0 \leqslant x \in E(\mathcal{N},\tau)$.

Proof. It is sufficient to show that there exists a symmetric functional $0 \leqslant \varphi_0 \in E[0,\tau(1))^*$ satisfying $\varphi(x) = \varphi_0(\mu(x))$ for all $0 \leqslant x \in E(\mathcal{N},\tau)$. Let $\mathcal{M}$ be the commutative von Neumann algebra $L_\infty[0,\tau(1))$, with trace given by integration. The algebra $\widetilde{\mathcal{M}}$ may be identified with the space of all measurable functions on $[0,\tau(1))$ which are bounded except on a set of finite measure. Since $\mathcal{M}$ does not contain any minimal projections, there exists a positive rearrangement-preserving algebra homomorphism $J:\widetilde{\mathcal{M}}\to\widetilde{\mathcal{N}}$ ([CS2, Lemma 4.1], [DDP1, Theorem 3.5]). Let $0\leqslant\varphi\in E(\mathcal{N},\tau)^*$ be symmetric. For $f\in E[0,\tau(1))$, define $\varphi_0(f):=\varphi(Jf)$. It is clear that $0\leqslant\varphi_0\in E[0,\tau(1))^*$ is symmetric. Moreover, if $0\leqslant x\in E(\mathcal{N},\tau)$, then $\mu(J(\mu(x)))=\mu(x)$ and hence $\varphi(x)=\varphi(J(\mu(x)))=\varphi_0(\mu(x))$. $\square$

4 Preliminaries on dilation and translation invariant states.

A construction of Dixmier traces τ_ω depends crucially on the choice of the "invariant mean" ω. In this section we recall and review the most important classes of such means. We denote by $\ell_\infty=\ell_\infty(\mathbb{N})$ the Banach space of all bounded sequences of complex numbers. By a state on a unital C^*-algebra we mean a positive linear functional with value 1 on the unit of the algebra. We recall that a positive linear functional $\mathbb{L}\in\ell_\infty^*$ is called a Banach limit if $\mathbb{L}$ is translation invariant and $\mathbb{L}(\mathbf{1})=1$ (here, $\mathbf{1}=(1,1,1\dots)$). A Banach limit $\mathbb{L}$ satisfies in particular $\mathbb{L}(\xi)=0$ for all $\xi\in c_0$ (= all sequences from ℓ_∞ converging to zero). We denote the collection of all Banach limits on ℓ_∞ by $BL(\mathbb{N})$. Note that $\|\mathbb{L}\|=1$ for all $\mathbb{L}\in BL(\mathbb{N})$.

We recall that sequence $\xi=\{\xi_n\}_{n=1}^\infty\in\ell_\infty$ is said to be *almost convergent* to $\alpha\in\mathbb{R}$, denoted $F\text{-}\lim_{n\to\infty}\xi_n=\alpha$ if and only if $\mathbb{L}(\xi)=\alpha$ for all $\mathbb{L}\in BL(\mathbb{N})$. The notion of an almost convergent sequence is due to G.G. Lorentz [Lor], who showed that the sequence $\{\xi_n\}_{n=1}^\infty$ is almost convergent to α if and only if the equality $\lim_{p\to\infty}\frac{\xi_n+\xi_{n+1}+\dots+\xi_{n+p-1}}{p}=\alpha$ holds uniformly for $n=1,2,\dots$. We denote by ac (respectively, ac_0) the set of all almost convergent (respectively, all almost convergent to 0) sequences from ℓ_∞. Clearly, ac and ac_0 are closed subspaces in ℓ_∞. We define the shift operator $T\colon\ell_\infty\to\ell_\infty$, the Cesàro operator $H:\ell_\infty\to\ell_\infty$ and dilation operators $D_n:\ell_\infty\to\ell_\infty$ for $n\in\mathbb{N}$ by formulas

$$T(x_1,x_2,x_3,\dots)=(x_2,x_3,x_4,\dots).$$
$$H(x_1,x_2,x_3,\dots)=(x_1,\frac{x_1+x_2}{2},\frac{x_1+x_2+x_3}{3},\dots),$$
$$D_n(x_1,x_2,x_3,\dots)=(\underbrace{x_1,\dots,x_1}_{n},\underbrace{x_2,\dots,x_2}_{n},\dots),$$

for all $x=(x_1,x_2,x_3,\dots)\in\ell_\infty$.

Each of the above operators is positive and leaves invariant the unit element $\mathbf{1}$ of the algebra ℓ_∞ and consequently is bounded with the norm equal to 1.

Moreover, $\{D_n\}_{n=1}^{\infty}$ is an abelian semigroup. The main tool in our construction of various classes of invariant means is the well known Markov-Kakutani fixed point theorem.

Theorem 4.1 (Markov-Kakutani). Let F be a locally convex Hausdorff space and let K be a non-empty convex compact subset of F. Let $\mathcal{T}$ be an abelian semigroup of linear continuous operators on F such that $S(K) \subseteq K$ for all $S \in \mathcal{T}$. Then there exists $x \in K$ such that $Sx = x$ for all $S \in \mathcal{T}$.

It is easy to see that the set of all fixed points from Theorem 4.1 forms a convex compact subset of K.

We shall be applying the Markov-Kakutani theorem in the setting when $F = (\ell_\infty)^*$, $(L_\infty(\mathbb{R}))^*$, $(L_\infty(\mathbb{R}_+^*))^*$ equipped with the weak $*$-topology. For simplicity of exposition we present the proofs only for the first case.

Lemma 4.2 ([DPSSS2, CPS2]). The following is true.
(i) $D_n T = T^n D_n \quad \forall\, n \geqslant 1$;
(ii) $HTx - THx \in c_0 \quad \forall\, x \in \ell_\infty$;
(iii) $HD_n x - D_n Hx \in c_0 \quad \forall\, x \in \ell_\infty$.

Proof. The proof of (i) is straightforward. For the proof of (ii) note that for all $x \in \ell_\infty$

$$|(HTx)_k - (THx)_k| = \left| \frac{1}{k+1} \frac{x_2 + \cdots + x_{k+1}}{k} - \frac{1}{k+1} x_1 \right| \leqslant \frac{2}{k+1} \|x\|_\infty,$$

which shows that $HTx - THx \in c_0$.

We now indicate the proof of (iii). Let $n \geqslant 1$ and $x \in \ell_\infty$. For $1 \leqslant k \in \mathbb{N}$ there exist $l \geqslant 1$ and $1 \leqslant r \leqslant n$ such that $k = (l-1)n + r$. Hence

$$(HD_n x)_k = \frac{1}{k} \sum_{i=1}^{k} (D_n x)_i = \frac{n}{k} \sum_{j=1}^{l-1} x_j + \frac{r}{k} x_l$$

and

$$(D_n Hx)_k = (Hx)_l = \frac{1}{l} \sum_{j=1}^{l} x_j.$$

Noting that, $nl - k = n - r \leqslant n$ and $rl - k \leqslant nl - k \leqslant n$, it follows that

$$\begin{aligned} |(HD_n x)_k - (D_n Hx)_k| &= \left| \frac{nl-k}{kl} \sum_{j=1}^{l-1} x_j + \frac{rl-k}{kl} x_l \right| \\ &\leqslant \frac{n}{k} \left(\frac{1}{l} \sum_{j=1}^{l-1} |x_j| \right) + \frac{n}{k} |x_l| \leqslant \frac{2n}{k} \|x\|_\infty. \end{aligned}$$

This shows that $HD_n x - D_n Hx \in \ell^{(1,\infty)} \subseteq c_0$. $\square$

Theorem 4.3. There exists a state $\tilde{\omega}$ on ℓ_∞ such that for all $n \geqslant 1$

$$\tilde{\omega} \circ T = \tilde{\omega} \circ H = \tilde{\omega} \circ D_n = \tilde{\omega}.$$

Proof. Let $K = \{0 \leqslant \varphi \in (\ell_\infty)^* : \varphi(\mathbf{1}) = 1,\ T^*\varphi = \varphi\}$. Since K contains ordinary Banach limits it is not empty. It is clear that K is convex and $*$-weakly compact. We claim that $D_n^*(K) \subseteq K$. Indeed, by Lemma 4.2(i) above we know that $T^*D_n^* = D_n^*(T^*)^n$, hence for $\varphi \in K$

$$T^*(D_n^*\varphi) = D_n^*(T^*)^n\varphi = D_n^*\varphi,$$

which implies that $D_n^*\varphi \in K$. Therefore we may apply Theorem 4.1 to the set K and the abelian semigroup $\{D_j^*\}_{j=1}^\infty$. Consequently the set

$$K_1 = \{0 \leqslant \varphi \in (\ell_\infty)^* : \varphi(\mathbf{1}) = 1,\ T^*\varphi = \varphi,\ D_n^*\varphi = \varphi,\ n \geqslant 1\}$$

is non-empty and again it is clear that K_1 is convex and $*$-weakly compact.

We now show that $H^*(K_1) \subseteq K_1$. To this end, first observe that $\varphi(z) = 0$ for all $z \in c_0$ and all $\varphi \in K_1$ (as $T^*\varphi = \varphi$). Given $\varphi \in K_1$ it follows from Lemma 4.2(iii) that

$$(D_n^*H^*\varphi)(x) - (H^*D_n^*\varphi)(x) = \varphi(HD_nx - D_nHx) = 0$$

for all $x \in \ell_\infty$ and so

$$D_n^*(H^*\varphi) = H^*(D_n^*\varphi) = H^*\varphi.$$

Similarly it follows from Lemma 4.2(ii) that $T^*(H^*\varphi) = H^*\varphi$ for all $\varphi \in K_1$. Consequently, $H^*(K_1) \subseteq K_1$. Applying Theorem 4.1 to the set K_1 and the semigroup $\{(H^*)^n\}_{n=0}^\infty$, we may conclude that there exists $\tilde{\omega} \in K_1$ such that $H^*(\tilde{\omega}) = \tilde{\omega}$, by which the proof is complete. □

We define the isomorphism $L : L_\infty(\mathbb{R}) \to L_\infty(\mathbb{R}_+^*)$ by $L(f) = f \circ \log$. Firstly, we define the Cesaro means (transforms) on $L_\infty(\mathbb{R})$ and $L_\infty(\mathbb{R}_+^*)$, respectively by:

$$H(f)(u) = \frac{1}{u}\int_0^u f(v)dv \quad \text{for} \quad f \in L_\infty(\mathbb{R}),\ u \in \mathbb{R}$$

and,

$$M(g)(t) = \frac{1}{\log t}\int_1^t g(s)\frac{ds}{s} \quad \text{for} \quad g \in L_\infty(\mathbb{R}_+^*),\ t > 0.$$

A brief calculation yields for $g \in L_\infty(\mathbb{R}_+^*)$,

$$LHL^{-1}(g)(r) = \frac{1}{\log r}\int_0^{\log r} g(e^u)du = \frac{1}{\log r}\int_1^r g(v)\frac{dv}{v} = M(g)(r),$$

i.e L intertwines the two means.

We shall now consider analogues of the operators T, D_n and H acting on $L_\infty(\mathbb{R})$ and $L_\infty(\mathbb{R}_+^*)$.

Definition 4.4. Let T_b denote translation by $b \in \mathbb{R}$, D_a denote dilation by $\frac{1}{a} \in \mathbb{R}_+^*$ and let P^a denote exponentiation by $a \in \mathbb{R}_+^*$. That is,

$$\begin{aligned} T_b(f)(x) &= f(x+b) \quad \text{for} \quad f \in L_\infty(\mathbb{R}), \\ D_a(f)(x) &= f\left(\frac{x}{a}\right) \quad \text{for} \quad f \in L_\infty(\mathbb{R}), \\ P^a(f)(x) &= f(x^a) \quad \text{for} \quad f \in L_\infty(\mathbb{R}_+^*). \end{aligned}$$

Some of the basic relations between these L_∞ spaces and their self-maps are provided for easy access by the following proposition, whose proof is similar to Lemma 4.2.

Proposition 4.5 ([CPS2]). $L_\infty(\mathbb{R})$ together with the self-maps, D_a, T_b, and H $(a > 0, b \in \mathbb{R})$ is related to $L_\infty(\mathbb{R}_+^*)$ together with the self-maps, P^a, D_a, and M $(a > 0)$ via the isomorphism

$$L : L_\infty(\mathbb{R}) \to L_\infty(\mathbb{R}_+^*)$$

and the following identities:
(1) $LD_{\frac{1}{a}}L^{-1} = P^a$ for $a > 0$,
(2) $LT_bL^{-1} = D_{(\exp(b))^{-1}}$ for $b \in \mathbb{R}$,
(3) $LHL^{-1} = M$,
(4) $D_aH = HD_a$ and $P^aM = MP^a$ for $a > 0$,
(5) $\lim_{t\to\infty}(HT_b - T_bH)f(t) = 0$ for $f \in L_\infty(\mathbb{R})$ and $b \in \mathbb{R}$,
(6) $\lim_{t\to\infty}(MD_a - D_aM)f(t) = 0$ for $f \in L_\infty(\mathbb{R}_+^*)$ and $a > 0$.

Proposition 4.6 ([CPS2]). If a continuous functional $\tilde{\omega}$ on $L_\infty(\mathbb{R})$ is invariant under the Cesaro operator H, the shift operator T_a or the dilation operator D_a then $\tilde{\omega} \circ L^{-1}$ is a continuous functional on $L_\infty(\mathbb{R}_+^*)$ invariant under M, the dilation operator D_a or P^a respectively. Conversely, composition with L converts an M, D_a or P^a invariant continuous functional on $L_\infty(\mathbb{R}_+^*)$ into an H, T_a or D_a invariant continuous functional on $L_\infty(\mathbb{R})$.

We denote by $C_0(\mathbb{R})$ (respectively, $C_0(\mathbb{R}_+^*)$) the continuous functions on $\mathbb{R}$ (respectively, $\mathbb{R}_+^*$) vanishing at infinity.

The proof of the following theorem is similar to that of Theorem 4.3.

Theorem 4.7 ([CPS2]). There exists a state $\tilde{\omega}$ on $L_\infty(\mathbb{R})$ satisfying the following conditions:
(1) $\tilde{\omega}(C_0(\mathbb{R})) \equiv 0$.
(2) If f is real-valued in $L_\infty(\mathbb{R})$ then

$$ess \liminf_{t\to\infty} f(t) \leqslant \tilde{\omega}(f) \leqslant ess \limsup_{t\to\infty} f(t).$$

(3) If the essential support of f is compact then $\tilde{\omega}(f) = 0$.
(4) For all $a > 0$ and $c \in \mathbb{R}$

$$\tilde{\omega} = \tilde{\omega} \circ T_c = \tilde{\omega} \circ D_a = \tilde{\omega} \circ H.$$

Combining Theorem 4.7 and Proposition 4.6, we obtain

Corollary 4.8. There exists a state ω on $L_\infty(\mathbb{R}_+^*)$ satisfying the following conditions:
(1) $\omega(C_0(\mathbb{R}_+^*)) \equiv 0$.
(2) If f is real-valued in $L_\infty(\mathbb{R}_+^*)$ then

$$ess \liminf_{t\to\infty} f(t) \leqslant \omega(f) \leqslant ess \limsup_{t\to\infty} f(t).$$

(3) If the essential support of f is compact then $\omega(f) = 0$.
(4) For all $a, c > 0$

$$\omega = \omega \circ D_c = \omega \circ P^a = \omega \circ M.$$

The results given in Theorem 4.7 and Corollary 4.8 allow one to exercise an alternative approach to the theory of Dixmier [Dix66CR] and Connes-Dixmier traces [Co1]. Whereas Dixmier's original approach is based on the use of dilation invariant functionals, we replace the latter with Banach limits (= translation invariant functionals) and make use of the well-developed theory of almost convergent sequences.

We introduce the following notation

$$\begin{aligned} BL(\mathbb{R}) = \{&\text{the set of all states } \tilde\omega \text{ on } L_\infty(\mathbb{R}) \text{ satisfying} \\ &\text{conditions (1)–(3) of Theorem 4.7} \\ &\text{such that } \tilde\omega \circ T_c = \tilde\omega \text{ for every } c \in \mathbb{R}\}, \\ D(\mathbb{R}_+^*) = \{&\text{the set of all states } \omega \text{ on } L_\infty(\mathbb{R}_+^*) \text{ satisfying} \\ &\text{conditions (1)–(3) of Corollary 4.8} \\ &\text{such that } \omega \circ D_c = \omega \text{ for every } c \in \mathbb{R}_+^*\}, \\ BL(\mathbb{R}_+) = \{&\text{the set of all states } \tilde\omega \text{ on } L_\infty(\mathbb{R}_+) \text{ satisfying} \\ &\text{conditions (1)–(3) of Corollary 4.8} \\ &\text{such that } \tilde\omega \circ T_c = \tilde\omega \text{ for every } c \in \mathbb{R}_+\}. \end{aligned}$$

The following simple remark plays an important role in the sequel. If $\omega \in D(\mathbb{R}_+^*)$, then $\mathbb{L} := \omega \circ L$ belongs to $BL(\mathbb{R})$. If $\mathbb{L} \in BL(\mathbb{R}_+)$, then $\tilde\omega := \mathbb{L} \circ L^{-1}$ belongs to $D(\mathbb{R}_+^*)$. Finally, note that the isomorphism $L : L_\infty(\mathbb{R}) \to L_\infty(\mathbb{R}_+^*)$ sends the space $C_b(\mathbb{R})$ of all bounded continuous functions on $\mathbb{R}$ onto the space $C_b(\mathbb{R}_+)$ of all bounded continuous functions on $\mathbb{R}_+$. Thus, one can reformulate all the results from Propositions 4.5, 4.6, Theorem 4.7 and Corollary 4.8 for the spaces of continuous bounded functions on $\mathbb{R}$ and $\mathbb{R}_+$.

5 Concrete constructions of singular functionals\traces.

5.1 Dixmier traces

If ω is a state on ℓ_∞ (respectively, on $L_\infty(\mathbb{R})$, $L_\infty(\mathbb{R}_+^*)$), then we shall frequently denote its value on the element $\{x_i\}_{i=1}^\infty$ (respectively, $f \in L_\infty(\mathbb{R})$, $L_\infty(\mathbb{R}_+^*)$) by $\omega\text{-}\lim_{i\to\infty} x_i$ (respectively, $\omega\text{-}\lim_{t\to\infty} f(t)$). Recall that Theorems 4.3, 4.7 and Corollary 4.8 guarantee the existence of translation and/or dilation invariant states on ℓ_∞, $L_\infty(\mathbb{R})$, $L_\infty(\mathbb{R}_+^*)$. For simplicity, we explain the construction of Dixmier traces for the ideal of compact operators $(\mathcal{L}^{(1,\infty)}(\mathcal{H}), \|\cdot\|_{(1,\infty)})$ defined in Section 2.1. First, we recall Definition 1.6.

Definition 5.1. Let ω be a D_2-invariant state on ℓ_∞. *Dixmier trace* of $T \in \mathcal{L}_+^{(1,\infty)}(\mathcal{H})$ is a number

$$\tau_\omega(T) := \omega\text{-}\lim_{N\to\infty} \frac{1}{\log(1+N)} \sum_{n=1}^{N} \mu_n(T).$$

Remark. We have deliberately chosen ω to satisfy only the dilation invariance assumption in the proof below, even though Dixmier originally imposed on ω the assumption of dilation and translation invariance. We shall discuss differences below.

Definitions 1.6 and 1.8 extend to Marcinkiewicz spaces $\mathcal{L}^{(1,\infty)}(\mathcal{N},\tau)$ and further to Marcinkiewicz spaces $M(\psi)(\mathcal{N},\tau)$, where $\psi \in \Omega$ satisfies condition (7). More precisely, fix an arbitrary state ω on $L_\infty(\mathbb{R}_+^*)$ satisfying conditions (1)–(3) of Corollary 4.8 which is $D_{\frac{1}{2}}$-invariant. Setting

$$\tau_\omega(x) := \omega\text{-}\lim_{t\to\infty} a(x,t), \quad 0 \leqslant x \in M(\psi)(\mathcal{N},\tau) \tag{10}$$

and repeating a slightly modified argument (see the details in [DPSS, p. 51]) from the proof of Propositions 1.7 and 3.4, we obtain an additive homogeneous functional on $M(\psi)(\mathcal{N},\tau)_+$, which extends to a symmetric functional on $M(\psi)(\mathcal{N},\tau)$ by linearity. In the sequel, we refer to any functional τ_ω defined in (10), where $\omega \in D(\mathbb{R}_+^*)$ as a Dixmier trace.

Importantly, we note that the duality between the dilation invariant functionals on $L_\infty(\mathbb{R}_+^*)$ and translation invariant functionals on $L_\infty(\mathbb{R})$ allows an alternative definition of Dixmier traces. For simplicity, we consider this definition only for the space $\mathcal{L}^{(1,\infty)}(\mathcal{N},\tau)$, where $(\mathcal{N},\tau)$ is an arbitrary semifinite von Neumann algebra.

Let $\mathbb{L}$ (respectively, $\mathbb{L}$) belong to $BL(\mathbb{R}_+)$ (respectively, $BL(\mathbb{N})$). We set

$$\left.\begin{aligned} F_{\mathbb{L}}(T) &:= \mathbb{L}\text{-}\lim_{t\to\infty} \frac{1}{\log(1+e^t)} \int_0^{e^t} \mu_s(T)ds, \\ F_{\mathbb{L}}(T) &:= \mathbb{L}\text{-}\lim_{N\to\infty} \frac{1}{\log(1+e^N)} \sum_{n=1}^{[e^N]} \mu_n(T), \end{aligned}\right. \quad T \in \mathcal{L}^{(1,\infty)}(\mathcal{N},\tau)$$

where $[e^N]$ is the integral part of e^N. This simple observation is in fact crucial, since it allows to approach the theory of Dixmier traces via classical tools provided by the theory of Banach limits.

Theorem 5.2. [[CPS2, DPSSS1, DPSSS2, LSS]] For every semifinite von Neumann algebra $(\mathcal{N},\tau)$ and arbitrary states $\mathbb{L} \in BL(\mathbb{R}_+)$ and $\mathbb{L} \in BL(\mathbb{N})$, the functionals $F_{\mathbb{L}}$ and $F_{\mathbb{L}}$ are fully symmetric functionals on $\mathcal{L}^{(1,\infty)}(\mathcal{N},\tau)$.

The following result shows that the class of Dixmier traces on $\mathcal{L}^{(1,\infty)}(\mathcal{N},\tau)$ coincides with the sets of functionals $\{F_{\mathbb{L}} \ : \ \mathbb{L} \in BL(\mathbb{N})\}$ and $\{F_{\mathbb{L}} \ : \ \mathbb{L} \in BL(\mathbb{R}_+)\}$. The proof of the first equality below follows from the remarks at the end of the preceding section.

Theorem 5.3 ([LSS, Theorems 2.3,6.2]). For every semifinite von Neumann algebra $(\mathcal{N},\tau)$, we have

$$\{\tau_\omega \mid \omega \in D(\mathbb{R}_+^*)\} = \{F_{\mathbb{L}} \mid \mathbb{L} \in BL(\mathbb{R}_+)\} = \{F_{\mathbb{L}} \mid \mathbb{L} \in BL(\mathbb{N})\}.$$

The detailed study of the class of concave functions $\psi \in \Omega$ for which analogues of Theorems 5.2 and 5.3 hold for similarly defined classes of fully symmetric functionals on Marcinkiewicz spaces $M(\psi)(\mathcal{N},\tau)$ is contained in [DPSSS1, DPSSS2, LSS].

5.2 Connes-Dixmier traces

We have shown in the preceding subsection that with every state $\omega \in D(\mathbb{R}_+^*)$ (respectively, $\mathbb{L} \in BL(\mathbb{R}_+), BL(\mathbb{R})$, $\mathbb{L} \in BL(\mathbb{N})$) there exists an associated Dixmier trace τ_ω (respectively, a fully symmetric functional $F_{\mathbb{L}}$, $F_{\mathbb{L}}$). It is possible to isolate various subsets in the sets of states $D(\mathbb{R}_+^*)$, $BL(\mathbb{R}_+)$, $BL(\mathbb{R})$, $BL(\mathbb{N})$ and relate with them corresponding subsets of traces. For example, let us consider the sets of all H-invariant (respectively, M-invariant) states on $L_\infty(\mathbb{R})$ (respectively, $L_\infty(\mathbb{R}_+^*)$). It is easy to see that

$$\begin{aligned} &\{\omega \ : \ \omega \ \text{ is a } H\text{-invariant state on } L_\infty(\mathbb{R}) \text{ (resp. } \ell_\infty)\} \\ &\quad \subsetneqq \{\omega : \ \ \omega \in BL(\mathbb{R}) \ \ (\text{resp. } BL(\mathbb{N}))\} \end{aligned} \tag{11}$$

and

$$\{\omega : \ \ \omega \text{ is an } M\text{-invariant state on } L_\infty(\mathbb{R}_+^*)\} \subsetneqq \{\omega : \ \ \omega \in D(\mathbb{R}_+^*)\}. \tag{12}$$

Indeed, suppose that $0 \leqslant \omega \in \ell_\infty^*$ is such that $\omega(\mathbf{1}) = 1$ and $\omega(Hx) = \omega(x)$, for every $x \in \ell_\infty$. To prove $\omega(Tx) = \omega(x)$, $x \in \ell_\infty$, it is sufficient to show that $\omega(HTx) = \omega(Hx)$, $x \in \ell_\infty$. However, a straightforward calculation yields

$$\begin{aligned}(HTx)_N - (Hx)_N &= \frac{x_2 + \ldots + x_{N+1}}{N} - \frac{x_1 + \ldots + x_N}{N} \\ &= \frac{x_{N+1} - x_1}{N} \to 0, \quad \text{as} \quad N \to \infty.\end{aligned}$$

Similarly, it can be shown that for every $x \in L_\infty(\mathbb{R})$ and $b \in \mathbb{R}$ we have

$$\lim_{t\to\infty} (HT_b x)(t) - (Hx)(t) = 0$$

which establishes (11) and the inclusion (12) follows from (11) via Corollary 4.8. Alain Connes in [Co1] suggested to work with the set of states on $L_\infty(\mathbb{R}_+^*)$, which is larger then the set on the left hand side of (12). Namely, let us consider the following class of states on $L_\infty(\mathbb{R}_+^*)$

$$CD(\mathbb{R}_+^*) := \{\tilde{\omega} = \gamma \circ M \ : \ \gamma \text{ is an arbitrary singular state on } C_b[0,\infty)\}.$$

It is still easy to verify that $CD(\mathbb{R}_+^*) \subsetneqq D(\mathbb{R}_+^*)$, and then infer the proper inclusion $CBL(\mathbb{R}_+) \subsetneqq BL(\mathbb{R}_+)$, where

$$CBL(\mathbb{R}_+) := \{\mathbb{L} = \gamma \circ H \ : \ \gamma \text{ is an arbitrary singular state on } C_b(\mathbb{R}_+)\},$$

from Proposition 4.5. We refer to the subclass of Dixmier traces

$$\{\tau_\omega \ : \omega \in CD(\mathbb{R}_+^*)\}$$

as the class of Connes-Dixmier traces. The following theorem shows that an analogue of Theorem 5.3 also holds for the class of Connes-Dixmier traces.

Theorem 5.4 ([LSS, Theorems 5.6, 6.2]). For every semi-finite von Neumann algebra $(\mathcal{N}, \tau)$, we have

$$\{\tau_\omega : \ \ \omega \in CD(\mathbb{R}_+^*)\} = \{F_{\mathbb{L}} : \ \ \mathbb{L} \in CBL(\mathbb{R}_+)\}.$$

We complete this subsection with the remark that there is another natural subclass of Dixmier traces which is associated with the subset of states on $L_\infty(\mathbb{R}_+^*)$ appearing in Corollary 4.8

$$\begin{aligned}\{\omega \in L_\infty(\mathbb{R}_+^*)^* \ : \ &\omega \text{ is an } M\text{-invariant} \\ &\text{and } P^a\text{-invariant state on } L_\infty(\mathbb{R}_+^*),\ a > 0\},\end{aligned}$$

or equivalently, with the set

$$\begin{aligned}\{\mathbb{L} \in L_\infty(\mathbb{R})^* \ : \ &\mathbb{L} \text{ is an } H\text{-invariant} \\ &\text{and } D_a\text{-invariant state on } L_\infty(\mathbb{R}),\ a > 0\}.\end{aligned}$$

The class of Dixmier traces associated with the latter set is further referred to as the class of maximally invariant Dixmier traces. Clearly, the latter class is contained in the class of Connes-Dixmier traces. Maximally invariant Dixmier traces are used in Sections 8 and 9.

6 Class of measurable elements.

In this section, we briefly review the notion of measurable operators introduced by A.Connes [Co1].

Definition 6.1. $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ is called *(Dixmier)-measurable* if $\tau_\omega(T)$ does not depend on the choice of $\omega \in D(\mathbb{R}_+^*)$.

Definition 6.2 ([Co1]). $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ is called *(Connes-Dixmier)-measurable* if $\tau_\omega(T)$ does not depend on the choice of $\omega \in CD(\mathbb{R}_+^*)$.

Remark 6.3. (i). It is obvious that the sets of measurable operators defined above are linear spaces, which are, in fact, closed subspaces of $\mathcal{L}^{(1,\infty)}(\mathcal{H})$. However, these subspaces are not order ideals, in other words, the fact that a self-adjoint operator A is measurable does not necessarily imply that A_+ and A_- are measurable operators. Example. Take a positive non-measurable diagonal operator $A = \mathrm{diag}\{a_1, a_2, a_3, \ldots\}$ from $\mathcal{L}^{(1,\infty)}(\mathcal{H})$. Define a diagonal operator B by $B = \mathrm{diag}\{a_1, -a_1, a_2, -a_2, \ldots\}$. Evidently, B is measurable, moreover $\tau_\omega(B) = 0$ for all ω. However, the positive and negative parts of B are not measurable.

(ii). Definitions of Dixmier and Connes-Dixmier measurable operators naturally extend [LSS] to Marcinkiewicz spaces $\mathcal{L}^{(1,\infty)}(\mathcal{N}, \tau)$, where $(\mathcal{N}, \tau)$ is an arbitrary semifinite von Neumann algebra, and further, to operator Marcinkiewicz spaces $M(\psi)(\mathcal{N}, \tau)$, for all ψ satisfying condition (5).

It is obvious, that every Dixmier-measurable operator is also Connes-Dixmier measurable. Our objective in the present section is to describe the classes of positive Connes-Dixmier measurable operators and positive Dixmier-measurable operators and to show that these two classes actually coincide. To this end, we will need two auxiliary results.

Theorem 6.4. [Hardy, section 6.8] Let $b(t)$ be a positive piecewise differentiable function such that $tb'(t) > -H$ for some $H > 0$ and all $t > C$, where C is a constant. Then

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t b(s)\,ds = A \quad \text{for some} \quad A \geqslant 0 \quad \text{if and only if} \quad \lim_{t\to\infty} b(t) = A.$$

Imitating the Lorentz definition of almost convergent sequences, a positive function $f \in C_b[0,\infty)$ is said to be *almost convergent* if all states from $BL(\mathbb{R}_+)$ take the same value on this function.

Theorem 6.5. [LSS, Theorem 3.3] If a function $f \in C_b[0,\infty)$ is almost convergent to a number A then, the following limit

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(s)\,ds$$

exists and is equal to A.

Proof. Suppose the result is false. Then there exists a constant $c \neq A$ such that $(Hf)(t_n) \to c$ for some sequence $t_n \uparrow \infty$. Take the unit ball B of $C_b([0,\infty))^*$ and consider the sequence of functionals $\sigma_{t_n}(x) = x(t_n)$, $n \geqslant 1$ from B. Since B is weak*-compact, this sequence has a limit point $V \in B$. It is easy to see that $V \geqslant 0$, $V(1) = 1$, $V(p) = \lim_{n\to\infty} p(t_n) = 0$ for every $p \in C_0[0,\infty)$ and also that $V(H(f)) = \lim_{n\to\infty} H(f)(t_n) = c$. Define the functional L on $L_\infty(\mathbb{R}_+)$ by setting $L(x) := V(H(x))$. It is easy to verify that L is a state from $BL(\mathbb{R}_+)$ and that $V(f) = c \neq A$. Thus, the supposition that the result does not hold is false. $\square$

The following theorem is the main result of this section.

Theorem 6.6. [LSS] A positive operator T from $\mathcal{L}^{(1,\infty)}(\mathcal{N},\tau)$ is Dixmier-measurable if and only if the limit

$$\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\,ds$$

exists.

Proof. The "if" part of the assertion is trivial. Now, fix an operator $T \in \mathcal{L}_+^{(1,\infty)}(\mathcal{N},\tau)$ such that for $g(t) := \frac{1}{\log(1+t)} \int_0^t \mu_s(T) ds$, we have $\tau_\omega(T) = \omega\text{-}\lim_{t\to\infty} g(t) = A \geqslant 0$ for every $\omega \in CD(\mathbb{R}_+^*)$. It follows from the remarks made in Section 5.2 that for all $\mathbb{L} \in BL(\mathbb{R}_+)$, we have $Tr_{\mathbb{L}}(T) := \mathbb{L}\text{-}\lim_{\lambda\to\infty} g(e^\lambda) = A$, and therefore, by Theorem 6.5, we obtain

$$\lim_{u\to\infty} \frac{1}{u} \int_0^u \left(\frac{1}{\log(1+e^\lambda)} \int_0^{e^\lambda} \mu_s(T)\,ds \right) d\lambda = A.$$

Setting,

$$b(\lambda) := \frac{1}{\log(1+e^\lambda)} \int_0^{e^\lambda} \mu_s(T)\,ds, \quad \lambda > 0$$

we have

$$\begin{aligned} \lambda b'(\lambda) &\geqslant \lambda \frac{d}{d\lambda} \left(\frac{1}{\log(1+e^\lambda)} \right) \int_0^{e^\lambda} \mu_s(T)\,ds \\ &= -\frac{\lambda e^\lambda}{(1+e^\lambda)\log^2(1+e^\lambda)} \int_0^{e^\lambda} \mu_s(T)\,ds \\ &\geqslant -\frac{\lambda}{\log(1+e^\lambda)} \cdot \frac{1}{\log(1+e^\lambda)} \int_0^{e^\lambda} \mu_s(T)\,ds \geqslant -||T||_{(1,\infty)}. \end{aligned}$$

Applying Theorem 6.4, we now infer that $\lim_{\lambda\to\infty} b(\lambda) = A$, and therefore $\lim_{t\to\infty} g(t) = A$. $\square$

We shall now show that a similar argument as in the proof above yields a stronger result.

Theorem 6.7 ([LSS]). A positive operator T from $\mathcal{L}^{(1,\infty)}(\mathcal{N},\tau)$ is Connes-Dixmier-measurable if and only if the limit

$$\lim_{t\to\infty}\frac{1}{\log(1+t)}\int_0^t \mu_s(T)\,ds$$

exists.

Proof. We need only to show the "only if" part. We shall use the notations $g(\cdot)$ and $b(\cdot)$ introduced in the proof of Theorem 6.6. Suppose that $T\in\mathcal{L}_+^{(1,\infty)}(\mathcal{N},\tau)$ satisfies the equality $A=\gamma\circ M(g)=\gamma\circ LHL^{-1}(g)$ for every state $\gamma\in C_b^*[0,\infty)$ vanishing on $C_0(0,\infty)$.

Note that if γ is dilation invariant state, then $\gamma\circ L$ is a translation invariant state. This remark (and the fact that $L: L_\infty(\mathbb{R})\to L_\infty(\mathbb{R}_+^*)$ is an isomorphism) show that $HL^{-1}(g)=Hb$ is almost convergent. Applying Theorem 6.5 to the function Hb, we see that the limit $\lim_{t\to\infty}HH(b)(t)$ exists. Assume, for a moment, that we have already verified the assumption of Theorem 6.4 for the function Hb. Then, we infer from that theorem that the limit $\lim_{t\to\infty}H(b)(t)$ also exists, and repeating the application of the same theorem (as in the proof of Theorem 6.6), we conclude that there exists the limit $\lim_{t\to\infty}b(t)$, and hence the limit $\lim_{\lambda\to\infty}g(\lambda)$.

It remains to verify the assumption of Theorem 6.4 for Hb. We have for all $\lambda\geqslant 1$

$$\lambda(Hb)'(\lambda)=\frac{\lambda b(\lambda)-\int_0^\lambda b(s)ds}{\lambda}\geqslant -\|b\|_\infty\geqslant -\|T\|_{(1,\infty)}.$$

□

Corollary 6.8. The set of all positive Dixmier measurable operators and the set of all positive Connes-Dixmier measurable operators coincide.

The following questions are open:

(i). Do the spaces of Dixmier measurable and Connes-Dixmier measurable operators coincide?

(ii). What is the description of the set of all (positive) operators, which are measurable with respect to the set of all maximally invariant Dixmier traces?

(iii). What is the description of the set of all (positive) operators measurable with respect to the set of all symmetric functionals?

For results extending Theorems 6.6 and 6.7 to Marcinkiewicz spaces $M(\psi)(\mathcal{N},\tau)$, with $\psi\in\Omega$ satisfying condition (6), we refer the reader to [LSS].

7 Norming properties of Dixmier and Connes-Dixmier functionals

A reader may have an impression that Dixmier and Connes-Dixmier traces form a very "thin" subset of the unit sphere of the dual space. Such an impression is wrong as established by Theorems 7.3 and 7.4 below. We shall need the following theorem of Sucheston.

Theorem 7.1 ([Su67AMM]). For $x \in \ell_\infty$

$$\sup_{\mathbb{L} \in BL(\mathbb{N})} \mathbb{L}(x) = \lim_{n\to\infty} \left(\sup_m \frac{1}{n} \sum_{j=1}^{n} x_{m+j} \right).$$

The following proposition easily follows from its "commutative" counterpart, which, in its turn, can be obtained from [KPS].

Proposition 7.2. Let $T \in \mathcal{L}^{(1,\infty)}(\mathcal{N},\tau)$. The distance from T to the subspace $\mathcal{L}_0^{(1,\infty)}(\mathcal{N},\tau)$ in the norm $||\cdot||_{(1,\infty)}$ is equal to

$$\rho(T) := \limsup_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\, ds.$$

Recall, that in the special case $\mathcal{L}^{(1,\infty)}(\mathcal{N},\tau) = \mathcal{L}^{(1,\infty)}(\mathcal{H})$, the space $\mathcal{L}_0^{1,\infty}(\mathcal{H})$ is the closed linear span in $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ of the set of all finite-dimensional operators.

Theorem 7.3. [LSS] Let $T \in \mathcal{L}^{(1,\infty)}(\mathcal{N},\tau)$. The distance from T to the subspace $\mathcal{L}_0^{(1,\infty)}(\mathcal{N},\tau)$ in the norm $||\cdot||_{(1,\infty)}$ is equal to $\sup\{\tau_\omega(|T|) \, : \, \omega \in D(\mathbb{R}_+^*)\}$.

Proof. It is sufficient to consider the case $T \geqslant 0$. We note first that

$$\begin{aligned}\sup\{\tau_\omega(T) \, : \, \omega \in D(\mathbb{R}_+^*)\} &= \sup\{Tr_{\mathbb{L}}(T) \, : \, \mathbb{L} \in BL(\mathbb{R}_+)\} \\ &= \sup\{Tr_{\mathbb{L}}(T) \, : \, \mathbb{L} \in BL(\mathbb{N})\}.\end{aligned}$$

It is clear that

$$\begin{aligned}q(T) &:= \sup_{\mathbb{L} \in BL(\mathbb{N})} \mathrm{Tr}_{\mathbb{L}}(T) \\ &\leqslant \limsup_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)\, ds \ (= \rho(T)).\end{aligned}$$

We have to prove the reverse inequality $q(T) \geqslant \rho(T)$. By Sucheston's theorem 7.1, it is enough to prove that $\forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall n \geqslant N \, \exists m \in \mathbb{N}$ such that

$$\frac{1}{n} \sum_{j=1}^{n} \frac{1}{m+j} \int\limits_0^{e^{m+j}} \mu_s(T)\, ds \geqslant \rho(T) - \epsilon.$$

For this purpose, it is enough to put $N = 1$ and to take m such that $\frac{m}{m+n} > \frac{\rho(T)-\epsilon}{\rho(T)-\epsilon/2}$ and

$$\frac{1}{\log(1+e^m)} \int_0^{e^m} \mu_s(T)\, ds > \rho(T) - \epsilon/2. \tag{13}$$

Then

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^{n} \frac{1}{m+j} \int\limits_0^{e^{m+j}} \mu_s(T)\, ds &\geqslant \frac{1}{n} \sum_{j=1}^{n} \frac{1}{m+n} \int\limits_0^{e^m} \mu_s(T)\, ds \\ &= \frac{m}{m+n} \cdot \frac{1}{m} \int\limits_0^{e^m} \mu_s(T)\, ds > \rho(T) - \epsilon. \end{aligned}$$

To verify that selection of m satisfying (13) is feasible, first we locate a sequence $1 \leqslant t_1 < t_2 \ldots \uparrow \infty$, such that

$$\lim_{k\to\infty} \frac{1}{\log(1+t_k)} \int_0^{t_k} \mu_s(T) ds > \rho(T) - \epsilon/4, \quad k \geqslant 1, \tag{14}$$

(this may be done due to Proposition 7.2). For every k, we define $m_k \in \mathbb{N}$, so that $e^{m_k - 1} \leqslant t_k \leqslant e^{m_k}$. Then

$$\begin{aligned} \frac{1}{\log(1+t_k)} \int_0^{t_k} \mu_s(T) ds &\leqslant \frac{1}{\log(1+e^{m_k-1})} \int_0^{e^{m_k}} \mu_s(T) ds \\ &= \frac{\log(1+e^{m_k})}{\log(1+e^{m_k-1})} \cdot \frac{1}{\log(1+e^{m_k})} \int_0^{e^{m_k}} \mu_s(T) ds. \end{aligned}$$

Since $\frac{\log(1+e^{m_k})}{\log(1+e^{m_k-1})} \to 1$, we see that (13) follows from (14). $\square$

A natural question is whether the norming property remains true for the class of Connes-Dixmier traces, as answered below.

Theorem 7.4. [LSS] Let $T \in \mathcal{L}^{(1,\infty)}(\mathcal{N}, \tau)$. The distance from T to the subspace $\mathcal{L}_0^{(1,\infty)}(\mathcal{N}, \tau)$ in the norm $||\cdot||_{(1,\infty)}$ is equivalent to $\sup_\omega \tau_\omega(T)$, where the supremum is taken over all singular states $\omega = \gamma \circ M$, where γ is a singular state on $C_b[0, \infty)$.

As in the preceding section, the results given in Theorems 7.3 and 7.4 admit an extension to Marcinkiewicz spaces $M(\psi)(\mathcal{N},\tau)$ with $\psi \in \Omega$ satisfying condition (6).

We finish this section with the comment that it is not clear yet, whether the difference in the results of Theorems 7.3 and 7.4 signify that the set of all Dixmier traces is different from the set of all Connes-Dixmier traces.

8 The Dixmier trace and asymptotics of zeta functions.

8.1 Background

The key role of the Dixmier trace in noncommutative geometry was discovered by Connes around 1990, [Co1]. Since then, it has become a cornerstone of noncommutative geometry. Notably, the Dixmier trace is used to define dimension, integration and has been used in physical applications, along with heat kernel type expansions, to define 'spectral actions' for noncommutative field theories, [Co1, Co2, Co3]. The Dixmier trace (or more precisely Dixmier *traces*) are a family of non-normal traces on the bounded operators on a separable Hilbert space $\mathcal{H}$ measuring the logarithmic divergence of the trace of a compact operator. There is an ideal of compact operators denoted $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ consisting precisely of those operators with finite Dixmier trace. Following [Co1] connections between Dixmier traces, zeta functions and heat kernel asymptotics were systematically studied in [CPS2].

Briefly, for an important special case, we show that for a positive compact operator T, the existence of the limit

$$\lim_{r\to\infty} \frac{1}{r}\mathrm{Trace}(T^{p+\frac{1}{r}})$$

implies that the operator T lies in an ideal $\mathcal{Z}_p$. The ideal $\mathcal{Z}_1$ is $\mathcal{L}^{(1,\infty)}(\mathcal{H})$, while for $p>1$ $\mathcal{Z}_p$ is strictly larger than $\mathcal{L}^{(p,\infty)}(\mathcal{H})$. (It is in fact precisely what is termed, in [LT2, Section 1.d], the p-convexification of $\mathcal{L}^{(1,\infty)}(\mathcal{H})$.) We then show that if $\lim_{r\to\infty} \frac{1}{r}\mathrm{Trace}(T^{p+\frac{1}{r}})$ exists it equals $p\mathrm{Trace}_\omega(T^p)$ for any state ω generating a Dixmier trace, Trace_ω. Thus we show that the asymptotics of the zeta function singles out the class of compact operators which have a finite Dixmier trace.

In fact the analogues of these statements are true for compact operators T in a semifinite von Neumann algebra $\mathcal{N}$ with faithful, normal, semifinite trace τ for which there are corresponding ideals $\mathcal{Z}_p(\mathcal{N})$ and $\mathcal{L}^{(p,\infty)}(\mathcal{N},\tau)$. Readers unfamiliar with ideal theory in such general algebras may restrict attention to the standard case of bounded operators on an infinite dimensional separable Hilbert space with its usual trace (denoted by 'Trace' here). Our reason for striving for generality stems from the emergence recently of applications of the semifinite von Neumann theory [BCPRSW, BeF, CM, CPRS2, CPRS3].

Our results follow primarily from (strengthened versions of) deep facts from [CPS2] and recent advances in the study of singular traces, some of which seem not to be well known. We also work in this paper with general Marcinkiewicz spaces and general 'Dixmier traces' as these spaces are already known to arise in the study of pseudodifferential operators [Ni].

8.2 Summary of the main results

We need some notation in order to present the results. We remark that in a semifinite von Neumann algebra $\mathcal{N}$ with faithful normal semifinite trace τ the $\tau-$compact operators are generated by projections P with $\tau(P) < \infty$. Suppose that T is a $\tau-$compact positive operator in $\mathcal{N}$. (If one has a semifinite spectral triple determined by an unbounded self adjoint operator D then one should think of T as $|D|^{-1}$ or $(1+D^2)^{-1/2}$.) For a given τ let τ_ω denote a Dixmier trace corresponding to an element $\omega \in \ell^*_\infty(\mathbb{N})$ or $\ell^*_\infty(\mathbb{R}_+)$. We remark that ω must satisfy some invariance properties which we will explain in detail in Section 3. By the zeta function of T we mean $\zeta(s) = \tau(T^s)$.

Consider the following hypothesis:
(*) Under the assumption that $\tau(T^s)$ exists for all $s > p$ suppose that $\lim_{r\to\infty} \frac{1}{r}\zeta(p+\frac{1}{r})$ exists.
It is then natural to ask, in view of [CPS2, Co1], the following question:
A. If hypothesis (*) holds then does it follow that $T \in \mathcal{L}^{(p,\infty)}$?
We prove that the answer to Question A is yes if $p = 1$ and no if $p > 1$. This leads to a second question:
B. For $p > 1$ what constraint does hypothesis (*) place on the singular values of T?

We remark that in contrast to the situation with the classical Schatten ideals it is not true that if $T \in \mathcal{L}^{(1,\infty)}$ then $T^{1/p} \in \mathcal{L}^{(p,\infty)}$. In fact there is a strictly smaller ideal inside $\mathcal{L}^{(1,\infty)}$ characterized by this property. We prove correspondingly that there is an ideal $\mathcal{Z}_p$ strictly larger than $\mathcal{L}^{(p,\infty)}$ with the property that if $T^{1/p} \in \mathcal{Z}_p$ then $T \in \mathcal{L}^{(1,\infty)}$. We also prove that if hypothesis (*) holds then $T \in \mathcal{Z}_p$.

This leads to the further question:
C. If hypothesis (*) holds how does the limit relate to the Dixmier trace of T^p?
In fact we show that for a certain class of Dixmier traces τ_ω

$$\lim_{r\to\infty} \frac{1}{r}\zeta(p+\frac{1}{r}) = p \lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)^p ds := p\tau_\omega(T^p).$$

Our methods then lead us to prove some stronger versions of several results in [CPS2]. In that paper we were forced to consider a subset of the set of all Dixmier traces determined by requiring invariance under a certain transformation group. In the new approach of this letter we can relax many of these invariance conditions.

Then, in view of [Co1, p 563] and the relationship of the zeta function to the heat kernel, it is natural to ask what hypothesis (*) implies concerning the small time asymptotics of the trace of the heat semigroup. (We note that hypothesis (*) implies that the heat semigroup $e^{-tT^{-2}}$, defined using the functional calculus, is trace class for all $t > 0$.) This matter is resolved in Theorem 9.1. Let

$F(\lambda) = \lambda^{-1}\tau(e^{-\lambda^{-2}T^{-2}})$, then under hypothesis (*) for $p = 1$ this function is bounded on $(0, \infty)$ and Theorem 9.1 says that for certain $\omega \in L_\infty((0,\infty))^*$, $\omega(F)$ is a multiple of the Dixmier trace $\tau_\omega(T)$.

Conversely we know that if $\lambda^{-1}\tau(e^{-\lambda^{-2}T^{-2}})$ has an asymptotic expansion in λ as $\lambda \to \infty$ then the leading term in this expansion precisely determines the first singularity of $\tau(T^s)$ as $\mathrm{Re}(s)$ decreases. In this case, using the results described above, we find that $T \in \mathcal{Z}_1$ and the residue of the zeta function is equal to the Dixmier trace of T.

Finally, in Section 6, we revisit a question raised in [CGS]. Namely, for T in some general ideal $\mathcal{I}$ (in the $\tau-$compact operators), which admits a Dixmier trace τ_ω, what are the minimal conditions on an algebra $\mathcal{A}$ such that the functional $a \to \tau_\omega(aT)$ on $\mathcal{A}$ is actually a trace? This question is important in the manifold reconstruction theorem of [Co1]. We find that the methods of this paper enable us to substantially generalize [CGS] (who answer the question only for $\mathcal{I} = \mathcal{L}^{(p,\infty)}$). We find that, for the same minimal conditions as in [CGS], there is a very large class of Marcinkiewicz ideals $\mathcal{I}$ including $\mathcal{I} = \mathcal{Z}_p$ for which $a \to \tau_\omega(aT)$ is a trace.

8.3 Preliminaries

In this subsection we generalize and strengthen some results from [CPS2].

Lemma 8.1. *For every $\psi \in \Omega$ satisfying* (7) *and every $1 > \alpha > 0$, there is $C = C(\alpha)$ such that $\psi(t) < Ct^\alpha$, $t > 0$.*

Recall that for any τ-measurable operator T, the distribution function of T is defined by

$$\lambda_t(T) := \tau(\chi_{(t,\infty)}(|T|)), \quad t > 0,$$

where $\chi_{(t,\infty)}(|T|)$ is the spectral projection of $|T|$ corresponding to the interval (t, ∞) (see [FK]). By Proposition 2.2 of [FK],

$$\mu_s(T) = \inf\{t \geqslant 0 \ : \ \lambda_t(T) \leqslant s\}.$$

We infer that for any τ-measurable operator T, the distribution function $\lambda_{(\cdot)}(T)$ coincides with the (classical) distribution function of $\mu_{(\cdot)}(T)$. From this formula and the fact that λ is right-continuous, we can easily see that for $t > 0$, $s > 0$

$$s \geqslant \lambda_t \Longleftrightarrow \mu_s \leqslant t.$$

Or equivalently,

$$s < \lambda_t \Longleftrightarrow \mu_s > t.$$

Using Remark 3.3 of [FK] this implies that:

$$\int_0^{\lambda_t} \mu_s(T) ds = \int_{[0,\lambda_t)} \mu_s(T) ds = \tau(|T|\chi_{(t,\infty)}(|T|)), \quad t > 0. \tag{15}$$

Lemma 8.2. *For $T \in M(\psi)$ $T \geqslant 0$ and any $\beta > 1$ there is a $C = C(\beta)$ such that $\lambda_{1/t}(T) < Ct^{\beta}$ for every $t > 0$.*

Remark. Since $\beta > 1$ could be arbitrary, it is obvious that the constant C could be replaced by 1 if t is sufficiently large.

In the sequel we will suppose that ψ possesses the following property

$$A(\beta) = \sup_{t>0} \frac{\psi(t^{\beta})}{\psi(t)} \to 1, \text{ if } \beta \downarrow 1. \tag{16}$$

Observe that if $\psi(t) = \log(1+t)^{\gamma}$, $\gamma > 0$, then condition (16) is satisfied.

Proposition 8.3. (cf. [CPS2, Proposition 2.4]) *For $T \in \mathcal{M}(\psi)$ positive let ω be D_2 and P^{α}-invariant, $\alpha > 1$ state on $L^{\infty}(\mathbb{R}_+^*)$. Then*

$$\tau_{\omega}(T) = \omega - \lim_{t\to\infty} \frac{1}{\psi(t))} \int_0^t \mu_s(T)ds = \omega - \lim_{t\to\infty} \frac{1}{\psi(t)} \tau(T\chi_{(\frac{1}{t},\infty)}(T))$$

and if one of the $\omega-$limits is a true limit then so is the other.

8.4 An alternative description of $\mathcal{L}^{(1,\infty)}$.

The zeta function of a positive compact operator T is given by $\zeta(s) = \tau(T^s)$ for real positive s on the assumption that there exists some s_0 for which the trace is finite. Note that it is then true that $\tau(T^s) < \infty$ for all $s > s_0$. In this subsection we will always assume $\tau(T^s) < \infty$ for all $s > 1$ and we are interested in the asymptotic behavior of $\zeta(s)$ as $s \to 1$.

Let us define the space

$$\mathcal{Z}_1 = \{T \in \mathcal{N} : \ \|T\|_{\mathcal{Z}_1} = \limsup_{p\downarrow 1} (p-1)\tau(|T|^p) < \infty\}.$$

Since we also have the other equivalent definition

$$\|T\|_{\mathcal{Z}_1} = \limsup_{p\downarrow 1}(p-1)(\int_0^{\infty} \mu_t(|T|)^p dt)^{1/p} = \limsup_{p\downarrow 1}(p-1)\|T\|_{L_p}$$

(recall that we use the notation L_p for the Schatten ideals in $(\mathcal{N},\tau)$) the ordinary properties of the semi-norm for $\|\cdot\|_{\mathcal{Z}_1}$ are immediate.

Theorem 8.4. (i) *Let $T \geqslant 0$, $T \in \mathcal{N}$ and $\limsup_{s\to 0} s\tau(T^{1+s}) = C < \infty$, then*

$$\limsup_{u\to\infty} \frac{1}{\ln u} \int_0^u \mu_t(T)dt \leqslant Ce.$$

(ii) *The spaces $\mathcal{Z}_1$ and $\mathcal{L}^{1,\infty}$ coincide. Moreover, if $\mathcal{N}$ is a type I factor with the standard trace, or else $\mathcal{N}$ is semifinite and the trace is non-atomic then denoting by $\mathcal{L}_0^{1,\infty}$ the closure of $L_1(\mathcal{N},\tau)$ in $\mathcal{L}^{1,\infty}$, we have for any $T \in \mathcal{C}_1$*

$$\mathrm{dist}_{\mathcal{L}^{1,\infty}}(T, \mathcal{L}_0^{1,\infty}) = \limsup_{u\to\infty} \frac{1}{\ln u} \int_0^u \mu_t(T)dt \leqslant e\|T\|_{\mathcal{Z}_1}$$

and $\|T\|_{\mathcal{Z}_1} \leqslant \|T\|_{1,\infty}$.

Proof. (i) By assumption for every $\epsilon > 0$ there is an $s_0 > 0$ such that for all $s \in [0, s_0]$

$$s\int_0^\infty \mu_t(T)^{1+s}dt \leqslant C + \epsilon. \tag{17}$$

Then, for $u \geqslant 1$ according to Hölder's inequality and (17) we have

$$\int_0^u \mu_t(T)dt \leqslant \left(\int_0^u \mu_t(T)^{1+s}dt\right)^{\frac{1}{1+s}} \left(\int_0^u 1^{\frac{1+s}{s}}dt\right)^{\frac{s}{1+s}} \leqslant$$

$$\left(\frac{s}{s}\int_0^\infty \mu_t(T)^{1+s}dt\right)^{\frac{1}{1+s}} u^{\frac{s}{1+s}} \leqslant ((C+\epsilon)/s)^{\frac{1}{1+s}} u^{\frac{s}{1+s}} \leqslant (C+\epsilon)\frac{1}{s}u^s.$$

Set $u_0 = e^{1/s_0}$ and for $u > u_0$ set $s = 1/\ln u (< s_0)$. Then $u = e^{\ln u}$ and by the previous inequality

$$\int_0^u \mu_t(T)dt \leqslant (C+\epsilon)\frac{1}{s}u^s = (C+\epsilon)\frac{e^{\ln u \frac{1}{\ln u}}}{\frac{1}{\ln u}} = (C+\epsilon)e\ln u.$$

That is we have the inequality

$$\frac{1}{\ln u}\int_0^u \mu_t(T)dt \leqslant (C+\epsilon)e \text{ for } u > u_0.$$

Since

$$\|T\|_{\mathcal{L}^{1,\infty}} = \sup_{1\leqslant u\leqslant\infty} \frac{1}{\ln(1+u)}\int_0^u \mu_t(T)dt$$

we conclude that $T \in \mathcal{L}^{1,\infty}$. Moreover, since $\epsilon > 0$ is arbitrary

$$\limsup_{u\to\infty} \frac{1}{\ln u}\int_0^u \mu_t(T)dt \leqslant eC.$$

Hence (i) and the embedding $\mathcal{Z}_1 \subset \mathcal{L}^{1,\infty}$ are established.

To complete the proof of $Z_1 = \mathcal{L}^{(1,\infty)}$, let us take an arbitrary $T \in \mathcal{L}^{1,\infty}$ and note that by the definition of the norm in the Marcinkiewicz space $\mathcal{L}^{1,\infty}$ we have $x \prec\prec \|T\|_{1,\infty}/(1+t)$. Since the spaces $L_p(\mathcal{N},\tau)$, $1 \leqslant p \leqslant \infty$, are fully symmetric operator spaces we have

$$\|T\|_p \leqslant \|T\|_{1,\infty}\|1/(1+t)\|_p,\ p > 1.$$

Taking the p-th power we get

$$\int_0^\infty \mu_t(T)^p\,dt \leqslant \|T\|_{1,\infty}^p \int_0^\infty 1/(1+t)^p dt = \|T\|_{1,\infty}^p \frac{1}{p-1}.$$

If now $p \downarrow 1$ we conclude that

$$\|T\|_{\mathcal{Z}_1} = \limsup_{p\downarrow 1}(p-1)\int_0^\infty \mu_t(T)^p\,dt \leqslant \|T\|_{1,\infty}.$$

Hence, $\mathcal{L}^{1,\infty} \subset \mathcal{Z}_1$. Due to the first part of the proof we infer that the spaces $\mathcal{Z}_1$ and $\mathcal{L}^{1,\infty}$ are coincident.

We omit the proof of the distance formula in (ii). □

Corollary 8.5. *Let $T \in \mathcal{N}$ be τ-compact and positive with $\tau(T^s) < \infty$ for all $s > 1$. If $\lim_{r\to\infty} \frac{1}{r}\tau(T^{1+\frac{1}{r}})$ exists then $T \in \mathcal{L}^{(1,\infty)}$.*

8.5 The case $p > 1$.

Our approach above to the study of $\mathcal{Z}_1$ allows us to generalize immediately. Let us define a class of spaces $\mathcal{Z}_q$, $q \geqslant 1$ by:

$$\mathcal{Z}_q = \{T \in \mathcal{N}_+ :\ \|T\|_{\mathcal{Z}_q} = \limsup_{p\downarrow q}((p-q)\tau(T^p))^{1/p} < \infty\}.$$

Setting $r = 1 + \frac{p-q}{q} = \frac{p}{q}$, we have

$$\begin{aligned}\|T\|_{\mathcal{Z}_q} &= \limsup_{p\downarrow q}((p-q)\tau(T^{q(1+(p-q)/q)}))^{1/p}\\ &= (q\limsup_{p\downarrow q}(p-q)/q\tau((T^q)^{(1+(p-q)/q)}))^{1/p}\\ &= q^{1/q}(\limsup_{r\downarrow 1}((r-1)\tau((T^q)^r))^{1/(qr)} = (q\|T^q\|_{\mathcal{Z}_1})^{1/q}.\end{aligned}$$

Now it is clear that $T \in \mathcal{Z}_q$ if and only if $T^q \in \mathcal{Z}_1$ and $\|T\|_{\mathcal{Z}_q} = (q\|T^q\|_{\mathcal{Z}_1})^{1/q}$.

We now state a few consequences of Theorem 8.4. The classical p-convexification procedure for an arbitrary Banach lattice X is described in [LT2, Section 1.d] and is sometimes termed power norm transformation. It is simply a direct generalization of the procedure of defining L_p-spaces from an L_1-space.

The proof of the first corollary below is immediate.

Corollary 8.6. (i) *There is a more convenient equivalent formula for the semi-norm $\|\cdot\|_{\mathcal{Z}_q}$ namely*

$$\|T\|_{\mathcal{Z}_q}^+ = \|T^q\|_{\mathcal{Z}_1}^{1/q},\ q \geqslant 1.$$

(ii) *The space* $\mathcal{Z}_q$ *coincides as a set with the* q*-convexification of the operator space* $\mathcal{L}^{1,\infty}$ *:*

$$\mathcal{L}_q^{1,\infty} = \{T \in \mathcal{N}_+ : \ \|T\|_{1,\infty}^q = \sup_{1<u<\infty} \left(\frac{\int_0^u \mu_t(T)^q dt}{\log(1+u)} \right)^{1/q} < \infty\}.$$

If $\mathcal{N}$ *is a type I factor with the standard trace, or else* $\mathcal{N}$ *is semifinite and the trace is non-atomic then the semi-norms* $\|\cdot\|_{\mathcal{Z}_q}$ *and* $\mathrm{dist}_{\mathcal{L}_q^{1,\infty}}(\cdot, \mathcal{L}_{q,0}^{1,\infty})$ *are equivalent. Here,* $\mathcal{L}_{q,0}^{1,\infty}$ *is the closure of* $L_1(\mathcal{N},\tau)$ *in* $\mathcal{L}_q^{1,\infty}$*.*

Corollary 8.7. *(i) An element* $T \in \mathcal{Z}_p$*,* $p \geqslant 1$*, iff* $T^p \in \mathcal{L}^{1,\infty}$*. Moreover*

$$\frac{1}{r}\int_0^\infty \mu_t(T)^{p+1/r} dt = \frac{1}{r}\tau(T^{p+1/r}) = p\frac{1}{pr}\tau(T^{p(1+1/pr)}). \tag{18}$$

and for $r > 0$ *the expression in (18) belongs to* $L^\infty(\mathbb{R}_+^*)$*.*
(ii) If $T \in \mathcal{L}^{p,\infty}$ *then* $T \in \mathcal{Z}_p$*.*
(iii) If T *is a positive in* $\mathcal{N}$ *such that* $\lim_{r\to\infty} \frac{1}{r}\tau(T^{p+\frac{1}{r}})$ *exists, then* $T \in \mathcal{Z}_p$*.*

Proof. The first statement is immediate from earlier results. To prove (ii) we remind the reader that $T \in \mathcal{L}^{p,\infty}$ iff $\mu_t(T) \leqslant C\min(1, t^{-1/p})$ for some $C < \infty$. Then as $r \to \infty$

$$\begin{aligned}\frac{1}{r}\int_0^\infty \mu_t(T)^{p+1/r} dt &\leqslant C\frac{1}{r}(1 + \int_1^\infty t^{-1-1/pr} dt) \\ &= C\frac{1}{r}(1 - prt^{-1/pr}|_1^\infty) = C\frac{(1+pr)}{r} < \infty.\end{aligned}$$

For (iii), we note that if $\lim_{r\to\infty} \frac{1}{r}\tau(T^{p+\frac{1}{r}})$ exists, then $T^p \in \mathcal{Z}_1$ and by (i) $T \in \mathcal{Z}_p$ □

In view of the preceding corollary we have the following implications

$$T \in \mathcal{L}^{p,\infty} \Longrightarrow T \in \mathcal{Z}_p,$$

$$T \in \mathcal{Z}_p \Longleftrightarrow T^p \in \mathcal{Z}_1 = \mathcal{L}^{1,\infty}.$$

Hence, everything which has been proved for $T \in \mathcal{Z}_1 = \mathcal{L}^{1,\infty}$ is automatically true for $S = T^p$ provided $T \in \mathcal{Z}_p$ or especially if $T \in \mathcal{L}^{p,\infty}$.

8.6 The space $\mathcal{Z}_p$, $p > 1$ is strictly larger than $\mathcal{L}^{p,\infty}$

We deduce the result in the title of this subsection by proving that the analogue of Theorem 8.4 does not hold when $p > 1$.

Proposition 8.8. *The assumption* $\sup_{r\geqslant 1} \frac{1}{r}\tau(T^{p+\frac{1}{r}}) < \infty$ *does not guarantee* $T \in \mathcal{L}^{(p,\infty)}$*.*

Proof. We use the notation $\mu_t(T) := x(t),\ t > 0$. The proof is based on the observation (see [KPS] and also detailed explanations in [Suk, Section 5]) that the ordinary norm

$$\|x\|_\psi = \sup_{t>0} \frac{\int_0^t x^*(s)ds}{\psi(t)}$$

in the Marcinkiewicz space $M(\psi)$ (here, $\psi \in \Omega$ as in Section 2) is equivalent to the quasi-norm

$$F_\psi(x) = \sup_{0<t<\infty} \frac{tx^*(t)}{\psi(t)}$$

provided that $\liminf_{t\to\infty} \frac{\psi(2t)}{\psi(t)} > 1$. For $\psi_p(t) = t^{1-1/p},\ p > 1$, the norm $\|\cdot\|_{\psi_p}$ and quasi-norm $F_p(\cdot) = F_{\psi_p}(\cdot)$ are equivalent. In other words, the norm of any element T from the ideal $\mathcal{L}^{(p,\infty)}$ is equivalent to $F_p(x)$. This is not the case for $\psi_0(t) := \ln(1+t)$ (that is the functional $F_0(\cdot) = F_{\psi_0}(\cdot)$ and the norm in $\mathcal{L}^{(1,\infty)}$ are not equivalent) and it is easy to locate a function $z(t) = z^*(t)$ such that $\|z\|_{\psi_0} < \infty$ but $F_0(z) = \sup_{t>0} z^*(t)t = \infty$. For example, we take $z(t) = n/2^{n^2}$ for $t \in (2^{(n-1)^2}, 2^{n^2}],\ n = 1, 2, ...,$ and $z(t) = 1$ for $t \in [0, 1]$. It is easy to verify that there exists $0 < C < \infty$ such that

$$\int_0^t z^*(s)ds \leqslant C\ln(1+t)$$

(that is $z \prec\prec C/(1+t)$) and at the same time

$$z(t)t|_{t=2^{n^2}} = z(2^{n^2})2^{n^2} = n,\ n = 1, 2, ...$$

(that is $F_0(z) = \infty$).

Observe that since $z \prec\prec C/(1+t)$, we have for every $\nu > 0$

$$\int_0^\infty z(t)^{1+\nu}dt \leqslant C\int_0^\infty (1/(1+t))^{1+\nu}dt = C/\nu < \infty.$$

Now, let us fix $p > 1$ and set $x(t) = z^{1/p}(t)$ for $t > 0$. The estimate above gives

$$s\int_0^\infty x^{p+s}(t)dt = p(s/p\int_0^\infty z(t)^{1+s/p}dt) \leqslant Cp < \infty.$$

Nevertheless,

$$F_p(x) = \sup_{0<t<\infty} x(t)t^{1/p} = (F_0(z))^{1/p} = \infty.$$

That is the condition $\sup_{r\geqslant 1} \frac{1}{r}\tau(T^{p+\frac{1}{r}}) < \infty$ does not imply $T \in \mathcal{L}^{(p,\infty)}$. $\square$

We remark that while $\mathcal{Z}_p,\ p > 1$ is the p-convexification of the ideal $\mathcal{L}^{1,\infty}$; in turn, the ideal $\mathcal{L}^{p,\infty}$ is the p-convexification of some subideal in $\mathcal{L}^{1,\infty}$, which is termed the 'small ideal' in [CPS2]. We will establish this latter fact in subsection 5.2.

8.7 Limits of zeta functions

Our earlier results enable us to considerably weaken the hypotheses in one of the main theorems of [CPS2]. First we recall the following preliminary result proved in [CPS2].

Proposition 8.9. (weak*-Karamata theorem) *Let $\tilde{\omega} \in L_\infty(\mathbb{R})^*$ be a dilation invariant state and let β be a real valued, increasing, right continuous function on $\mathbb{R}_+$ which is zero at zero and such that the integral $h(r) = \int_0^\infty e^{-\frac{t}{r}} d\beta(t)$ converges for all $r > 0$ and $C = \tilde{\omega} - \lim_{r\to\infty} \frac{1}{r} h(r)$ exists. Then*

$$\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r} h(r) = \tilde{\omega} - \lim_{t\to\infty} \frac{\beta(t)}{t}.$$

The classical Karamata theorem has a similar statement with the $\tilde{\omega}$ limits replaced by ordinary limits.

In the following we will take $T \in \mathcal{L}^{(1,\infty)}$ positive, $||T|| \leqslant 1$ with spectral resolution $T = \int \lambda dE(\lambda)$. We would like to integrate with respect to $d\tau(E(\lambda))$; unfortunately, these scalars $\tau(E(\lambda))$ are, in general, all infinite. To remedy this situation, we instead must integrate with respect to the increasing (negative) real-valued function $N_T(\lambda) = \tau(E(\lambda) - 1)$ for $\lambda > 0$. Away from 0, the increments $\tau(\triangle E(\lambda))$ and $\triangle N_T(\lambda)$ are, of course, identical. The following theorem is a strengthened version of Theorem 3.1 of [CPS2] made possible by Proposition 8.3.

Theorem 8.10. *For $T \in \mathcal{L}^{(1,\infty)}$ positive, $||T|| \leqslant 1$ let ω be a D_2-dilation and P^α-invariant, $\alpha > 1$ state on $L^\infty(\mathbb{R}_+^*)$. Let $\tilde{\omega} = \omega \circ L$ where L is given in Section 3, then we have:*

$$\tau_\omega(T) = \tilde{\omega} - \lim \frac{1}{r} \tau(T^{1+\frac{1}{r}}).$$

If $\lim_{r\to\infty} \frac{1}{r}\tau(T^{1+\frac{1}{r}})$ exists then

$$\tau_\omega(T) = \lim_{r\to\infty} \frac{1}{r} \tau(T^{1+\frac{1}{r}})$$

for an arbitrary dilation invariant functional $\omega \in L^\infty(\mathbb{R}_+^*)^*$.

Proof. The proof is just a minor rewriting of the corresponding argument in [CPS2]. By Proposition 4.6, the state $\tilde{\omega}$ is dilation invariant and by Theorem 8.4(i) $h(r) = \frac{1}{r}\tau(T^{1+\frac{1}{r}}) \in L^\infty(\mathbb{R}_+)$. So, we can apply the weak*-Karamata theorem. First write $\tau(T^{1+\frac{1}{r}}) = \int_{0^+}^1 \lambda^{1+\frac{1}{r}} dN_T(\lambda)$. Thus setting $\lambda = e^{-u}$

$$\tau(T^{1+\frac{1}{r}}) = \int_0^\infty e^{-\frac{u}{r}} d\beta(u)$$

where $\beta(u) = \int_u^0 e^{-v} dN_T(e^{-v}) = -\int_0^u e^{-v} dN_T(e^{-v})$. Since the change of variable $\lambda = e^{-u}$ is strictly decreasing, β is, in fact, nonnegative and increasing. By the weak*-Karamata theorem applied to $\tilde{\omega} \in L^\infty(\mathbb{R})^*$

$$\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r} \tau(T^{1+\frac{1}{r}}) = \tilde{\omega} - \lim_{u\to\infty} \frac{\beta(u)}{u}.$$

Next with the substitution $\rho = e^{-v}$ we get:

$$\tilde{\omega} - \lim_{u\to\infty} \frac{\beta(u)}{u} = \tilde{\omega} - \lim_{u\to\infty} \frac{1}{u} \int_{e^{-u}}^1 \rho dN_T(\rho). \tag{19}$$

Set $f(u) = \frac{\beta(u)}{u}$. We want to make the change of variable $u = \log t$ or in other words to consider $f \circ \log = Lf$. This is permissable by the discussion in Section 3 which tells us that if we start with a functional $\omega \in L^\infty(\mathbb{R}_+^*)^*$ as in the theorem we may replace it by the functional $\tilde{\omega} = \omega \circ L$ which is dilation invariant with

$$\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r} \tau(T^{1+\frac{1}{r}}) = \tilde{\omega} - \lim_{u\to\infty} \frac{\beta(u)}{u}$$
$$= \tilde{\omega} - \lim_{u\to\infty} f(u) = \omega - \lim_{t\to\infty} Lf(t) = \omega - \lim_{t\to\infty} \frac{1}{\log t} \int_{1/t}^1 \lambda dN_T(\lambda).$$

Now, by Proposition 8.3 applied to $\psi(t) = \log(1+t) \sim \log t$

$$\omega - \lim_{t\to\infty} \frac{1}{\log t} \int_{1/t}^1 \lambda dN_T(\lambda) = \omega - \lim_{t\to\infty} \frac{1}{\log t} \tau(\chi_{(\frac{1}{t},1]}(T)T) = \tau_\omega(T).$$

This completes the proof of the first part of the theorem.

The proof of the second part is similar. Using the classical Karamata theorem we obtain the following analogue of (19):

$$\lim_{r\to\infty} \frac{1}{r} \tau(T^{1+r}) = \lim \frac{\beta(u)}{u} = \lim_{u\to\infty} \frac{1}{u} \int_{e^{-u}}^1 \rho dN_T(\rho).$$

Making the substitution $u = \log t$ on the right hand side we have by Proposition 8.3

$$\lim_{u\to\infty} \frac{1}{u} \int_{e^{-u}}^1 \rho dN_T(\rho) = \lim_{t\to\infty} \frac{1}{\log t} \int_{\frac{1}{t}}^1 \lambda dN_T(\lambda)$$
$$= \tau_\omega(T) = \lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(T) ds.$$

□

We now deduce some corollaries of the discussion above. Retaining the notation as in the previous theorem we let ω be a D_2-dilation and P^α-invariant,

$\alpha > 1$ state on $L^\infty(\mathbb{R}_+^*)$. Let $\tilde{\omega} = \omega \circ L$. The assumption that $\frac{1}{r}\zeta(T^{1+\frac{1}{r}})$ is bounded in r means that, by Theorem 8.4, $T \in \mathcal{Z}_1 = \mathcal{L}^{1,\infty}$. Then by Theorem 8.10

$$\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\zeta(T^{1+\frac{1}{r}}) = \tau_\omega(T).$$

Consequently using (18) if either $T \in \mathcal{Z}_p$ or if $T \in \mathcal{L}^{p,\infty}$, $p > 1$ we have the formulae

$$\begin{aligned} \tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\zeta(T^{p+\frac{1}{r}}) &= \tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\tau(T^{p+1/r}) \\ &= p\tilde{\omega} - \lim_{pr\to\infty} \frac{1}{pr}\tau(T^{p(1+1/pr)}) = p\tau_\omega(T^p) \end{aligned} \tag{20}$$

where the last step uses dilation invariance of $\tilde{\omega}$, which is guaranteed by our choice of ω. The equation (20) together with Theorem 8.10 tell us that if one of the limits in the previous equality is true then so are the others. In particular, if $\lim_{r\to\infty} \frac{1}{r}\zeta(T^{p+\frac{1}{r}})$ exists, then $T \in \mathcal{Z}_p$ and

$$\lim_{r\to\infty} \frac{1}{r}\zeta(T^{p+\frac{1}{r}}) = p \lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(T^p) ds.$$

9 The heat semigroup formula

9.1 Asymptotics of the trace of the heat semigroup

Throughout this section $T \geqslant 0$. For $q \in \mathbb{R}_+$ we define $e^{-T^{-q}}$ as the operator that is zero on $\ker T$ and on $\ker T^\perp$ is defined in the usual way by the functional calculus. We remark that if $T \geqslant 0$, $T \in \mathcal{Z}_p$ for some $p \geqslant 1$ then $e^{-tT^{-q}}$ is trace class for all $t > 0$. This is because if $x \in E$, where $(E, \|\cdot\|_E)$ is any fully symmetric (or symmetric) space then

$$\|x\|_E \geqslant \|x^*(t)\chi_{[0,s]}(t)\|_E \geqslant x^*(s)\|\chi_{[0,s]}\|_E = x^*(s)\varphi(s),$$

where $\varphi(\cdot)$ is the fundamental function of E. Consequently, $x^*(s) \leqslant \|x\|_E/\varphi(s)$. For $E = \mathcal{Z}_p = \mathcal{L}_p^{1,\infty}$ (see Corollary 8.6(ii)) the fundamental function is $\varphi(s) = (s/log(1+s))^{1/p}$ Hence, for every $t > 0$

$$\mu_s(e^{-tT^{-q}}) = e^{-t/(\mu_s(T))^q} \leqslant e^{-tC(s/log(1+s))^{q/p}} \leqslant e^{-tCs^{q/p-\epsilon}}$$

for some $C > 0$ all $0 < p, q$ and $0 < \epsilon < q/p$. Thus $\tau(e^{-tT^{-q}}) < \infty$ for $q > 0$ (since $\epsilon > 0$ is arbitrary).

Theorem 9.1. (cf [CPS2]) *If $T \geqslant 0$, $T \in \mathcal{Z}_p$, $1 \leqslant p < \infty$ then, choosing ω to be DPM invariant and $\tilde{\omega}$ to be related with ω as in Remark 3.6, we have for $q > 0$*

$$\omega - \lim_{\lambda\to\infty} \frac{1}{\lambda}\tau(e^{-T^{-q}\lambda^{-q/p}}) = \frac{1}{q}\Gamma(p/q)\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\zeta(p + \frac{1}{r}) = \frac{p}{q}\Gamma(p/q)\tau_\omega(T^p).$$

9.2 The $L^{p,\infty}$-case and the 'small' ideal.

As $T \in \mathcal{L}^{p,\infty}$ means that $\mu_t(T)t^{1/p} < C < \infty$ and $\mu_t(T^p) = \mu_t(T)^p$ we conclude that $\mu_t(T^p)t < C^p < \infty$. That is $T \in \mathcal{L}^{p,\infty} \Longrightarrow S = T^p \in \mathcal{I}$ where $\mathcal{I}$ is the so called 'small' subideal of $\mathcal{L}^{1,\infty}$ identified in [CPS2]. Recall that $\mathcal{I}$ is specified by the condition on the singular values of $T \geqslant 0, T \in \mathcal{L}^{1,\infty}$: $\mu_s(T) \leqslant C/s$ for some constant $C > 0$. In subsection 4.1 [CPS2] we proved the following result by a direct argument that avoids the use of the zeta function. If ω is M invariant and satisfies conditions (1),(2),(3) of Theorem 3.4 and $T \in \mathcal{I}$ then

$$\omega - \lim_{\lambda \to \infty} \lambda^{-1} \tau(e^{-\lambda^{-2} T^{-2}}) = \Gamma(3/2)\tau_\omega(T).$$

We may now apply this stronger result of [CPS2] to operators $S \in \mathcal{I}$ where $S = T^p$ and $T \in \mathcal{L}^{p,\infty}$ to obtain the equality

$$\omega - \lim_{\lambda \to \infty} \lambda^{-1} \tau(e^{-\lambda^{-2} S^{-2}}) = \Gamma(3/2)\tau_\omega(S).$$

Hence we obtain the following result

$$\textit{If } T \in \mathcal{L}^{p,\infty} \textit{ then } \omega - \lim_{\lambda \to \infty} \lambda^{-1} \tau(e^{-\lambda^{-2} T^{-2p}}) = \Gamma(3/2)\tau_\omega(T^p). \tag{21}$$

Note that we have obtained this result under weaker conditions on ω than the more general Theorem 9.1 where $T \in \mathcal{Z}_p$. It would be interesting to understand an example in noncommutative geometry where $\mathcal{Z}_p$ arises naturally. We remark that in classical geometric examples such as differential operators on manifolds it is $\mathcal{L}^{p,\infty}$ $p \geqslant 1$ and the 'small ideal' $\mathcal{I}$ that arise naturally.

10 Application to spectral triples

Throughout this Section the following assumptions hold. We let $\mathcal{D}$ be an unbounded self adjoint densely defined operator on $\mathcal{H}$ affiliated to $\mathcal{N}$ (this amounts to $(1 + \mathcal{D}^2)^{-1} \in \mathcal{N}$). We suppose that $\mathcal{A}$ is a *-algebra in $\mathcal{N}$ consisting of operators a such that $[\mathcal{D}, a]$ is bounded and refer to the triple $(\mathcal{D}, \mathcal{A}, \mathcal{N})$ as a semifinite spectral triple.

Denote for brevity $\mathcal{M}^\psi := M(\psi)(\mathcal{N}, \tau)$ with ψ as in Section 4, satisfying (7) and (16). As in Corollary 8.6, we consider the following p-convexification of $\mathcal{M}^\psi$

$$\mathcal{M}^{\psi,p} := \{T \in \mathcal{N}_+ : \ \|T\|_{\psi,p} = \sup_{1<u<\infty} \frac{(\int_0^u \mu_t(T)^p dt)^{1/p}}{\psi^{1/p}(u)} < \infty\}, \quad p > 1.$$

We let τ_ω be a Dixmier trace on $\mathcal{M}^\psi$ corresponding to a suitable singular state ω. Suppose that $(1+\mathcal{D}^2)^{-p/2} \in \mathcal{M}^\psi$, or equivalently that $(1+\mathcal{D}^2)^{-1/2} \in \mathcal{M}^{\psi,p}$.

In applications of noncommutative geometry the functional φ_ω on $\mathcal{A}$ given by $\varphi_\omega(a) = \tau_\omega(a(1+D^2)^{-p/2})$ plays a key role. In particular it is of interest to know if this functional is a trace on $\mathcal{A}$. In [CGS] this question was answered in the affirmative for the case of $(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{p,\infty}$. Their proof generalizes to our setting. In particular, it holds under the weaker assumption $(1+\mathcal{D}^2)^{-1/2} \in \mathcal{Z}_p$.

Theorem 10.1. *Under the immediately preceding hypotheses we have*

$$\varphi_\omega(ab) = \varphi_\omega(ba) \qquad a, b \in \mathcal{A}.$$

The proof is an extension of the approach in [CGS]. We need four preliminary facts. Some may be proved in a similar way to the corresponding results in [CGS].

Lemma 10.2. *Given a spectral triple $(\mathcal{D}, \mathcal{A}, \mathcal{N})$ we have*
(i) For $a, b \in \mathcal{N}$ the Hölder inequality

$$\tau_\omega(ab) \leqslant \tau_\omega(|a|^p)^{1/p} \tau(|b|^q)^{1/q}$$

for $p, q \geqslant 1$, $\frac{1}{p} + \frac{1}{q} = 1$, holds.
(ii) For any r with $0 < r < 1$ and $a \in \mathcal{A}$ the operator $[(1+\mathcal{D}^2)^{r/2}, a]$ is bounded and satisfies

$$||[(1+\mathcal{D}^2)^{r/2}, a]|| \leqslant C||[\mathcal{D}, a]||$$

where the constant $C > 0$ does not depend on a.
(iii) Let $T \in \mathcal{M}^\psi$ and $f(t) = \mu_t(T)$ so that f is a bounded decreasing function on $(0, \infty)$ from $M(\psi)$, then $f^\alpha \in L_1(\mathbb{R}_+)$ for every $\alpha > 1$.
(iv) The statement of the theorem (for $(1+\mathcal{D}^2)^{-p/2} \in \mathcal{M}^\psi$) is implied by

$$\tau_\omega(|[(1+\mathcal{D}^2)^{-p/2}, a]|) = 0 \text{ for all } a \in \mathcal{A}.$$

Proof. (i) We have by the Hölder inequality for function spaces

$$\int_0^t \mu_s(ab)ds \leqslant \int_0^t \mu_s(a)\mu_s(b)ds \leqslant (\int_0^t \mu_s(a)^p ds)^{1/p} (\int_0^t \mu_s(b)^q ds)^{1/q}$$

Dividing by $\psi(t)$ and applying the functional ω we get

$$\tau_\omega(ab) \leqslant \omega \left[\left(\frac{\int_0^t \mu_s(a)^p ds}{\psi(t)} \right)^{1/p} \left(\frac{\int_0^t \mu_s(b)^q ds}{\psi(t)} \right)^{1/q} \right]$$

$$\leqslant \omega \left(\frac{\int_0^t \mu_s(a)^p ds}{\psi(t)} \right)^{1/p} \omega \left(\frac{\int_0^t \mu_s(b)^q ds}{\psi(t)} \right)^{1/q} = \tau_\omega(|a|^p)^{1/p} \tau_\omega(|b|^q)^{1/q}$$

using Hölder inequality for states on abelian C^*-algebras. We omit the proof for $p = 1$, $q = \infty$. $\square$

Choose r with $0 < r < 1$ such that $k = p/r \in \mathbb{N}$. Following [CGS], we see that the proof of the theorem rests on the identity (for $k \in \mathbb{N}$)

$$[a,(1+\mathcal{D}^2)^{-kr/2}] = \sum_{j=1}^{k}(1+\mathcal{D}^2)^{-jr/2}[(1+\mathcal{D}^2)^{r/2},a](1+\mathcal{D}^2)^{(j-k-1)r/2}$$

where we are using part (ii) of the Lemma to give boundedness of $[(1+\mathcal{D}^2)^{r/2},a]$. We now apply the previous identity to obtain:

$$\tau_\omega(|[a,(1+\mathcal{D}^2)^{-p/2}]|) = \tau_\omega(|[a,(1+\mathcal{D}^2)^{-kr/2}]|)$$

$$\leqslant \sum_{j=1}^{k}\tau_\omega[|(1+\mathcal{D}^2)^{-jr/2}[(1+\mathcal{D}^2)^{r/2},a](1+\mathcal{D}^2)^{(j-k-1)r/2}|]$$

Hence choosing $p_j = \frac{2p}{r(2j-1)}$, $q_j = \frac{2p}{r(2k-2j+1)}$ and applying part (i) of the Lemma,

$$\tau_\omega(|[a,(1+\mathcal{D}^2)^{-p}]|) \leqslant ||[(1+\mathcal{D}^2)^{r/2},a]||$$
$$\times \sum_{j=1}^{k}(\tau_\omega((1+\mathcal{D}^2)^{-p_j jr/2}))^{1/p_j}(\tau_\omega((1+\mathcal{D}^2)^{(j-k-1)q_j r/2}))^{1/q_j}$$

The exponents $p_j jr/2$ and $(j-k-1)q_j r/2$ are larger than p so using part (iii) of the Lemma, the Dixmier trace in the last two terms vanishes. Now use part (iv) of the Lemma to complete the proof of the Theorem.

11 Lidskii type formula for Dixmier traces

A semifinite analogue of the classical Lidskii theorem stated in terms of the (so-called) Brown spectral measure μ_T of $T \in \mathcal{N}$ asserts [Brown] that

$$\tau(T) = \int_{\sigma(T)\setminus\{0\}} \lambda d\mu_T(\lambda).$$

In the case, when $\mathcal{N} = \mathcal{L}(\mathcal{H})$ and τ is the standard trace Tr the equality above reduces to the classical case asserting that the trace $Tr(T)$ of an arbitrary trace class operator T is given by the sum $\sum_{n\geqslant 1}\lambda(T)$, where $\{\lambda_n(T)\}_{n\geqslant 1}$ is the sequence of eigenvalues of T, arranged in decreasing order of absolute values of λ_n and counting multiplicities. Note, that in the case $T \geqslant 0$, the equality $\mathrm{Tr}(T) = \sum_{n\geqslant 1}\lambda_n(T)$ follows immediately from the spectral theorem for compact operators. If $T^* = T$, then again, by the spectral theorem we can select the orthonormal basis of $\mathcal{H}$ consisting of eigenvalues of T and still infer Lidskii's theorem without any difficulty. Here it is worth observing that the assumption

$T^* = T$ belongs to the ideal $\mathcal{L}^1(\mathcal{H})$ of all trace class operators on $\mathcal{H}$ implies the absolute convergence of the series $\sum_{n\geqslant 1} |\lambda_n(T)|$. The latter fact guarantees the convergence of the series $\sum_{n\geqslant 1} \lambda_n(T)$ in whatever ordering of the set of all eigenvalues for T is chosen (in particular, for the decreasing ordering of absolute values of T).

The core difference of this situation with the setting of Dixmier traces consists in the fact that the series $\sum_{n\geqslant 1} |\lambda_n(T)|$ diverges for every normal $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H}) \setminus \mathcal{L}^1(\mathcal{H})$. Therefore, even though for a given $T = T^* \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ we define $\tau_\omega(T)$ as the difference $\tau_\omega(T_+) - \tau_\omega(T_-)$ where each number is computed according to the definition $\omega\text{-}\lim_{N\to\infty} \frac{1}{\log(1+N)} \sum_{n=1}^{N} \lambda_n(T_\pm)$, it is by no means clear that we have

$$\tau_\omega(T) = \omega\text{-}\lim_{N\to\infty} \frac{1}{\log(1+N)} \sum_{n=1}^{N} \lambda_n(T)$$

for the special enumeration of the set $\{\lambda_n(T)\}_{n\geqslant 1}$ given by the decreasing order of absolute values of $|\lambda_n(T)|$; or for that matter for *any* enumeration of this set. This difficulty becomes even more pronounced in the case of a general semifinite von Neumann albegra.

Note that the restriction $\mu_t(T) \leqslant \frac{C}{t}$, $t \geqslant 1$ imposed on $T \in \mathcal{L}^{(1,\infty)}(\mathcal{N}, \tau)$ in the theorem below, implies that T belongs to the "small" ideal, see Section 9.2.

Theorem 11.1 ([AzSu]). (i). If $(\mathcal{N}, \tau)$ is a semifinite von Numann albegra and $T \in \mathcal{L}^{(1,\infty)}(\mathcal{N}, \tau)$ satisfies $\mu_t(T) \leqslant \frac{C}{t}$, $t \geqslant 1$ for some $C > 0$, then

$$\tau_\omega(T) = \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_{\lambda \notin \frac{1}{t}G} \lambda d\mu_T(\lambda).$$

(ii). Let T be a compact operator on a Hilbert space $\mathcal{H}$, such that $\mu_n(T) \leqslant \frac{C}{n}$, $n \geqslant 1$ for some $C > 0$. Let $\lambda_1, \lambda_2, \ldots$ be the list of eigenvalues of the operator T counted with multiplicities such that $|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots$. Then

$$\begin{aligned} \tau_\omega(T) &= \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} \sum_{\lambda \in \sigma(T),\ \lambda \notin \frac{1}{t}G} \lambda \mu_T(\lambda) \\ &= \omega\text{-}\lim_{N\to\infty} \frac{1}{\log(1+N)} \sum_{i=1}^{N} \lambda_i, \end{aligned} \tag{22}$$

where $\mu_T(\lambda)$ is the algebraic multiplicity of the eigenvalue λ and G is an arbitrary bounded neighborhood of $0 \in \mathbb{C}$.

Here, we emphasize the fact that G is an *arbitrary* bounded neighborhood of 0. In the case $\mathcal{N} = \mathcal{L}(\mathcal{H})$, formula (22) says that we need to compute the sum of all eigenvalues $\lambda_j(T)$ which do not belong to the "squeezed" neighborhood $\frac{1}{t}G$ as $t \to \infty$ and then to apply ω-limit. It is interesting to emphasize the special case of formula (22) for measurable operators belonging to the ideal $N(\psi_1)$. The result below should be compared with the results of Theorems 6.6 and 6.7.

Corollary 11.2. If T is a (Connes-Dixmier) measurable operator satisfying the assumption of Theorem 11.1 (i) (respectively, (ii)), then

$$\tau_\omega(T) = \lim_{t\to\infty} \frac{1}{\log(1+t)} \int_{\lambda \notin \frac{1}{t}G} \lambda d\mu_T(\lambda)$$

$$\left(\text{resp.} = \lim_{N\to\infty} \frac{1}{\log(1+N)} \sum_{i=1}^{N} \lambda_i\right),$$

for every $\omega \in CD(\mathbb{R}_+^*)$.

Fix an orthonormal basis in $\mathcal{H}$ and identify every element $x \in \mathcal{L}(\mathcal{H})$ with its matrix $(x_{ij})_{i,j=1}^{\infty}$. It is well-known [GK] that the triangular truncation operator $\mathfrak{T}$ given by

$$\mathfrak{T}(x) = \begin{cases} x_{ij}, & i \geqslant j \\ 0, & i < j \end{cases}$$

acts boundedly from the trace class $\mathcal{L}^1(\mathcal{H})$ into $\mathcal{L}^{(1,\infty)}(\mathcal{H})$. Noting that $\mathfrak{T}(x) - \operatorname{diag}(x)$ is quasinilpotent for every $x \in \mathcal{L}^1(\mathcal{H})$, we obtain the following

Corollary 11.3. The operator $\mathfrak{T}(x)$ is Connes-Dixmier measurable for every $x \in \mathcal{L}^1(\mathcal{H})$, moreover $\tau_\omega(\mathfrak{T}(x)) = 0$, $\omega \in CD(\mathbb{R}_+^*)$.

References

[AzSu] N. Azamov, F. Sukochev, *A Lidskii type formula for Dixmier traces*, C.R. Acad. Sci. Paris, Ser. I, **340** (2005), 107-112.

[BCPRSW] M.T. Benameur, A.L. Carey, J. Phillips, A. Rennie, F.A. Sukochev, K.P. Wojciechowski, *An analytic approach to spectral flow in von Neumann algebras*, Analysis, Geometry and Topology of Elliptic Operators. Papers in Honor of Krzysztof P Wojciechowski, 297–352, World Sci. Publ., Singapore, 2006.

[BeF] M-T. Benameur, T. Fack, *Type II Noncommutative Geometry. I. Dixmier Trace in von Neumann Algebras*, Adv. Math. **199** (2006), 29–87.

[Brown] L.G. Brown, *Lidskii theorem in the type II case*, Geometric methods in operator algebras (Kyoto, 1983), 1–35, Pitman Res. Notes Math. Ser., 123, Longman Sci. Tech., Harlow, 1986.

[CGS] F. Cipriani, E. Guido, S. Scarlatti, *A remark on trace properties of K-cycles*, J. Operator Theory **35** (1996), 179–189.

[CPRS2] A.L. Carey, J. Phillips, A. Rennie, F. Sukochev, *The local index theorem in semifinite von Neumann algebras I. Spectral Flow*, Advances in Math to appear.

[CPRS3] A.L. Carey, J. Phillips, A. Rennie, F. Sukochev, *The Local Index Theorem in semifinite von Neumann algebras II: the even case*, Advances in Math to appear.

[CPS2] A.L. Carey, J. Phillips, F. A. Sukochev, *Spectral Flow and Dixmier Traces*, Advances in Math. **173** (2003), 68-113.

[CRSS] A. Carey, A. Rennie, A. Sedaev and F. Sukochev, *The Dixmier trace and asymptotics of zeta functions,* J. Funct. Anal. **249** (2007), no. 2, 253–283.

[Co1] A. Connes, *Noncommutative Geometry*, Acad. Press, San Diego, 1994.

[Co2] A. Connes, *Geometry From the Spectral Point of View*, Lett. Math. Phys., **34** (1995), 203-238.

[Co3] A. Connes, *The action functional in noncommutative geometry*, Commun. Math. Phys. **117** (1988), 673-683.

[CM] A. Connes, H. Moscovici, *The Local Index Formula in Noncommutative Geometry*, GAFA **5** (1995), 174-243.

[CaS] A. L. Carey and F. A. Sukochev, *Dixmier traces and some applications to noncommutative geometry,* (Russian) Uspekhi Mat. Nauk **61** (2006), no. 6(372), 45–110; translation in Russian Math. Surveys **61** (2006), no. 6, 1039–1099.

[CS1] V.I. Chilin, F.A. Sukochev, *Symmetric spaces over semifinite von Neumann algebras.* (Russian) Dokl. Akad. Nauk SSSR **313** (1990), no. 4, 811–815; translation in Soviet Math. Dokl. **42** (1991), no. 1, 97–101.

[CS2] V.I. Chilin, F.A. Sukochev, *Weak convergence in non-commutative symmetric spaces,* J. Operator Theory **31** (1994), 35-65.

[Dix66CR] J. Dixmier, *Existence de traces non normales*, (French) C. R. Acad. Sci. Paris **262** (1966) 1107–1108.

[DDP1] P.G. Dodds, T.K. Dodds, B. de Pagter, *Non-commutative Banach function spaces*, Math. Z. **201** (1989), 583-597.

[DDP2] P.G. Dodds, T.K. Dodds, B. de Pagter, *Fully symmetric operator spaces*, Integr. Equat. Oper. Th. **15** (1992), 942-972.

[DPSSS1] P.G. Dodds, B. de Pagter, F.A. Sedaev, E.M. Semenov, F.A. Sukochev, *Singular symmetric functionals*, (Russian) Zap. Nauchn. Sem. S.-Peterburg., Otdel. Mat. Inst. Steklov. (POMI) **290** (2002), Issled. po Linein. Oper. i Teor. Funkts. 30, 42–71, 178; translation in J. Math. Sci. (N. Y.) **124**, (2004), no. 2, 4867–4885.

[DPSSS2] P.G. Dodds, B. de Pagter, F.A. Sedaev, E.M. Semenov, F.A. Sukochev, *Singular symmetric functionals and Banach limits with additional invariance properties*, (Russian) Izv. Ross. Akad. Nauk, Ser. Mat. **67** (2003), no. 6, 111–136.

[DPSS] P. G. Dodds, B. de Pagter, E. M. Semenov, F. A. Sukochev *Symmetric functionals and singular traces*, Positivity **2** (1998), 47-75.

[FK] T. Fack, H. Kosaki, *Generalised s-numbers of τ-measurable operators*, Pacific J. Math., **123** (1986), 269–300.

[GK] I. C. Gohberg, M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators.*, Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18 American Mathematical Society, Providence, R.I. 1969.

[Hardy] G.H. Hardy, *Divergent Series*, Oxford, Clarendon Press, 1949.

[KS] N. Kalton, F. Sukochev, *Rearrangement invariant functionals with applications to traces on symmetrically normed ideals*, Canad. Math. Bull. **51** (2008), no. 1, 67–80.

[KPS] S.G. Kreĭn, Yu.I. Petunīn, E.M Semenov, *Interpolyatsiya lineĭnykh operatorov.* (Russian). "Nauka", Moscow, 1978; *Interpolation of linear operators.* Translated by J. Szűcs. Translations of Mathematical Monographs, 54. American Mathematical Society, Providence, R.I., 1982.

[LT] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces. I. Sequence spaces,* Results in Mathematics and Related Areas, 92. Springer-Verlag, Berlin-New York, 1977.

[LT2] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces. II. Function spaces,* Results in Mathematics and Related Areas, 97. Springer-Verlag, Berlin-New York, 1979.

[LSS] S. Lord, A. Sedaev, F. Sukochev, *Dixmier traces as singular symmetric functionals and applications to measurable operators*, J. Funct. Analysis **224** (2005), no. 1, 72–106.

[Lor] G.G. Lorentz, *A contribution to the theory of divergent sequences*, Acta. Math. **80** (1948), 167–190.

[Ni] F. Nicola, *Trace functionals for a class of pseudo-differential operators in* $\mathbb{R}^n$*,* Math. Physics, Anal. and Geometry **6**, 89-105 (2003).

[Su67AMM] L. Sucheston, *Banach limits*, Amer. Math. Monthly **74** (1967), 308–311.

[Suk] F. A. Sukochev, *Operator estimates for Fredholm Modules*, Canad.J.Math. **52** (2000), 849-896.

[Wo84KT] M. Wodzicki, *Noncommutative residue. I. Fundamentals* K-theory, arithmetic and geometry (Moscow, 1984–1986), 320–399, Lecture Notes in Math., 1289, Springer, Berlin, 1987.