Spectral Action in NCG

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The question :

Why NCG in Physics ?

Spectral theory and quantum physics

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gravitational field
Equation of motion for matter in the field

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gravitational field

Equation of motion for matter in the field

Problems with

physical meaning of coordinates x^μ

Equation of field

The question :

Assume general covariance, then

given the tetrad-field $e^I_\mu(x)$: a solution,

(I for inertial reference frame)

after a diffeomorphism ϕ (active or passive),

$$e'^I_\nu(\phi(x)) = \frac{\partial x^\mu}{\partial \phi(x)^\nu}\, e^I_\mu(x) \quad \text{is another solution}$$

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What about Invariance of laws in all reference frames ?

When Relativity became General, points disappeared !

Only fields on fields exist : NO GIVEN SPACETIME.

Moral : Quantum forces and GR $\longrightarrow$ Spectral approach

The question :

Another argument :

Observables and notion of infinitesimals

Cannot coexist in classical spacetime.

Can coexist in algebraic setting

Infinitesimal are just compact operators !

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Another argument :

$\sum_{n \in \mathbb{N}^*} \frac{1}{n}$ diverges logaritmically like $\int \frac{1}{r}$

Dixmier trace is not a gadget

The spectral action :

$$(\mathcal{A}, \mathcal{H}, \mathcal{D})$$

dimension d

- J , reality operator (Tomita) $\longleftrightarrow$ charge conjugation
- $\chi \longleftrightarrow$ chirality

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Manifolds : $\int_M scal(x)\sqrt{det(g_x)}\, dx$

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What could be an action ?

Manifolds : $\int_M scal(x)\sqrt{det(g_x)}\, dx$

$$\mathcal{S}(\mathcal{D}, \Lambda, \Phi) := \mathrm{Tr}\left(\Phi(\mathcal{D}/\Lambda)\right)$$

$\Phi > 0$ even function
($\Phi =$ step function, number of eigenvalues of $\mathcal{D}^2 < \Lambda^2$)

Computation of $\mathcal{S}(\mathcal{D}, A, \Phi)$

Inner fluctuations : A one-form in $\Omega^1_{\mathcal{D}}(\mathcal{A})$

$$A = \sum_{finite} a_i\,[\mathcal{D},\ b_i], \quad a_i,\ b_i \in \mathcal{A}$$

$$\mathcal{D} \longleftrightarrow \mathcal{D}_A := \mathcal{D} + \widetilde{A},$$
$$\widetilde{A} := A + \epsilon J A J^{-1}$$

$\epsilon = \pm 1$ defined by $\mathcal{D}J = \epsilon J \mathcal{D}$.

Commutative Case :

$$\mathcal{D}_A = \mathcal{D}, \quad \forall A = A^*$$

Computation of $\mathcal{S}(\mathcal{D}, A, \Phi)$

Depends only on the spectrum
Gauge invariant

1. Heat Kernel Approach :

Computation of $\mathcal{S}(\mathcal{D}, A, \Phi)$

Depends only on the spectrum
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1. Heat Kernel Approach :

   Constraints on Φ :

   - Step function not allowed !
   - Distributional approach (Estrada–Gracia-Bondia–Varilly)
   - $\Phi \in \mathcal{S}(\mathbb{R}^+)$ is a Laplace transform of

   $\hat{\psi} \in \mathcal{S}(\mathbb{R}^+) = \{\, g \in \mathcal{S}(\mathbb{R}) \,:\, g(x) = 0,\, x \leq 0 \,\}$ (Nest–Vogt–Werner)

   Then Φ has analytic extension on the right complex plane :

$$\Phi(z) = (-1)^m \int_0^\infty e^{-tz}\, t^m\, \hat{\psi}(t)\, dt, \quad Re(z) > 0.$$

$$\text{Tr}\left(\Phi(\mathcal{D}/\Lambda)\right) = \sum_{k=0}^{d} \Phi_k\, a_k\, \Lambda^{d-k} + \mathcal{O}(\Lambda^{-1})$$

Computation of $\mathcal{S}(\mathcal{D}, A, \Phi)$

$$\mathrm{Tr}\left(\Phi(\mathcal{D}/\Lambda)\right) = \sum_{k=0}^{d} \Phi_k \, a_k \, \Lambda^{d-k} + \mathcal{O}(\Lambda^{-1})$$

a_k : Seeley-De Witt coefficients :
on a manifold, $e^{-t\mathcal{D}^2}$ is tracial with smooth Schwartz kernel $k_t(x, y)$

$$k_t(x, y) \simeq_{t\to 0} \frac{1}{(4\pi \, t)^{d/2}} \sqrt{det(g_x)} \sum_{n\geq 0} k_n(x, y) \, t^n \, e^{-d(x,y)^2/4t}$$

$$\begin{aligned}\Phi_k &= \frac{1}{\Gamma(m-k)} \int_0^\infty \Phi(t) \, t^{m-1-k} \, dt, \quad k = 0, 1, \dots, m-1 \\ &= (-1)^k \, \phi^{(k-m)}(0), \quad k = m, \dots, d.\end{aligned}$$

$m = d/2$
Different entrance points :

Heat kernels, Zeta functions, Dixmier traces

Computation of $\mathcal{S}(\mathcal{D}, A, \Phi)$

2. Zeta function approach

Computation of $\mathcal{S}(\mathcal{D}, A, \Phi)$

2. Zeta function approach

   • **Commutative case** : **Pseudodifferential operators**

   $P \in \Psi DO$ of order q

$$s \in \mathbb{C} \longrightarrow \zeta_P(s) = \mathrm{Tr}(P|\mathcal{D}|^{-s})$$

is holomorphic for $\mathrm{Re}(s) > q + d$,
with at most poles at integers $k \leq q + d$.
Leading residue is

$$\mathrm{Res}_{s=d+q}\ \zeta_P(s) = c_1\, \mathrm{Tr}_{Dix}(P) = c_2 \int_{S^*(M)} \sigma_P$$

(Guillemin, Wodzicki,..., with $\mathcal{D}^2 = \triangle$)

Computation of $\mathcal{S}(\mathcal{D}, A, \Phi)$

• **Abstract case** : Subleading residues

$(\mathcal{A}, \mathcal{H}, \mathcal{D})$ dim d :

$$0 < \mathrm{Tr}_{Dix}(a|\mathcal{D}|^{-d}) < \infty, \quad \forall a \in \mathcal{A}^+$$

When $\mathcal{D}$ non invertible,

$$\mathcal{D} \longleftrightarrow \mathcal{D} + P_0, \quad P_0 : \text{Proj. on Ker}\mathcal{D}$$

Construction of ΨDO :

$$OP^0 = \{\, T : t \to e^{it|\mathcal{D}|}\, T\, e^{-it|\mathcal{D}|} \in C^\infty\big(\mathbb{R}, \mathcal{B}(\mathcal{H})\big) \,\}$$
$$OP^\alpha = \{\, T : T|\mathcal{D}|^{-\alpha} \in OP^0 \,\}, \quad \alpha \in \mathbb{R}$$

Computation of $\mathcal{S}(\mathcal{D}, A, \Phi)$

Definition : $T \in \Psi DO$

$$\exists\, d \in \mathbb{Z}, \text{such that } \forall N \in \mathbb{N},$$

$$\begin{aligned}
&\exists\, p \in \mathbb{N}_0\\
&\exists\, P \in \text{Alg}\{\,\mathcal{A}, J\mathcal{A}J^{-1}, \mathcal{D}, |\mathcal{D}|\,\}\\
&\exists\, R \in OP^{-N}\\
&P|\mathcal{D}|^{-2p} \in OP^{d}\\
&T = P|\mathcal{D}|^{-2p} + R
\end{aligned}$$

(p, P, R depend on N).

Idea : work modulo large N.

Computation of $\mathcal{S}(\mathcal{D}, A, \Phi)$

Properties :

- ΨDO is an algebra.
- $⨍ T := \mathrm{Res}_{s=0}\, \zeta_{\mathcal{D}}^{T}(s) = \mathrm{Res}_{s=0}\, \mathrm{Tr}\left(T\,|\mathcal{D}|^{-s}\right), \quad T \in \Psi DO$

$a|\mathcal{D}|^{-(d+\epsilon)} \in \mathcal{L}^1(\mathcal{H})$, $\forall a \in \mathcal{A}$, $\forall \epsilon > 0$

$⨍$ is a trace on ΨDO.

3. Spectrum dimension

$$Sd(\mathcal{A}, \mathcal{H}, \mathcal{D}) := \{\, \text{poles of } \mathrm{Tr}\left(T|\mathcal{D}|^{-s}\right) : T \in \Psi DO \,\}$$

Example : M compact spin^c Riemannian manifold of dimension d
$\mathcal{A} = C^\infty(M)$, $\mathcal{H} = L^2(Spineurs)$, $\mathcal{D} = -i\gamma^\mu \partial_\mu$

$$Sd(\mathcal{A}, \mathcal{H}, \mathcal{D}) = \{\, d - k : k \in \mathbb{N}_0 \,\}.$$

Computation of $\mathcal{S}(\mathcal{D}, A, \Phi)$

Remark :

Carey–Rennie–Sedaev–Sukochev (2006) :

$0 \leq T \in \mathcal{B}(\mathcal{H})$, $T^s \in \mathcal{L}^1(\mathcal{H})$ for all $s > 1$, then, if $l = \lim_{\epsilon \to 0} \epsilon \,\text{Tr}(T^{1+\epsilon})$ exists, then $\text{Tr}_{Dix}(T) = l$

Conclusion :

$$\begin{aligned} \mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) &= \sum_{0<k\in Sd^+} \Phi_k \, \Lambda^k ⨍ |\mathcal{D}_A|^{-k} + \Phi(0)\, \zeta_{D_A}(0) + \mathcal{O}(\Lambda^{-1}) \\ &= \ ? \end{aligned}$$

NC-torus of dimension d

$$\mathcal{A} := \{\, a = \sum_{k\in\mathbb{Z}^d} a_k\, U_k \;:\; (a_k)_k \in \mathcal{S}(\mathbb{Z}^d)\,\}$$

$$U_k\, U_q = e^{-ik.\Theta q}\, U_q\, U_k$$

Θ skew symmetric real $d \times d$-matrix.

U_k are unitaries

$\tau(a) := a_0$ is a trace giving by GNS $\mathcal{H}_\tau$

$$\mathcal{H} := \mathcal{H}_\tau \otimes \mathbb{C}^{2^m}, \quad m = \lfloor d/2 \rfloor$$

Natural derivations : $\delta_\mu, \quad \mu \in \{\, 1, \ldots, d\,\}$

$$\delta_\mu U_k := ik_\mu\, U_k$$

Dirac operator

$$\mathcal{D} := -i\delta_\mu \otimes \gamma^\mu$$

Reality operator : $J_0(a) := a^*$, define C by

$C\gamma^\mu = -\epsilon\gamma^\mu C,\ \epsilon = \pm 1,\ C^2 = \pm 1$

$$J = J_0 \otimes C$$

NC-torus of dimension d

$(\mathcal{A}, \mathcal{H}, \mathcal{D}, J)$ is a regular spectral triple

Hermitian one-form $\quad A := \sum_i a_i[\mathcal{D}, b_i],\ a_i,\ b_i \in \mathcal{A}$

$$\begin{aligned}\mathcal{D}_A &= \mathcal{D} + A + \epsilon JAJ^{-1} \\ &= -i\big(\delta_\mu + L(A_\mu) - R(A_\mu)\big) \otimes \gamma^\mu, \quad A_\mu = -{A_\mu}^*\end{aligned}$$

Theorem (Essouabri-B.I.-Levy-Sitarz, J. Noncommut. Geom. 2008)
1) Sd(Nc-torus)$= \{\, d-k \ :\ k \in \mathbb{N}_0 \,\}$ and all poles are simple.
2) $\zeta_D(0) =$dim Ker$(\mathcal{D})$.

NC-torus of dimension d

HIC SUNT DRACONES

TRUE if Θ is badly approximable :

Diophantine condition : $\exists u \in \mathbb{Z}^d$, $\exists \delta > 0$

$$|q.\Theta u - p| > c|q|^{-\delta}, \quad \forall 0 \neq q \in \mathbb{Z}^d, \quad \forall p \in \mathbb{Z}$$

Such $\Theta u's$ have zero Lebesgue measure.

Due to J : Control holomorphy of Hurwitz–Epstein Zeta functions.

NC-torus of dimension d

Result :

$$s \in \mathbb{C} \to f_a(s) := \sum_{0 \neq k \in \mathbb{Z}^d} \frac{P(k)}{||k||^s} \, e^{i2\pi \, k.a}$$

$a \in \mathbb{R}^d$, $P \in \mathbb{C}[x_1, \dots, x_d]$ homogeneous polynomial of degree p,
$||k|| = \sqrt{k_1{}^2 + \dots k_d{}^2}$.
- When $a \in \mathbb{Z}^d$, f_a has meromorphic extension to $\mathbb{C}$.
f_a not entire $\Longleftrightarrow$ $\text{Res}_{s=d+p} \, f_a(s) = \int_{u \in S^{d-1}} P(u) \, dS(u) \neq 0$
- When $a \in \mathbb{R}^d \setminus \mathbb{Z}^d$, f_a has meromorphic extension to $\mathbb{C}$.
- When Θ is diophantine, for any $q > 0$,

$$g(s) := \sum_{l \in (\mathbb{Z}^d)^q} c(l) \, f_{\Theta(\sum_i l_i)}(s), \quad c(l) \in \mathcal{S}(\mathbb{Z}^d)^q$$

has only one pole on $s = d + p$ with

$$\underset{s=d+n}{\text{Res}} \, g(s) = c \int_{u \in S^{d-1}} P(u) \, dS(u)$$

NC-torus of dimension d

Examples :

$$\mathrm{Res}_{s=0} \sum_{k\in\mathbb{Z}^2} \frac{k_i k_j}{||k||^{s+4}} = \delta_{i,j}\pi$$

$$\mathrm{Res}_{s=0} \sum_{k\in\mathbb{Z}^4} \frac{k_i k_j k_l k_m}{||k||^{s+6}} = (\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})\frac{\pi^2}{12}$$

etc with more similar results ...
Appear in ζ-regularization, multiplicative anomaly, Casimir effect, ...

Chamseddine–Connes (2006) extended :
For any one-form A, the constant term in Λ

$$\zeta_{\mathcal{D}_A}(0) - \zeta_{\mathcal{D}}(0) = -\int\!\!\!\!\!- \log(1 + \widetilde{A}\mathcal{D}^{-1})$$
$$= \sum_{k=1}^{d} \frac{(-1)^k}{k} \int\!\!\!\!\!- (\widetilde{A}\mathcal{D}^{-1})^k$$

NC-torus of dimension d

Result

$d = 2$:

$$\mathcal{S}(\mathcal{D}_A, \Lambda, \phi) = 4\pi\, \phi_2\, \Lambda^2 + \mathcal{O}(\Lambda^{-2})$$

$d = 4$:

$$\mathcal{S}(\mathcal{D}_A, \Lambda, \phi) = 8\pi^2\, \phi_4\, \Lambda^4 - \tfrac{4\pi^2}{3}\, \tau(F_{\mu\nu} F^{\mu\nu}) + \mathcal{O}(\Lambda^{-2})$$

More generally, $\forall d \geq 1$,

$$\mathcal{S}(\mathcal{D}_A, \Lambda, \phi) = \sum_{k=0}^{d} \phi_{d-k}\, c_{d-k}(A)\, \Lambda^{d-k} + \mathcal{O}(\Lambda^{-1})$$

$c_{d-1}(A) = 0$, $c_{d-k}(A) = 0$ for k odd (d odd $\Rightarrow c_0(A) = 0$)

Conjecture : Noncommutative coefficients of $\mathcal{D} + \widetilde{A} \simeq$ coefficients of $\mathcal{D} + A$ for the commutative torus

NC-torus of dimension d

Remarks :

- No tadpole here : $⨍ \widetilde{A}\mathcal{D}^{-1} = 0$
- For general spectral triples (with simple dimension spectrum)

$⨍$ can be defined with $\mathcal{D}$ or $\mathcal{D}_A = \mathcal{D} + \widetilde{A}$:

$$⨍ P = \text{Res}_{s=0} \text{ Tr}\left(P|\mathcal{D}_A|^{-s}\right), \quad P \in \Psi DO$$

- Top term (cosmological term) : $⨍ |\mathcal{D}_A|^{-d} = ⨍ |\mathcal{D}|^{-d}$
- Constant term depends on :

dim Ker ($\mathcal{D}_A$)- dim Ker ($\mathcal{D}$)

Beyond Diophantine condition

Case $d = 2$, $\quad \Theta = \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \theta \in \mathbb{R}$

Let $f : [1, \infty[\to]0, \infty[$, continuous, $x^2 f(x)$ non-increasing

$$\mathcal{K}(f) := \{\, \theta \in \mathbb{R} \,:\, |\theta - \tfrac{p}{q}| < f(q), \text{ for infinitely many rational numbers } \tfrac{p}{q} \,\}$$

Such θ are termed f-approximable.
(Not valid for all rational $\frac{p}{q}$ since $(\theta q)_{q \geq 1}$ are dense in $[0, 1]$ when $\theta \notin \mathbb{Q}$)

Result (Jarnik 1953)
For each f, there exists an uncountable set $\mathcal{K}(f)$ of real numbers $\frac{1}{2\pi}\theta$, f-approximable but not $c\,f$-approximable for any $0 < c < 1$.
$\mathcal{K}(f)$ has zero Lebesgue measure if $\sum_{q=1}^{\infty} q\, f(q)$ converges and full Lebesgue measure otherwise.

Consequences : Tuning f, $\exists\, a, b \in \mathcal{A}$ such that correction term

$$\mathrm{Tr}\left(L(a)\, R(b)\, e^{-t\mathcal{D}^2}\right) - \left(\cdots\right)_{Dioph}$$

- is not exponentially small
- is not $\mathcal{O}(\frac{1}{t})$ like for $\frac{1}{2\pi}\theta \in \mathbb{Q}$
- is of arbitrary order !

Extension to non-compact case

M non compact connected complete Riemannian spin^c manifold with bounded curvature and control of its heat kernel $K_t(.,.)$

$$\sup_{p\in M}\int_0^\infty t^k\, e^{-t} K_t(p,p)\, dt < \infty, \quad \forall k > \tfrac{d}{2} - 1$$

$$\sup_{p\in M}\int_m^\infty \tfrac{e^{-t}}{\sqrt{t}}\, K_t(p,p)\, dt < c\, m^{-(d-1)/2}, \quad \forall m \in [0,\, 1]$$

Valid :

- M has Ricci curvature bounded from below.
- Positive injectivity radius and control of isoperimetric constants of balls of a given radius.
- Bounded geometry.

M has a smooth isometric proper action α of $\mathbb{R}^l$

Example : Moyal planes $M = \mathbb{R}^{2n}$, $\alpha =$ translation

$$f \star_\hbar g(x) := \int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} f(x - \tfrac{\Theta}{\hbar} u)\, g(x - v)\, e^{-i\, u\cdot v}\, du\, dv$$

Extension to non-compact case

Result (Gayral, B.I. & Várilly, JFA 2006) : When $f \in C_c^\infty(M)$

$$\fint L_f |\mathcal{D}|^{-d} = \fint M_f |\mathcal{D}|^{-d} = c \int_M f$$

Remarks :
- When M is compact, α being proper must be periodic.
- When M is non compact, $\mathcal{D}$ has a continuous spectrum but $L_f(\mathcal{D} - \lambda)^{-1}$ is compact.

Result (Gayral, B.I.,Gracia-Bondía, Schücker & Várilly, CMP 2004) :

Moyal planes are spectral triples.

Spatial regularization

Choose $\rho \in \mathcal{A}$

$$\mathcal{S}(\mathcal{D}_A, \Lambda, \Phi, \rho) := \mathrm{Tr}\left(\rho\, \Phi(\mathcal{D}/\Lambda)\right) \tag{1}$$

Result (Gayral,I., JMP 2005)
$\mathcal{S}(\mathcal{D}_A, \Lambda, \Phi, \rho)$ has same coefficients as before with

$$c_k(A) \longleftrightarrow \int_{\mathbb{R}^d} \rho(x)\, c_k(A)(x)\, dx$$

Remark :
(1) is not satisfactory : two many choices !
Chamseddine–Connes (2006) : Dilaton field ϕ

$$\mathcal{D} \longleftrightarrow e^{-\phi}\, \mathcal{D}\, e^{-\phi}, \quad \phi = \phi^* \in Z(\mathcal{A})$$

$$N(\Lambda) = \dim \{\, \mathcal{D}^2 \leq \Lambda^2 \,\} \longleftrightarrow N(\phi) = \dim \{\, \mathcal{D}^2 \leq \phi^2 \,\}$$

THE question

PART II

NOT **RIGHT**

even

POSSIBLY **WRONG**

Computation of $\mathcal{S}(\mathcal{D}_A, \Lambda, \Phi)$

Conclusion of part I :

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = \sum_{0<k\in Sd^+} \Phi_k \, \Lambda^k \fint |\mathcal{D}_A|^{-k} + \Phi(0)\, \zeta_{D_A}(0) + \mathcal{O}(\Lambda^{-1})$$
$$= \;?$$

LOCALITY : USUAL TRACE COMPUTED VIA RESIDUES !

Computation of $\mathcal{S}(\mathcal{D}_A, \Lambda, \Phi)$

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = \sum_{0<k\in Sd^+} \Phi_k \, \Lambda^k \fint |\mathcal{D}_A|^{-k} + \Phi(0)\, \zeta_{D_A}(0) + \mathcal{O}(\Lambda^{-1}).$$

Only few examples :

Standard models : CHAMSEDDINE–CONNES–MARCOLLI,
KASTLER–B.I–SCHÜCKER–STEPHAN
GRACIA-BONDIA–VARILLY

Moyal planes : GAYRAL–B.I–VASSILEVICH

Noncommutative torus : ESSOUABRI–B.I–LEVY–SITARZ

Isospectral deformations of toric manifolds : GAYRAL–B.I–VARILLY

Moyal planes + gauge theory : GROSSE–WULKENHAAR

How to proceed ?

Hochschild cohomology approach

For $n \in \mathbb{N}^*$ and $a_i \in \mathcal{A}$, define

$$\phi_n(a_0, \cdots, a_n) := \fint a_0[\mathcal{D}, a_1]\mathcal{D}^{-1} \cdots [\mathcal{D}, a_n]\mathcal{D}^{-1}.$$

$da := [\mathcal{D}, a]$. On the universal n-forms $\Omega_u^n(\mathcal{A})$

$$\int_{\phi_n} a_0 da_1 \cdots da_n := \phi_n(a_0, a_1, \cdots, a_n).$$

$b - B$ bicomplex defined by Connes : b is the Hochschild coboundary map

$$b\phi(a_0, \dots, a_{n+1}) := \sum_{j=0}^{n} (-1)^j \phi(a_0, \dots, a_j a_{j+1}, \dots, a_{n+1}) + (-1)^{n+1}\phi(a_{n+1}a_0, a_1, \dots, a_n),$$

$$B_0\phi_n(a_0, a_1, \dots, a_{n-1}) := \phi_n(1, a_0, \dots, a_{n-1}).$$

$B := NB_0$, where $N := 1 + \lambda + \ldots + \lambda^n$ is the cyclic skewsymmetrizer on the n-cochains and $\lambda\phi(a_0, \dots, a_n) := (-1)^n \phi(a_n, a_0, \dots, a_{n-1})$.

Spectral action in 3-dimension

Result : If $(\mathcal{A}, \mathcal{D}, \mathcal{H})$ is a spectral triple of dimension 3, then

$$b\phi_1 = -\phi_2, \quad b\phi_2 = 0, \quad b\phi_3 = 0,$$
$$B\phi_1 = 0, \quad B\phi_2 = 0, \quad B\phi_3 = 3B_0\phi_3.$$

Remark : When $X \in OP^{-3}$, in $⨍ X$ one can use $[\mathcal{D}^{-1}, \mathcal{A}] = 0$.

Spectral action in 3-dimension

Scale invariant term of spectral action
Chamseddine-Connes 2006

$$\zeta_{\mathcal{D}_A}(0) - \zeta_{\mathcal{D}}(0) = -\fint A\mathcal{D}^{-1} + \tfrac{1}{2}\fint A\mathcal{D}^{-1}A\mathcal{D}^{-1} - \tfrac{1}{3}\fint A\mathcal{D}^{-1}A\mathcal{D}^{-1}A\mathcal{D}^{-1}.$$

For any spectral triple of dimension 3 and any one-form A,

$$\zeta_{\mathcal{D}_A}(0) - \zeta_{\mathcal{D}}(0) = -\tfrac{1}{2}\int_{N\phi_1} A + \tfrac{1}{2}\int_{\phi_2}(dA + A^2) - \tfrac{1}{2}\int_{\phi_3}(AdA + \tfrac{2}{3}A^3)$$
$$\simeq \text{ Tadpole } + \text{ Y.M. } + \text{ C.S. in dim 4.}$$

$SU_q(2)$

Let G Galilean group, P Poincaré group
$\forall \epsilon > 0$, there exists A_i, B_i, $i \in \{1, \dots, 10\}$ basis in Lie algebras of G and P such that

$$[A_i, A_j] =: {C_{ij}}^k A_k, \qquad [B_i, B_j] =: {C'_{ij}}^k B_k,$$
$$|{C_{ij}}^k - {C'_{ij}}^k| < \epsilon, \quad \forall i, j, k.$$

Impossible for $SU(2)$!
Woronowicz (1987) : Family of $SU_q(2)$ homeomorphic but not isomorphic pseudogroups.
Algebraic context : Quantum groups.
Idea : $q \to 1$, $C\big(SU_q(2)\big)$ goes to $C(S^3)$.
Podleś (1987) : quantum spheres S_q.
Plenty of deformations :

Hopf algebra and their representations
Differential calculi

Can NCG be useful ?

$SU_q(2)$

After many attempts (..., CHAKRABORTY-PAL,...),
DABROWSKI-LANDI-SITARZ-van SUIJLEKOM-VARILLY
$0 < q < 1$
$\mathcal{A} := \mathcal{A}\big(SU_q(2)\big)$: Polynomials in a, a^*, b, b^*

$$ba = q\,ab, \quad b^*a = q\,ab^*, \quad bb^* = b^*b, \quad a^*a + q^2\,b^*b = 1, \quad aa^* + bb^* = 1,$$

$U = \left(\begin{smallmatrix} a & b \\ -qb^* & a^* \end{smallmatrix}\right)$ is unitary.

Representations : Rep $SU_q(2) \simeq$ Rep $SU(2)$.
Thus spherical harmonics and spin representations :
Hilbert space $\mathcal{H} = \mathcal{H}^{\uparrow} \oplus \mathcal{H}^{\downarrow}$
$|j\mu n{\uparrow}\rangle$ with $j = 0, \frac{1}{2}, 1, \ldots,\ \mu = -j, \ldots, j$ and $n = -j^+, \ldots, j^+$
$|j\mu n{\downarrow}\rangle$ for $j = \frac{1}{2}, 1, \ldots,\ \mu = -j, \ldots, j$ and $n = -j^-, \ldots, j^-$
here $x^{\pm} := x \pm \frac{1}{2}$

$$|j\mu n\rangle\!\rangle := \left(\begin{smallmatrix} |j\mu n\uparrow\rangle \\ |j\mu n\downarrow\rangle \end{smallmatrix}\right)$$

Little asymmetry : $|0, \mu, n, \downarrow\rangle = |j, \mu, \pm(j + \frac{1}{2}), \downarrow\rangle = 0$

$SU_q(2)$

Dirac operator $\mathcal{D}$: Chosen to get equivariance of spectral triple for left + right action of $\mathcal{U}_q(su(2))$: usual Dirac on S^3 with round metric and same spectrum :

$$\mathcal{D}\,|j\mu n\rangle\rangle := \begin{pmatrix} 2j+\frac{3}{2} & 0 \\ 0 & -2j-\frac{1}{2} \end{pmatrix} |j\mu n\rangle\rangle.$$

Reality operator J

$$J\,|j,\mu,n,\uparrow\rangle := i^{2(2j+\mu+n)}\,|j,-\mu,-n,\uparrow\rangle, \quad J\,|j,\mu,n,\downarrow\rangle := i^{2(2j-\mu-n)}\,|j,-\mu,-n,\downarrow\rangle$$

thus it satisfies $J^{-1} = -J = J^*$ and $\mathcal{D}J = J\mathcal{D}$.
Remark : For modular conjugation J of Tomita, then $\mathcal{A}\big(SU_q(2)\big)^\circ \simeq \mathcal{A}\big(SU_{\frac{1}{q}}(2)\big)$ so $\mathcal{D}$ equivariant under $\mathcal{U}_{\frac{1}{q}}(su(2))$ as well, so $\mathcal{D}$ is a scalar : no-go result (no 3^+-summable spectral triple on $SU_q(2)$ for modular J.)

Remarks on representations

$$\pi(a)\,|j\mu n\rangle\!\rangle = \alpha^+_{j\mu n}\,|j^+\mu^+n^+\rangle\!\rangle + \alpha^-_{j\mu n}\,|j^-\mu^+n^+\rangle\!\rangle$$

$$\alpha^+_{j\mu n} = \sqrt{q^{\mu+n-\frac{1}{2}}}\,\sqrt{[j+\mu+1]}\begin{pmatrix} q^{-j-\frac{1}{2}}\frac{\sqrt{[j+n+\frac{3}{2}]}}{[2j+2]} & 0 \\ q^{\frac{1}{2}}\frac{\sqrt{[j-n+\frac{1}{2}]}}{[2j+1][2j+2]} & q^{-j}\frac{\sqrt{[j+n+\frac{1}{2}]}}{[2j+1]} \end{pmatrix}$$

with $[x] := \frac{q^x - q^{-x}}{q-q^{-1}}$.
Similar relations for $\alpha^-_{j\mu n}$ and b, a^*, b^*.
Off-diagonal terms : $\quad \alpha^+_{j\mu n\uparrow\downarrow} = \mathcal{O}(q^{2j+1}), \quad \alpha^-_{j\mu n\uparrow\downarrow} = \mathcal{O}(q^{2j})$.

$L_q\,|j\mu n\rangle\!\rangle := q^j\,|j\mu n\rangle\!\rangle$, $\qquad \mathcal{K}_q$ ideal generated by L_q.

Result : $\mathcal{K}_q \subset OP^{-\infty}$ and for $\underline{\pi} := \pi \mod K_q$ all off diagonals disappear and $\underline{\pi}$ is a representation of $\mathcal{A} \mod K_q$.

Under $⨍$ no difference between π and $\underline{\pi}$.

$SU_q(2)$

$(\mathcal{A}, \mathcal{D}, \mathcal{H})$ is a regular spectral triple up to

commutant property : $[\pi(a), J\pi(b)J^{-1}] \in \mathcal{K}_q$

first order condition : $\big[\,[\mathcal{D}, \pi(a)], J\pi(b)J^{-1}\,\big] \in \mathcal{K}_q$.

Definition :

$$\underline{\pi}(a) := a_+ + a_-, \quad \underline{\pi}(b) := b_+ + b_-$$

$$a_+ \,|j\mu n\rangle\!\rangle := q_{j^+ + \mu^+} \begin{pmatrix} q_{j^+ + n^+ + 1} & 0 \\ 0 & q_{j^+ + n} \end{pmatrix} |j^+ \mu^+ n^+\rangle\!\rangle,$$

$$a_- \,|j\mu n\rangle\!\rangle := q^{2j+\mu+n+\frac{1}{2}} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} |j^- \mu^+ n^+\rangle\!\rangle,$$

$$b_+ \,|j\mu n\rangle\!\rangle := q^{j+n-\frac{1}{2}} q_{j^+ + \mu^+} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} |j^+ \mu^+ n^-\rangle\!\rangle,$$

$$b_- \,|j\mu n\rangle\!\rangle := -q^{j+\mu} \begin{pmatrix} q_{j^+ + n} & 0 \\ 0 & q_{j^- + n} \end{pmatrix} |j^- \mu^+ n^-\rangle\!\rangle.$$

where $q_x := \sqrt{1 - q^{2x}}$.

Grading : Due to the shift $j \to j^\pm$:

Degree 1 : $a_+,\ b_+,\ a_-^*,\ b_-^*$, Degree -1 : $a_-,\ b_-,\ a_+^*,\ b_+^*$.

If $X :=$ *-algebra $\{\, a_\pm,\ b_\pm \,\}$, $T \in X$,

$T^\circ :=$ 0-degree part of T for this grading.

$SU_q(2)$

2 other representations : On the basis $\{\,\varepsilon_n : n \in \mathbb{Z}\,\}$ of $\ell^2(\mathbb{Z})$

$$\pi_\pm(a)\,\varepsilon_n := q_{n+1}\,\varepsilon_{n+1}, \qquad \pi_\pm(b)\,\varepsilon_n := \pm q^n\,\varepsilon_n.$$

$\mathcal{A}$ is a Hopf-*-algebra for a certain coproduct :

$$^*\text{-homomorphism } r : X \to \pi_+(\mathcal{A}) \otimes \pi_-(\mathcal{A})$$

$$\begin{aligned} r(a_+) &:= \pi_+(a) \otimes \pi_-(a), & r(a_-) &:= -q\,\pi_+(b) \otimes \pi_-(b^*), \\ r(b_+) &:= -\pi_+(a) \otimes \pi_-(b), & r(b_-) &:= -\pi_+(b) \otimes \pi_-(a^*). \end{aligned}$$

Symbol map : *-homomorphism $\sigma : \pi_\pm(\mathcal{A}) \to C^\infty(S^1)$ defined for $z \in S^1$ on the generators by

$$\sigma\big(\pi_\pm(a)\big)(z) := z, \quad \sigma\big(\pi_\pm(b)\big)(z) = 0.$$

Noncommutative integrals : $\displaystyle ⨍ T := \operatorname*{Res}_{s=0} \zeta_{\mathcal{D}}^T(s)$ where $\zeta_{\mathcal{D}}^T(s) := \operatorname{Tr}(T|\mathcal{D}|^{-s})$.

$SU_q(2)$

Theorem (DLSvSV) : The dimension spectrum of $\big(\mathcal{A}(SU_q(2)), \mathcal{H}, D\big)$ is $\{1, 2, 3\}$. Its KO-dimension is 3. Moreover, for $T \in$ algebra $\{\, \delta^k(\pi(\mathcal{A})) \,:\, k \in \mathbb{N}\,\}$

$$\textstyle\fint T|\mathcal{D}|^{-3} = 2(\tau_1 \otimes \tau_1)\big(r(T)^\circ\big),$$

$$\textstyle\fint T|\mathcal{D}|^{-2} = 2\big(\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1\big)\big(r(T)^\circ\big),$$

$$\textstyle\fint T|\mathcal{D}|^{-1} = \big(2\,\tau_0 \otimes \tau_0 - \tfrac{1}{2}\,\tau_1 \otimes \tau_1\big)\big(r(T)^\circ\big),$$

$$\textstyle\fint F\,T|\mathcal{D}|^{-3} = 0, \qquad \fint F\,T|\mathcal{D}|^{-2} = 0,$$

$$\textstyle\fint F\,T|\mathcal{D}|^{-1} = \big(\tau_0 \otimes \tau_1 - \tau_1 \otimes \tau_0\big)\big(r(T)^\circ\big),$$

with for $x \in \pi_\pm(\mathcal{A})$, $\tau_0(x) := \lim_{N\to\infty}\big(\operatorname{Tr}_N x - (N+1)\,\tau_1(x)\big)$, $\tau_1(x) := \frac{1}{2\pi}\int_0^{2\pi} \sigma(x)(e^{i\theta})\,d\theta$, and $\operatorname{Tr}_N x = \sum_{n=0}^{N}\langle \varepsilon_n, x\,\varepsilon_n\rangle$.

Computation of spectral action of $SU_q(2)$

$$\mathcal{S}(\mathcal{D}_{\mathcal{A}}, \Phi, \Lambda) = \sum_{1\leq k\leq 3} \Phi_k \, \Lambda^k ⨍ |D_{\mathcal{A}}|^{-k} + \Phi(0)\, \zeta_{D_{\mathcal{A}}}(0), \qquad (2)$$

$\Phi_k = \frac{1}{2} \int_0^\infty \Phi(t)\, t^{k/2-1}\, dt.$

Only 4 terms !

Totally different from commutative case of S^3 : terms in $\Lambda^3, \Lambda^1, \Lambda^{-1}, \Lambda^{-3}, \ldots$

Conclusion : no way to compare spectral action when $q \to 1$!

Tadpole : $⨍ A\mathcal{D}^{-1} \neq 0.$

$$⨍ a[\mathcal{D}, a^*]\mathcal{D}^{-1} = \frac{q^2+3}{2(q^2-1)}, \qquad ⨍ a^*[\mathcal{D}, a]\mathcal{D}^{-1} = \frac{3q^2-1}{2(q^2-1)}.$$

No tadpole on S^3 !

Top degree in (2) : $⨍ |\mathcal{D}_A|^{-3} = ⨍ |\mathcal{D}| = 2,\ \forall A \in \Omega^1(\mathcal{A})$ (general property.)

Computation of spectral action of $SU_q(2)$

Full computations of spectral action has been obtained for any A.

$$\fint |D_{\mathbb{A}}|^{-3} = 2\,,$$
$$\fint |D_{\mathbb{A}}|^{-2} = -8\, T_a(A),$$
$$\fint |D_{\mathbb{A}}|^{-1} = -\tfrac{1}{2} + 4\left(T_a(A^2) + T_a^2(A) - T(A)\right),$$

$\zeta_{D_{\mathbb{A}}}(0) = P_L(A) + P_Q(A) + P_C(A)$ where

$$P_L(A) = -4\, TP(A) + T_a(A)\,,$$
$$P_Q(A) = 2\, T(A^2) + 4\, T_a(A)\big(T(A) - T_a(A^2)\big)\,,$$
$$P_C(A) = -\tfrac{4}{3}\, T_a(A^3)\,.$$

Computation of spectral action of $SU_q(2)$

$$\mathcal{S}(\mathcal{D}_{a^*da}, \Phi, \Lambda) \,=\, 2\,\Phi_3\,\Lambda^3 - 8\,\Phi_2\,\Lambda^2 + \big(-\tfrac{1}{2} + \tfrac{8}{1-q^2}\big)\,\Phi_1\,\Lambda^1 + \big(\tfrac{4q^4}{1-q^4} - \tfrac{12q^2}{1-q^2} - \tfrac{13}{3}\big)\,\Phi(0).$$

Depends on q !
Has pole at roots of unity !

Main difficulty : algebraic use of the basis !

Differential calculus on $SU_q(2)$

Remarkable 1-form :

$$\pi(a^*)\, d\pi(a) + q^2\,\pi(b)\, d\pi(b^*) + q^2\,\pi(a)\, d\pi(a^*) + q^2\,\pi(b^*)\, d\pi(b) = \xi(\mathcal{D}),$$

where $\xi(s) = q\frac{[2s]-2s}{[s+\frac{1}{2}][s-\frac{1}{2}]}$.
In the $q = 0$ limit, this is exactly $F = \text{sign}(\mathcal{D})$.

$$F = \tfrac{1}{1-q^2}\,\left(\underline{\pi}(a^*)\, d\underline{\pi}(a) + q^2\,\underline{\pi}(b)\, d\underline{\pi}(b^*) + q^2\,\underline{\pi}(a)\, d\underline{\pi}(a^*) + q^2\,\underline{\pi}(b^*)\, d\underline{\pi}(b)\right),$$

Since $[F, \underline{\pi}(\mathcal{A})] = 0$, F is a central one-form up to $OP^{-\infty}$.

But : The order one-calculus is not universal (up to $OP^{-\infty}$).
F is not closed since, forgetting $\underline{\pi}$,

$$da^*\, da + q^2\, da\, da^* + q^2\, db^*\, db + q^2\, db\, db^* = -(1+q^2).$$

Differential calculus on $SU_q(2)$

Idea : to work modulo some ideal :
Let Ψ_0^0 the algebra $\{\, \delta^k\big(\pi(\mathcal{A})\big)$ and $\delta^k\big([\mathcal{D},\, \pi(\mathcal{A})]\big), k \in \mathbb{N}\,\}$.
$\exists$ pseudodifferential operators T of order ≤ 0 such that

$$\begin{aligned}\int\!\!\!\!\!\!- \, T\,T'\,|D|^{-2} = \int\!\!\!\!\!\!- \, T'\,T\,|D|^{-2} = \int\!\!\!\!\!\!- \, T\,T'\,|D|^{-3} = \int\!\!\!\!\!\!- \, T'\,T\,|D|^{-3} \\ = 0, \qquad \forall T' \in \Psi_0^0(\mathcal{A}).\end{aligned} \tag{3}$$

In fact, if $R =^*$-ideal generated by $\{\, a_-,\, b_- b_+,\, b_- b_+^*\,\}$
$\mathcal{R} = R + RF$ is a *-ideal in $\Psi^0(\mathcal{A})$ satisfies (2) and invariant by F, d.
Up to $\mathcal{R}$,

$$\begin{gathered} a\,da \simeq da\,a, \quad a^*\,da \simeq -da^*\,a, \quad b\,da \simeq q\,da\,b, \\ b^*\,da \simeq q\,da\,b^*, \quad a\,da^* \simeq -da\,a^*, \ldots \end{gathered}$$

Any selfadjoint one-form A can be presented as :
$$A \simeq xda - da^*\,x^* + y\,db - db^*\,y^* \text{ with } x,\, y \in \mathcal{A}.$$
This reduces computations

Differential calculus on $SU_q(2)$

Open question for higher forms :

What is the junk ?

$$\mathcal{J} := \{ \sum_i d\pi(x_i)\, d\pi(y_i) \;:\; \sum_i \pi(x_i)\, d\pi(y_i) = 0,\, x_i,\, y_i \in \mathcal{A} \}$$

necessary to compute 2-forms

$$\Omega^2_{\mathcal{D}}(\mathcal{A}) := \{ \sum_i d\pi(x_i)\, d\pi(y_i) \;:\; x_i,\, y_i \in \mathcal{A} \} \,/\, \mathcal{J}\,.$$

Idea : A "nice" Dirac operator should select a "nice" differential calculus.

Conclusion and perspectives

Difficulty to compare : wall between classical and quantum $SU(2)$.

Spectral action has rather peculiar properties.

Podleś spheres S_q : sulbalgebras of $SU_q(2)$ and $S_q \simeq SU_q(2) \,/\, U(1)$.

Some uniqueness of the differential calculus exists in some cases.

Quantum field theory on these triples will be to poor !

Generalization to 4-dimensional quantum Euclidean sphere S_q^4 (D'ANDREA-DABROWSKI-LANDI) : isospectral deformation of S^4 with $U_q\big(so(5)\big)$-invariance.

Interest : canonical instanton bundle on S_q^4.

Towards $SU_q(2)$-instantons on S_q^4.