Rademacher averages in noncommutative symmetric spaces

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Khintchine inequalities

- A Rademacher sequence is a sequence $(\varepsilon_k)_{k\geq 1}$ of independent $\{-1,1\}$-valued random variables on some probability space (Ω, dm), such that $m\big(\{\varepsilon_k = 1\}\big) = m\big(\{\varepsilon_k = -1\}\big) = 1/2$ for any $k \geq 1$.

For instance, consider the Cantor group $\Omega = \{-1,1\}^\infty$ with Haar measure dm and set

$$\varepsilon_k(\theta) \,=\, \theta_k, \qquad \theta = (\theta_k)_{k\geq 1} \in \Omega.$$

Then for any $k \neq j$,

$$\int_\Omega \varepsilon_k \varepsilon_j \, dm \,=\, \Big(\int_\Omega \varepsilon_k \, dm\Big)\Big(\int_\Omega \varepsilon_j \, dm\Big) = 0$$

and $(\varepsilon_k)_k$ is an orthonormal sequence in $L^2(\Omega)$. Hence

$$\Big\|\sum_k \alpha_k \varepsilon_k\Big\|_2 \,=\, \Big(\sum_k |\alpha_k|^2\Big)^{\frac{1}{2}}, \qquad \alpha_1, \ldots, \alpha_n \in \mathbb{C}.$$

- **Inequalities (K).** For any $1 \leq p < \infty$, there exist constants $A_p, B_p > 0$ such that for any $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$,

$$A_p \Big(\sum_k |\alpha_k|^2\Big)^{\frac{1}{2}} \,\leq\, \Big\|\sum_k \alpha_k \varepsilon_k\Big\|_p \,\leq\, B_p \Big(\sum_k |\alpha_k|^2\Big)^{\frac{1}{2}}.$$

Hilbert spaces

- For any Banach space X and $x_1, \dots, x_n$ in X, set

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}_p(X)} = \Big(\int_\Omega \Big\|\sum_k \varepsilon_k(\theta) x_k\Big\|_X^p \, dm(\theta)\Big)^{\frac{1}{p}}.$$

Equivalently,

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}_p(X)} = \Big(\frac{1}{2^n} \sum_{\theta_k = \pm 1} \Big\|\sum_{k=1}^n \theta_k x_k\Big\|_X^p\Big)^{\frac{1}{p}}.$$

The basic question is: **how to estimate these norms?**

- Assume that $X = H$ is a Hilbert space. As in the scalar case,

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}_2(H)} = \Big(\sum_k \|x_k\|^2\Big)^{\frac{1}{2}}$$

that for $x_1, \dots, x_n$ in H.

Theorem (Kwapien, 1972). Let X be a Banach space. If there exist constants $0 < C_1 < C_2$ such that for any $n \geq 1$ and any $x_1, \dots, x_n$ in X, we have

$$C_1\Big(\sum_k \|x_k\|^2\Big)^{\frac{1}{2}} \leq \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}_2(X)} \leq C_2\Big(\sum_k \|x_k\|^2\Big)^{\frac{1}{2}},$$

then X is isomorphic to a Hilbert space.

Khintchine inequalities on L^p

- Let (Σ, μ) be a measure space, let $1 \leq p < \infty$, and consider $X = L^p(\Sigma)$. Then we have an equivalence

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}_1(L^p(\Sigma))} \approx \Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_{L^p(\Sigma)}$$

for finite families $x_1, \ldots, x_n$ in $L^p(\Sigma)$.

Here the symbol $\approx$ means that there exists two constants $0 < C_1 < C_2$ such that for any $n \geq 1$ and for any $x_1, \ldots, x_n$ in $L^p(\Sigma)$, we have

$$C_1 \Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_p \leq \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}_1(L^p)} \leq C_2 \Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_p.$$

Proof. By Khintchine's inequality, we have

$$A_1 \Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}} \leq \int_\Omega \Big|\sum_k \varepsilon_k(\theta) x_k\Big|\, dm(\theta)$$

on Σ. Applying the L^p-norm and convexity, this implies

$$\begin{aligned} A_1 \Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_p &\leq \Big\|\int_\Omega \Big|\sum_k \varepsilon_k(\theta) x_k\Big|\,\Big\|_p \\ &\leq \int_\Omega \Big\|\sum_k \varepsilon_k(\theta) x_k\Big\|_p\, dm(\theta)\,. \end{aligned}$$

On the other hand, by Hölder and Fubini, we have

$$\begin{aligned}\int_\Omega \Big\|\sum_k \varepsilon_k(\theta)x_k\Big\|_p dm(\theta) &\le \Big(\int_\Omega \Big\|\sum_k \varepsilon_k(\theta)x_k\Big\|_p^p dm(\theta)\Big)^{\frac{1}{p}} \\ &= \Big\|\Big(\int_\Omega \Big|\sum_k \varepsilon_k(\theta)x_k\Big|^p dm(\theta)\Big)^{\frac{1}{p}}\Big\|_p.\end{aligned}$$

By Khintchine's inequality again, we have

$$\Big(\int_\Omega \Big|\sum_k \varepsilon_k(\theta)x_k\Big|^p dm(\theta)\Big)^{\frac{1}{p}} \le B_p\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}$$

on Σ. Hence

$$\Big\|\Big(\int_\Omega \Big|\sum_k \varepsilon_k(\theta)x_k\Big|^p dm(\theta)\Big)^{\frac{1}{p}}\Big\|_p \le B_p\Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_p.$$

Finally,

$$\int_\Omega \Big\|\sum_k \varepsilon_k(\theta)x_k\Big\|_p dm(\theta) \le B_p\Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_p.$$

Khintchine-Kahane inequalities on Banach spaces

- Let $1 \leq p < \infty$. Then on any Banach space X, we have an equivalence

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}_p(X)} \approx \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}_2(X)}$$

for finite families $x_1, \ldots, x_n$ in X.

- In the sequel, we shall write $\big\|\cdots\cdots\big\|_{\mathrm{Rad}(X)}$ instead of any of the norms $\big\|\cdots\cdots\big\|_{\mathrm{Rad}_p(X)}$.

Banach lattices with a concavity

- Let Λ be a Banach lattice, and let $1 < p, q < \infty$. We say that Λ is p-convex if

$$\Big\| \Big(\sum_k |x_k|^p \Big)^{\frac{1}{p}} \Big\|_\Lambda \lesssim \Big(\sum_k \|x_k\|_\Lambda^p \Big)^{\frac{1}{p}}$$

for finite families $x_1, \ldots, x_n$ in Λ.

We say that E is q-concave if

$$\Big(\sum_k \|x_k\|_\Lambda^q \Big)^{\frac{1}{q}} \lesssim \Big\| \Big(\sum_k |x_k|^q \Big)^{\frac{1}{q}} \Big\|_\Lambda.$$

for finite families $x_1, \ldots, x_n$ in Λ.

Theorem. Assume that Λ is q-concave for some $q < \infty$. Then we have an equivalence

$$\Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_{\mathrm{Rad}(\Lambda)} \approx \Big\| \Big(\sum_k |x_k|^2 \Big)^{\frac{1}{2}} \Big\|_\Lambda$$

for finite families $x_1, \ldots, x_n$ in Λ.

Proof. Similar to the one for L^p-spaces. Use q-concavity instead of Fubini.

Khintchine inequalities on non-commutative L^p-spaces

- Instead of $L^p(\Sigma)$, let us consider a **noncommutative L^p-space**

$$X = L^p(M)$$

associated to a von Neumann algebra M.

- Recall that on a commutative space $L^p(\Sigma)$, $|x|^2 = \overline{x}x$. Thus for $x \in L^p(M)$, which is an operator, one may try to replace $|x|^2$ by the positive operator x^*x, and hence

$$\Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\| \qquad \text{by} \qquad \Big\|\Big(\sum_k x_k^* x_k\Big)^{\frac{1}{2}}\Big\|.$$

We could as well consider xx^* in the place of x^*x (we are in a non-commutative context!) and then consider

$$\Big\|\Big(\sum_k x_k x_k^*\Big)^{\frac{1}{2}}\Big\|$$

instead.

Theorem (Lust-Piquard, 1986, Lust-Piquard/Pisier, 1991).

(1) For $2 \leq p < \infty$ and $x_1, \ldots, x_n$ in $L^p(M)$,

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(L^p(M))} \approx \max\Big\{\Big\|\Big(\sum_k x_k^* x_k\Big)^{\frac{1}{2}}\Big\|_{L^p(M)}, \Big\|\Big(\sum_k x_k x_k^*\Big)^{\frac{1}{2}}\Big\|_{L^p(M)}\Big\}.$$

(2) For $1 \leq p \leq 2$ and $x_1, \ldots, x_n$ in $L^p(M)$,

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(L^p(M))} \approx \inf\Big\{\Big\|\Big(\sum_k y_k^* y_k\Big)^{\frac{1}{2}}\Big\|_{L^p(M)} + \Big\|\Big(\sum_k z_k z_k^*\Big)^{\frac{1}{2}}\Big\|_{L^p(M)}\Big\},$$

where the infimum runs over all finite families $(y_k)_k$ and $(z_k)_k$ in $L^p(M)$ such that $x_k = y_k + z_k$ for any $k \geq 1$.

- **Warning!** In general,

$$\Big\|\Big(\sum_k x_k^* x_k\Big)^{\frac{1}{2}}\Big\|_{L^p(M)} \not\approx \Big\|\Big(\sum_k x_k x_k^*\Big)^{\frac{1}{2}}\Big\|_{L^p(M)}$$

when $p \neq 2$.

Indeed, take $M = B(\ell^2)$ with the usual trace, the corresponding noncommutative L^p-spaces are the Schatten spaces S^p. Then take the matrix units $x_k = E_{k1}$. Then

$$x_k^* x_k = E_{11} \qquad \text{and} \qquad x_k x_k^* = E_{kk}.$$

Hence

$$\sum_{k=1}^{n} x_k^* x_k = \begin{bmatrix} n & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{bmatrix}$$

whereas

$$\sum_{k=1}^{n} x_k x_k^* = \begin{bmatrix} 1 & 0 & 0 & \cdots & \\ 0 & \ddots & & & \\ 0 & & 1 & & \\ \vdots & & & 0 & \\ & & & & \ddots \end{bmatrix}$$

Hence

$$\Big\| \Big(\sum_{k=1}^{n} x_k^* x_k \Big)^{\frac{1}{2}} \Big\|_{S^p} = n^{\frac{1}{2}} \qquad \text{and} \qquad \Big\| \Big(\sum_{k=1}^{n} x_k x_k^* \Big)^{\frac{1}{2}} \Big\|_{S^p} = n^{\frac{1}{p}}.$$

Columns, rows, and the duality max/inf

- Let $1 \leq p < \infty$ and let p' be its conjugate number ($p^{-1}+p'^{-1} = 1$).

For any $x_1, \ldots, x_n$ in $L^p(M)$,

$$\Big\|\Big(\sum_{k=1}^{n} x_k^* x_k\Big)^{\frac{1}{2}}\Big\|_{L^p(M)} = \left\|\begin{bmatrix} x_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ x_n & 0 & \cdots & 0 \end{bmatrix}\right\|_{L^p(M_n(M))}$$

and

$$\Big\|\Big(\sum_{k=1}^{n} x_k x_k^*\Big)^{\frac{1}{2}}\Big\|_{L^p(M)} = \left\|\begin{bmatrix} x_1 & \cdots & x_n \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}\right\|_{L^p(M_n(M))}.$$

Consider the spaces

$$A = \left\{ \begin{bmatrix} \vdots & 0 & \cdots \\ x_k & \vdots & \\ \vdots & 0 & \cdots \end{bmatrix} \oplus \begin{bmatrix} \cdots & y_k & \cdots \\ 0 & \cdots & 0 \\ \vdots & & \vdots \end{bmatrix} \ : \ x_k,\, y_k \in L^p(M) \right\}$$

regarded as a subspace of $L^p(M_n(M)) \overset{\infty}{\oplus} L^p(M_n(M))$, and

$$B = \left\{ \begin{bmatrix} \vdots & 0 & \cdots \\ x'_k & \vdots & \\ \vdots & 0 & \cdots \end{bmatrix} \oplus \begin{bmatrix} \cdots & y'_k & \cdots \\ 0 & \cdots & 0 \\ \vdots & & \vdots \end{bmatrix} : x'_k,\, y'_k \in L^{p'}(M) \right\}$$

regarded as a subspace of $L^{p'}(M_n(M)) \overset{1}{\oplus} L^{p'}(M_n(M))$.

The column and row spaces on $L^p(M_n(M))$ are 1-complemented. Since $L^p(M_n(M))^* = L^{p'}(M_n(M))$, this implies that

$$A^* = B \qquad \text{isometrically.}$$

The susbspace $A_{\text{diag}} \subset A$ of all pairs of matrices for which $x_k = y_k$ is just $L^p(M)^n$ equipped with the norm

$$\|(x_1, \ldots, x_n)\| = \max\Big\{ \Big\| \Big(\sum_k x_k^* x_k \Big)^{\frac{1}{2}} \Big\|_{L^p(M)}, \Big\| \Big(\sum_k x_k x_k^* \Big)^{\frac{1}{2}} \Big\|_{L^p(M)} \Big\}.$$

Then is dual space A^*_{diag} is isometric to the quotient

$$B/A^{\perp}_{\text{diag}}.$$

Furthermore, $A^{\perp}_{\text{diag}} \subset B$ is the space of all pairs of matrices for which

$$x'_k + y'_k = 0, \qquad k = 1, \ldots, n.$$

This shows that A_{diag}^* is isometric to $L^{p'}(M)^n$ equipped with the norm

$$\|(x_1',\ldots,x_n')\| = \inf\Big\{\Big\|\Big(\sum_k u_k^* u_k\Big)^{\frac{1}{2}}\Big\|_{L^{p'}(M)} + \Big\|\Big(\sum_k v_k v_k^*\Big)^{\frac{1}{2}}\Big\|_{L^{p'}(M)}\Big\},$$

where the infimum runs over all finite families $(u_k)_k$ and $(v_k)_k$ in $L^{p'}(M)$ such that $x_k' = u_k + v_k$ for any $k \geq 1$.

Symmetric spaces

- For a measurable function $f\colon (0,\infty) \to \mathbb{C}$, let

$$\lambda_f(s) = \mathrm{mes}\big\{t > 0 \,:\, |f(t)| > s\big\} \qquad s > 0.$$

This is the distribution function of $|f|$. Then if $\lambda_f \not\equiv \infty$, let

$$f^*(t) = \inf\big\{s > 0 \,:\, \lambda_f(s) \le t\big\}, \qquad t > 0.$$

This is the decreasing rearrangement of $|f|$.

- A Banach lattice $E \subset L^1(0,\infty) + L^\infty(0,\infty)$ is called **symmetric** if for any f, g as above,

$$\big[g \in E \quad \text{and} \quad f^* = g^*\big] \implies \big[f \in E \quad \text{and} \quad \|f\|_E = \|g\|_E\big].$$

- We assume throughout that E is either separable or is the dual space of a separable symmetric space.

Noncommutative symmetric spaces

- We consider a semifinite von Neumann algebra M equipped with a normal, faithful, semifinite trace τ.

Assume that $M \subset B(H)$. For any "τ-measurable" operator x on H, consider

$$\lambda_x(s) = \tau\big(\chi_{(s,\infty)}(|x|)\big), \qquad s > 0,$$

where $\chi_{(s,\infty)}$ is the indicator function of (s, ∞) to which we apply the Borel functional calculus of $|x|$. Then consider

$$\mu(x) \colon t \mapsto \inf\big\{s > 0 \,:\, \lambda_x(s) \leq t\big\}, \qquad t > 0.$$

This is the singular value function of x.

- The **noncommutative symmetric space** associated to E and M is

$$E(M) = \{x \,:\, \mu(x) \in E\},$$

equipped with

$$\|x\|_{E(M)} = \|\mu(x)\|_E.$$

“Classical” noncommutative Khintchine inequalities on $E(M)$

- For any $x_1, \ldots, x_n$ in $E(M)$, set

$$\big\|(x_k)_k\big\|_{\max} \,=\, \max\Big\{\Big\|\Big(\sum_k x_k^* x_k\Big)^{\frac{1}{2}}\Big\|_{E(M)},\, \Big\|\Big(\sum_k x_k x_k^*\Big)^{\frac{1}{2}}\Big\|_{E(M)}\Big\}$$

and

$$\big\|(x_k)_k\big\|_{\inf} \,=\, \inf\Big\{\Big\|\Big(\sum_k y_k^* y_k\Big)^{\frac{1}{2}}\Big\|_{E(M)} + \Big\|\Big(\sum_k z_k z_k^*\Big)^{\frac{1}{2}}\Big\|_{E(M)}\Big\},$$

where the infimum runs over all finite families $(y_k)_k$ and $(z_k)_k$ in $E(M)$ such that

$$x_k = y_k + z_k, \qquad k \geq 1.$$

Theorem (Lust-Piquard/Xu, 2006).

(1) Assume that E is 2-concave. Then for $x_1, \ldots, x_n$ in $E(M)$,

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(E(M))} \,\approx\, \big\|(x_k)_k\big\|_{\inf}.$$

(2) Assume that E is 2-convex and q-concave for some $q < \infty$. Then for $x_1, \ldots, x_n$ in $E(M)$,

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(E(M))} \approx \big\|(x_k)_k\big\|_{\max}.$$

General question. Give a equivalent description of

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(E(M))}$$

whenever E is q-concave for some $q < \infty$.

A simple example

Consider $E = L^p(0,\infty) \cap L^q(0,\infty)$ with $1 < p < 2 < q < \infty$. Let R be the hyperfinite II_1-factor, i.e.

$$R = \overline{\bigcup_n \big(M_{2^n}, \tau_n\big)}^{w^*},$$

where $\tau_n = 2^{-n} tr \colon M_{2^n} \to \mathbb{C}$ is the normalized trace.

Let

$$M = R \oplus B(\ell^2).$$

We have $S^p \subset S^q$ and $L^q(R) \subset L^p(R)$. Hence

$$E(M) \ = \ L^q(R) \oplus S^p.$$

Consider the two subspaces

$$Y = L^q(R) \oplus (0) \qquad \text{and} \qquad Z = (0) \oplus S^p$$

of $E(M)$. Then:

(i) For $x_1, \ldots, x_n$ in Z,

$$\Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_{\mathrm{Rad}(E(M))} \approx \big\|(x_k)_k\big\|_{\inf} \not\approx \big\|(x_k)_k\big\|_{\max};$$

(ii) For $x_1, \ldots, x_n$ in Y,

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(E(M))} \approx \big\|(x_k)_k\big\|_{\max} \not\approx \big\|(x_k)_k\big\|_{\inf}.$$

Indeed, for any $m \geq 1$,

$$S_m^q \subset L^q(R) \qquad \text{completely isometrically.}$$

$\hookrightarrow$ **Natural question**

Assume that E is q-concave for some $q < \infty$. Do we have estimates

$$\big\|(x_k)_k\big\|_{\inf} \lesssim \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(E(M))} \lesssim \big\|(x_k)_k\big\|_{\max}$$

for $x_1, \ldots, x_n$ in $E(M)$?

Other averages

- For any $x_1, \ldots, x_n$ in $E(M)$, we set

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_E \;:=\; \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{E(L^\infty(\Omega)\overline{\otimes}M)}.$$

Recall that for any p, $\big\|\sum_k \varepsilon_k \otimes x_k\big\|_{\mathrm{Rad}_p(E(M))}$ is the norm of

$$\sum_k \varepsilon_k \otimes x_k$$

in $L^p(\Omega; E(M))$. Now we are interested in its norm in $\underline{E(L^\infty(\Omega)\overline{\otimes}M)}$.

$\hookrightarrow$ Can we estimate these norms $\big\|\cdots\cdots\big\|_E$?

$\hookrightarrow$ Can we compare

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(E(M))} \qquad \text{and} \qquad \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_E \;?$$

- Let $1 \leq p < \infty$ and consider $E = L^p$. Then

$$L^p\big(L^\infty(\Omega)\overline{\otimes}M\big) \;=\; L^p(\Omega; L^p(M)).$$

Hence in this case,

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}_p(L^p(M))} \;=\; \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_E.$$

Commutative case

• **Boyd indices I.** We let p_E and q_E be the lower and upper Boyd indices of E. In general,

$$1 \leq p_E \leq q_E \leq \infty$$

and for $E = L^p$, $p_E = q_E = p$. These indices measure a certain resemblance with L^p-spaces.

If E is q-concave, then $q_E \leq q$.

If E is p-convex, then $p_E \geq p$.

More information later!

Proposition. Fix a symmetric space E. Then:

(1) $q_E < \infty$ if and only if for any *commutative* M, we have an equivalence

$$\Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_E \approx \Big\| \Big(\sum_k |x_k|^2 \Big)^{\frac{1}{2}} \Big\|_{E(M)}$$

for finite families $x_1, \ldots, x_n$ in $E(M)$.

(2) E is q-concave for some $q < \infty$ if and only if for any *commutative* M, we have an equivalence

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(E(M))} \approx \Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_{E(M)}$$

for finite families $x_1, \ldots, x_n$ in $E(M)$.

References: Lindenstrauss/Tzafriri, Astashkin/Braverman, Maurey.

- Consider $E = L^{r,\infty}$, with $1 \leq r < \infty$. Then $q_E = r$ but E is q-concave for no finite q. Consequently, for some commutative M, we have

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_E \not\approx \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(E(M))}$$

for $x_1, \ldots x_n$ in $E(M)$.

- The question to determine when

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_E \approx \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(E(M))}$$

in the non commutative case is widely open.

See: P. Dodds, F. Sukochev, *Contractibility of the linear group in Banach spaces of measurable operators*, Integral Equations and Operator Theory 26 (1996), 305-337.

Main results I

The semifinite von Neumann algebra M is now arbitrary.

Theorem 1.

(1) If $q_E < \infty$, then for $x_1, \ldots, x_n$ in $E(M)$, we have

$$\|(x_k)_k\|_{\inf} \lesssim \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_E.$$

(2) If $p_E > 1$ and $q_E < \infty$, then for $x_1, \ldots, x_n$ in $E(M)$, we have

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_E \lesssim \|(x_k)_k\|_{\max}$$

Nota. The problem whether (2) holds true when $p_E = 1$ is open.

Theorem 2.

(1) If E is 2-concave or if $q_E < 2$, then for $x_1, \ldots, x_n$ in $E(M)$,

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_E \approx \|(x_k)_k\|_{\inf}$$

(2) Assume that $q_E < \infty$. If E is 2-convex or if $p_E > 2$, then for $x_1, \ldots, x_n$ in $E(M)$,

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_E \approx \|(x_k)_k\|_{\max}$$

Main results II

- Haagerup/Pisier (1993) and Pisier have obtained Khintchine inequalities for norms of double sums

$$\Big\|\sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij}\Big\|_{\mathrm{Rad}^2(L^p(M))},$$

where $(x_{ij})_{i,j\geq 1}$ is a finite family of some noncommutative $L^p(M)$.

- For any $(x_{ij})_{1\leq i,j\leq N}$ in $E(M)$, we set

$$\big\|(x_{ij})_{i,j}\big\|_{\max} = \max\Big\{\Big\|\Big(\sum_{i,j} x_{ij}^* x_{ij}\Big)^{\frac{1}{2}}\Big\|_{E(M)}, \Big\|\Big(\sum_{i,j} x_{ij} x_{ij}^*\Big)^{\frac{1}{2}}\Big\|_{E(M)}, \\ \big\|[x_{ij}]\big\|_{E(M_N(M))}, \big\|[x_{ji}]\big\|_{E(M_N(M))}\Big\},$$

and

$$\big\|(x_{ij})_{i,j}\big\|_{\inf} = \inf\Big\{\Big\|\Big(\sum_{i,j} a_{ij}^* a_{ij}\Big)^{\frac{1}{2}}\Big\|_{E(M)} + \Big\|\Big(\sum_{i,j} b_{ij} b_{ij}^*\Big)^{\frac{1}{2}}\Big\|_{E(M)} + \\ \big\|[c_{ij}]\big\|_{E(M_N(M))} + \big\|[d_{ji}]\big\|_{E(M_N(M))}\Big\},$$

where the infimum runs over all $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ in $E(M)$ such that

$$x_{ij} = a_{ij} + b_{ij} + c_{ij} + d_{ij}, \qquad i, j \geq 1.$$

Then we set:

$$\Big\|\sum_{i,j}\varepsilon_i\otimes\varepsilon_j\otimes x_{ij}\Big\|_E = \Big\|\sum_{i,j}\varepsilon_i\otimes\varepsilon_j\otimes x_{ij}\Big\|_{E(L^\infty(\Omega)\overline{\otimes}L^\infty(\Omega)\overline{\otimes}M)}.$$

Theorem 3.

(1) If E is 2-concave or if $q_E < 2$, then for x_{ij} in $E(M)$,

$$\Big\|\sum_{i,j}\varepsilon_i\otimes\varepsilon_j\otimes x_{ij}\Big\|_E \approx \big\|(x_{ij})_{i,j}\big\|_{\mathrm{inf}}$$

(2) Assume that $q_E < \infty$. If E is 2-convex or if $p_E > 2$, then for x_{ij} in $E(M)$,

$$\Big\|\sum_{i,j}\varepsilon_i\otimes\varepsilon_j\otimes x_{ij}\Big\|_E \approx \big\|(x_{ij})_{i,j}\big\|_{\max}$$

- The case when $E = L^p$ (for $1 \le p < \infty$) corresponds to the results by Haagerup/Pisier and Pisier mentioned above.

Interpolation

- Let $1 \leq p \leq q \leq \infty$. We say that $\mathbf{E} \in \mathbf{Int}(\mathbf{L^p}, \mathbf{L^q})$ provided that for any

$$T\colon L^p(0,\infty) + L^q(0,\infty) \longrightarrow L^p(0,\infty) + L^q(0,\infty)$$

such that $T\colon L^p \to L^p$ and $T\colon L^q \to L^q$ boundedly, then

$$T\colon E \longrightarrow E \qquad \text{boundedly.}$$

Theorem (Ovcinnikov, 1970, Arazy, 1978, P. Dodds/T. Dodds/De Pagter, 1992).

Assume $E \in \mathrm{Int}(L^p, L^q)$ and let

$$T\colon L^p(M) + L^q(M) \longrightarrow L^p(N) + L^q(N)$$

be any linear operator such that

$$T\colon L^p(M) \longrightarrow L^p(N) \qquad \text{and} \qquad T\colon L^q(M) \longrightarrow L^q(N)$$

are bounded. Then T maps $E(M)$ into $E(N)$, with an estimate

$$\|T\colon E(M) \to E(N)\| \leq C \max\big\{\|T\|_{p\to p}\,,\, \|T\|_{q\to q}\big\},$$

- Our general assumption ensures that $E \in \mathrm{Int}(L^1, L^\infty)$ and that for any $T\colon L^p(0,\infty) + L^q(0,\infty) \to L^p(0,\infty) + L^q(0,\infty)$, we have

$$\|T\colon E \to E\| \leq \max\bigl\{\|T\|_{1\to 1}\,,\ \|T\|_{\infty\to\infty}\bigr\}.$$

In this case, the above Theorem holds true with $C = 1$.

- **Boyd indices II.** For any measurable $f\colon (0,\infty) \to \mathbb{C}$, set

$$[\sigma_t f](s) = f\Bigl(\frac{s}{t}\Bigr), \qquad t, s > 0.$$

Then each σ_t is bounded on $E(0,\infty)$ and

$$\|\sigma_t\colon E \longrightarrow E\| \leq \max\{1, t\}.$$

For any $1 \leq p < \infty$,

$$\bigl\|\sigma_t f\bigr\|_p^p = \int_0^\infty \Bigl|f\Bigl(\frac{s}{t}\Bigr)\Bigr|^p\, ds \ = t\|f\|_p^p,$$

hence

$$\|\sigma_t\colon L^p \to L^p\| = t^{\frac{1}{p}}.$$

By definition, p_E is the smallest p such that

$$\forall\, t > 1, \qquad \|\sigma_t\colon E \longrightarrow E\| \geq t^{\frac{1}{p}},$$

and q_E is the biggest q such that

$$\forall\, t < 1, \qquad \|\sigma_t \colon E \longrightarrow E\| \geq t^{\frac{1}{q}}.$$

- We deduce that

$$E \in \mathrm{Int}(L^p, L^q) \implies p \leq p_E \leq q_E \leq q.$$

- Conversely, **Boyd's Theorem** (1971) asserts that $E \in \mathrm{Int}(L^p, L^q)$ provided that

$$1 \leq p < p_E \leq q_E < q \leq \infty.$$

Starting Lemma

- The proof of the first part of Theorem 1 is an adaptation of the Lust-Piquard/Pisier proof of the Khintchine inequalities on noncommutative L^1-spaces.

The starting point of the latter is the following equivalence property (Pisier, 1978).

Let $\mathbb{T}$ be the unit circle, let $e_n \in L^\infty(\mathbb{T})$ defined by $e_n(t) = e^{int}$ for $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. Then for any Banach space X, we have

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)} \approx \Big\|\sum_k e_{3^k} \otimes x_k\Big\|_{L^1(\mathbb{T};X)}$$

for finite families $x_1, \ldots, x_n$ in X.

Lemma. For any E as before, we have an equivalence

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{E(L^\infty(\Omega)\overline{\otimes}M)} \approx \Big\|\sum_k e_{3^k} \otimes x_k\Big\|_{E(L^\infty(\mathbb{T})\overline{\otimes}M)}$$

for finite families $x_1, \ldots, x_n$ in $E(M)$.

Proof.

One can replace e_{3^k} by $\eta_k\colon t \mapsto \cos(3^k t)$.

Recall that $E \in \mathrm{Int}(L^1, L^\infty)$.

Then it suffices to show that for any $n \geq 1$, there exist two linear mappings

$$T\colon L^1(\Omega; L^1(M)) + L^\infty(\Omega)\overline{\otimes} M \ \longrightarrow\ L^1(\mathbb{T}; L^1(M)) + L^\infty(\mathbb{T})\overline{\otimes} M$$

and

$$S\colon L^1(\mathbb{T}; L^1(M)) + L^\infty(\mathbb{T})\overline{\otimes} M \ \longrightarrow\ L^1(\Omega; L^1(M)) + L^\infty(\Omega)\overline{\otimes} M$$

such that

$$\|T\|_{1\to 1} \leq 1, \qquad \|T\|_{\infty\to\infty} \leq 1,$$

$$\|S\|_{1\to 1} \leq 1, \qquad \|S\|_{\infty\to\infty} \leq 1,$$

and for any $x_1, \ldots, x_n$ in $L^1(M) + M$,

$$T\Big(\sum_{k=1}^{n} \varepsilon_k \otimes x_k\Big) \ = \ \sum_{k=1}^{n} \eta_k \otimes x_k$$

and

$$S\Big(\sum_{k=1}^{n} \eta_k \otimes x_k\Big) \ = \ \sum_{k=1}^{n} \varepsilon_k \otimes x_k.$$

We consider

$$K(\theta,t) \;=\; \prod_{k=1}^{n}\bigl(1+\varepsilon_k(\theta)\,\eta_k(t)\bigr), \qquad \theta\in\Omega,\ t\in\mathbb{R}.$$

Then $K(\theta,t)\geq 0$ for any θ,t. For any $A\subset\{1,\ldots,n\}$, set

$$\varepsilon_A=\prod_{k\in A}\varepsilon_k \qquad \text{and} \qquad \eta_A=\prod_{k\in A}\eta_k,$$

with the conventions $\varepsilon_\emptyset=1$ and $\eta_\emptyset=1$. Then

$$K(\theta,t) \;=\; \sum_A \varepsilon_A(\theta)\,\eta_A(t),$$

hence

$$\forall\,\theta,\quad \int_{\mathbb{T}} K(\theta,t)\,\tfrac{dt}{2\pi} \;=1 \qquad \text{and} \qquad \forall\,t,\quad \int_{\Omega} K(\theta,t)\,dm(\theta) \;=1.$$

We define

$$[T(f)](t) \;=\; \int_{\Omega} K(\theta,t)f(\theta)\,dm(\theta)$$

and

$$[S(g)](\theta) \;=\; \int_{\mathbb{T}} K(\theta,t)g(t)\,\tfrac{dt}{2\pi}\,.$$

These maps are contractive both on L^1 and L^∞. Indeed,

$$\|T\|_{1\to1} \;=\; \sup_{\theta}\bigl\|K(\theta,\cdot\,)\bigr\|_{L^1(\mathbb{T})}, \quad \text{etc...}$$

Now for any $x \in L^1(M) + M$ and any $k = 1, \ldots, n$,

$$T(\varepsilon_k \otimes x) = \sum_A \Big(\int_\Omega \varepsilon_A \varepsilon_k \, dm \Big) \, \eta_A \otimes x$$

Moreover,

$$\begin{aligned} \int_\Omega \varepsilon_A \varepsilon_k \, dm &= 1 \quad \text{if } A = \{k\}, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Consequently, $T(\varepsilon_k \otimes x) = \eta_k \otimes x$.

Likewise,

$$\begin{aligned} \int_{\mathbb{T}} \eta_A \eta_k &= \tfrac{1}{2} \quad \text{if } A = \{k\}, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Hence $S(\eta_k \otimes x) = \frac{1}{2} \, \varepsilon_k \otimes x$.

Other variables

In all the above statements (except the previous "starting" lemma), one can replace the Rademacher variables by one of the following sequences of commutative or noncommutative variables:

Gaussian variables. A sequence $g_1, \ldots, g_n, \ldots$ of independent standard Gaussian variables.

Steinhaus variables. The sequence of functions $U_n(t) = e^{it_n}$ on $\mathbb{T}^\infty$, where $t = (t_n)_{n\geq 1}$.

Free generators. The sequence $(\lambda(c_n))_{n\geq 1}$ on $VN(G)$, where $G = \mathbb{F}_\infty$ is the free group generated by a sequence $(c_n)_{n\geq 1}$, and $\lambda\colon G \to VN(G) \subset B(\ell^2_G)$ is the left regular representation.

Fermions. A sequence $(w_n)_{n\geq 1}$ of operators defined as

$$w_n = v_n + v_n^*,$$

where the v_n's satisfy the Canonical Anticommutation Relations:

$$v_i v_j^* + v_j^* v_i = \delta_{ij} \quad \text{and} \quad v_i v_j + v_j v_i = 0.$$