Tensor products of Banach spaces

Marcel de Jeu April 13, 2022

Today: $C_0(S)$-spaces. $\mathcal{L}_\infty$-spaces. $L_1(\mu)\hat{\otimes}_\varepsilon X$

Context:

Projective	Injective
1°. respects quotients (2.5)	. does not respect quotients
2°. does not respect subspaces	. respects subspaces (p. 47)
3°. if X is $\mathcal{L}_{1,1+\varepsilon}$ for every $\varepsilon>0$, then $\cdot\hat{\otimes}_\pi X$ respects subspaces isometrically (proof of 2.20)	. if X is $\mathcal{L}_{\infty,1+\varepsilon}$ for every $\varepsilon>0$, then $\cdot\hat{\otimes}_\varepsilon X$ respects quotients (today: 3.6(1))
Part. case: $L_1(\mu)$ for arbitrary μ.	Part. case: $C_0(S)$ for S loc cpt Hd.
4°. μ σ-finite and complete:	. μ finite and complete
$L_1(\mu)\hat{\otimes}_\pi X \cong L_1(\mu,X)$ (2.19)	$L_1(\mu)\hat{\otimes}_\varepsilon X \cong$ completion of Pettis integrable functions (3.13) today
5°. $C_0(S)\hat{\otimes}_\pi X \cong\ ?$	. $C_0(S)\hat{\otimes}_\varepsilon X \cong C_0(S,X)$ p. 49/50 S loc cpt Hd today

Start with: $C_0(S) \hat{\otimes}_\varepsilon X \cong C_0(S, X)$ isometrically
S loc cpt Hd.

Recall: $\mathrm{Map}(S, \mathbb{F}) \otimes X \hookrightarrow \mathrm{Map}(S, X)$
(p. 11) $\quad f \otimes x \longmapsto \{ s \mapsto f(s) x \}$

So, if S is topological space: $J : C_0(S) \otimes X \longrightarrow C_0(S, X)$
$f \otimes x \longmapsto \{ s \mapsto f(s) x \}$

<u>Claim:</u> if $u \in C_0(S) \otimes_\varepsilon X$, then $\| J u \| = \varepsilon(u)$.

<u>Proof:</u> If $u \in Y \otimes Z$, $u = \sum y_i \otimes z_i$, then

$$\varepsilon(u) = \sup_{y^* \in N} \left\{ \| \sum y^*(y_i) z_i \|_Z \right\} \text{ if } N \subset Y^*$$

is norming for Y. Here: $\{ ev_s : s \in S \}$ is norming for $C_0(S)$. Hence, if $u = \sum f_i \otimes x_i$:

$$\begin{aligned} \varepsilon(u) &= \sup_{s \in S} \| \sum_i ev_s(f_i) \cdot x_i \|_X \\ &= \sup_{s \in S} \| \sum_i f_i(s) x_i \|_X \\ &= \| J u \| \end{aligned}$$

$\square$

Standard fact (in a few moments): $J(C_0(S) \otimes X)$ is dense in $C_0(S, X)$. (Even: $J(C_c(S) \otimes X)$ is dense in $C_0(S, X)$).

The following is now immediate

<u>Thm 1</u> Let S be loc cpt Hd, X Banach. Then the map $f\otimes x \mapsto \{s \mapsto f(s)x\}$ yields an isometric isomorphism $\tilde{J}$ between $C_0(S)\hat{\otimes}_\varepsilon X$ and $C_0(S,X)$.

<u>Corollary 2</u>: Let S_1, S_2 be loc. cpt Hd. Then the map $f_1\otimes f_2 \longmapsto \{(s_1,s_2)\mapsto f_1(s_1)f_2(s_2)\}$ yields an isometric isomorphism between

$C_0(S_1)\hat{\otimes}_\varepsilon C_0(S_2)$ and $C_0(S_1\times S_2)$.

<u>Proof</u>: For general topological spaces S_1 and S_2, $\Psi: C_0(S_1, C_0(S_2)) \xrightarrow{\sim} C_0(S_1\times S_2)$ isometrically as follows: if $\varphi\in C_0(S_1, C_0(S_2))$, let

$[\Psi(\varphi)](s_1,s_2) = [\varphi(s_1)](s_2)$ (exercise)

So if S_1 and S_2 are loc cpt Hd, then $\Psi\circ\tilde{J}$ is an isometric isomorphism between $C_0(S_1)\hat{\otimes}_\varepsilon C_0(S_2)$ and $C_0(S_1\times S_2)$.

If $f_1 \otimes f_2 \in C_0(S_1) \otimes C_0(S_2)$ then

$$[\Phi \circ \tilde{J}(f_1 \otimes f_2)](s_1, s_2) = \left([\tilde{J}(f_1 \otimes f_2)](s_1) \right)(s_2)$$

$$= \left(f_1(s_1) f_2 \right)(s_2) = f_1(s_1) f_2(s_2)$$

$\square$

<u>Remark</u>: special case of Thm 1:

$$c_0 \hat{\otimes}_\varepsilon X = C_0(\mathbb{N}) \hat{\otimes}_\varepsilon X \cong C_0(\mathbb{N}, X) = c_0(X) \qquad (2.2)$$

Still needed: density result. See Appendix for:

<u>Prop A.1</u> Let S be loc cpt Hd, $f_1, \dots, f_n \in C_0(S, X)$ and $\varepsilon > 0$. Then there exist $\varphi_1 \dots \varphi_m \in C_c(S)$ and points $s_1 \dots s_m$, $t_1 \dots t_m \in S$ such that:

1° $0 \leq \varphi_i \leq 1$ $(i = 1, \dots, m)$

2° $\sum_{i=1}^{m} \varphi_i \leq 1$

3° $\varphi_i(s_j) = \delta_{ij}$

4° $\| f_j - \sum_{i=1}^{m} \varphi_i f_j(t_i) \| < \varepsilon$ $(j = 1, \dots, n)$.

If S is cpt, 2° can be replaced with

$\sum_{i=1}^{m} \varphi_i = 1$

Remarks

(a) Take $n=1$: $J(C_c(S)\otimes X)$ is dense in $C_0(S,X)$.

(b) $\mathrm{Span}\{\varphi_1 \ldots \varphi_m\} \cong \ell_\infty^m$ isometrically.

Namely: if ~~$\varphi \in \mathrm{Span}$~~ $\varphi = \sum_{i=1}^m a_i \varphi_i$, let

$X(\varphi) = (ev_{s_1}(\varphi), \ldots, ev_{s_m}(\varphi)) = (a_1, \ldots, a_m)$.

Then $X: \mathrm{Span}\{\varphi_1 \ldots \varphi_n\} \cong \ell_\infty^m$ algebraically,

and in addition:

$|\varphi(s)| \leq \sum |a_i| \varphi_i(s) \leq (\max_i |a_i|) \sum_i \varphi_i(s)$

$\leq \max_i |a_i|$, so $\|\varphi\| \leq \|X(\varphi)\|$.

and for all $i=1, \ldots, m$:

$\|\varphi\| \geq |\varphi(s_i)| = |a_i|$, so $\|\varphi\| \geq \|X(\varphi)\|$.

Will use this to show that $C_0(S)$ is $\mathcal{L}_{\infty, 1+\varepsilon}$-space

for all $\varepsilon > 0$.

Definition: X is an $\mathcal{L}_{\infty,\lambda}$-space if, for all finite

dimensional $M \subset X$ there exist f.d. $N \supset M$

and $T : N \to \ell_\infty^{\dim N}$ invertible s.t.

$\|T\| \cdot \|T^{-1}\| \leq \lambda$ (i.e. $\exists N \supseteq M$ such that

$d(N, \ell_\infty^{\dim N}) < \lambda$ where d is Banach-Mazur

distance).

Need:

<u>Theorem A.3</u> Let X be a Banach space. Suppose

$\{x_1, \dots, x_n\} \subset X$ is linearly independent, and $\varepsilon > 0$.

Then there exists $\delta > 0$ such that, for all

$\{y_1, \dots, y_n\} \subset X$ with $\max_i \|y_i - x_i\| < \delta$, and

all f.d. $M \supset \operatorname{span}\{y_1, \dots, y_n\}$:

1° $\{y_1, \dots, y_n\}$ is linearly independent.

2° $\exists N \supset \operatorname{span}\{x_1, \dots, x_n\}$ with $\dim N = \dim M$

and $d(M, N) < 1 + \varepsilon$.

Combining this with Prop A1 and Remark (b)

on p. 5 this yields:

<u>Thm 3</u> If S is loc cpt Hd, then $C_0(S)$ is an $\mathcal{L}_{\infty,1+\varepsilon}$-space for all $\varepsilon > 0$.

<u>Proof</u>: Let $M \subset C_0(S)$ be given with $\dim M < \infty$. Say $\{f_1, \dots, f_n\}$ is a basis. Pick $\delta > 0$ for this basis as in Theorem A2 and next apply Proposition A1 for $\varepsilon = \delta$. In the notation of Proposition A1, let $g_j = \sum_{i=1}^{m} f_j(t_i)\varphi_i$, so $\|f_j - g_j\| < \delta$ $(j = 1, \dots, n)$. Then Theorem A2 shows that, for all f.d. $M' \supset \operatorname{span}\{g_1 \dots g_n\}$ there exists $N \supset \operatorname{span}\{f_1, \dots, f_n\} = M$ with

$$d(N, M') < 1 + \varepsilon.$$

Apply this with $M' = \operatorname{span}\{\varphi_1 \dots \varphi_m\} \underset{\text{isometrically}}{\cong} \ell^m_\infty$.

$\square$

Relevance ~~for~~ of $\mathcal{L}_{\infty,\lambda}$-spaces for $\mathcal{Q}_\lambda$ is as follows:

Thm 4 Let $Q: Z \to Y$ be a quotient operator

(a) If X is an $\mathcal{L}_{\infty,\lambda}$-space for some $\lambda>1$, then

$I \otimes_\varepsilon Q: X \hat{\otimes}_\varepsilon Z \to X \hat{\otimes}_\varepsilon Y$ is surjective

(so: $X \hat{\otimes}_\varepsilon Y \cong X \hat{\otimes}_\varepsilon Z /_{\ker I \otimes_\varepsilon Q}$ ~~...~~ linear homeomorphism)

(b) If X is an $\mathcal{L}_{\infty, 1+\varepsilon}$-space for all $\varepsilon>0$, then

$I \otimes_\varepsilon Q: X \hat{\otimes}_\varepsilon Z \to X \hat{\otimes}_\varepsilon Y$ is a quotient

operator

(so: $X \hat{\otimes}_\varepsilon Y \cong X \hat{\otimes}_\varepsilon Z /_{\ker I \otimes_\varepsilon Q}$ isometrically).

For this we need the following consequences of

Lemma A4:

Lemma 5: X, Y Banach, $Q: X \to Y$ bounded

(a) If, for all $\varepsilon>0$ and all y in a dense

subset of Y, there exists $x \in X$ such that

$\|y - Qx\| < \varepsilon$ and $\|x\| \le (1+\varepsilon)\|y\|$, then Q is surjective

(b) If, in addition $\|Q\| \leq 1$, then $\hat{Q}$ is a quotient operator.

Proof of Theorem 4

Want to apply Lemma 5 to $I \otimes_\varepsilon Q : X \hat{\otimes}_\varepsilon Z \to X \hat{\otimes}_\varepsilon Y$ with dense subspace $X \otimes Y$ of $X \hat{\otimes}_\varepsilon Y$.

So let $u \in X \otimes Y$ and $\varepsilon > 0$. Pick fin. dim M such that $u \in M \otimes Y$ and next select $N \supset M$ s.t. $\dim N = n < \infty$ and $T : N \to \ell_\infty^n$ with $\|T\| . \|T^{-1}\| < \lambda$. May assume $\|T\| \leq 1$ & $\|T^{-1}\| < \lambda$.

Then $T \otimes_\varepsilon I : N \hat{\otimes}_\varepsilon Y \to \ell_\infty^n \hat{\otimes}_\varepsilon Y$ is an isomorphism (linear homeomorphism).

Since $\ell_\infty^n \hat{\otimes}_\varepsilon Y = C(\{1, \ldots, n\}) \hat{\otimes}_\varepsilon Y$

$\cong C(\{1, \ldots, n\}, Y)$ isometrically

via $\sum_{i=1}^n e_i \otimes y_i \mapsto \{ i \mapsto y_i \}$

we have $\varepsilon\left(\sum_{i=1}^{n} e_i \otimes y_i\right) = \max_i \|y_i\|$ in $\ell_\infty^n \hat{\otimes}_\varepsilon Y$

and likewise in $\ell_\infty^n \hat{\otimes}_\varepsilon Z$. Hence the quotient operator

$Q : Z \to Y$ induces a quotient operator

$I \otimes_\varepsilon Q : \ell_\infty^n \hat{\otimes}_\varepsilon Z \to \ell_\infty^n \hat{\otimes}_\varepsilon Y$.

It is easy to see (cf. Corollary A 5.4) that then

there exists $w \in \ell_\infty^n \hat{\otimes}_\varepsilon Z$ with $I \hat{\otimes}_\varepsilon Q(w) = T \otimes_\varepsilon I(u)$

and $\varepsilon(w) \leq (1+\varepsilon)\, \varepsilon(T \otimes_\varepsilon I(u))$.

Put $v = T^{-1} \otimes_\varepsilon I(w) \in N \hat{\otimes}_\varepsilon Z \subset X \hat{\otimes}_\varepsilon Z$.

Then:

- $(I \otimes_\varepsilon Q)v = (I \otimes_\varepsilon Q)(T^{-1} \otimes_\varepsilon I)(w) = (T^{-1} \otimes_\varepsilon Q)(w)$

$$= (T^{-1} \otimes_\varepsilon I)(I \otimes_\varepsilon Q)(w)$$

$$= (T^{-1} \otimes_\varepsilon I)(T \otimes_\varepsilon I)u$$

$$= u$$

and

- $\varepsilon(v) \leq \|T^{-1} \otimes_\varepsilon I\| \, \varepsilon(w) = \|T^{-1}\| \, \varepsilon(w) \leq \lambda (1+\varepsilon) \cdot \varepsilon(T \otimes_\varepsilon I u)$

$\|T\| = 1$

$\leq \lambda(1+\varepsilon)\,\varepsilon(u)$

We conclude:

- $I \otimes_\varepsilon \lambda Q \left(\frac{v}{\lambda}\right) = u$
- $\varepsilon\left(\frac{v}{\lambda}\right) \leq (1+\varepsilon)\,\varepsilon(u)$.

Lemma 5(a) now implies that $I \otimes_\varepsilon \frac{Q}{\lambda}$ is surjective, hence so is $I \otimes_\varepsilon Q = \frac{1}{\lambda} I \otimes_\varepsilon Q$.

This establishes (a). As to (b), we apply Lemma 5(b), again for the dense subset $X \otimes Y$ of $X \hat{\otimes}_\varepsilon Y$. Certainly $\|I \otimes_\varepsilon Q\| = \|Q\| \leq 1$. ~~The condition in Lemma 5(a) is satisfied if u ...~~ For general $u \in X \otimes Y$ the proof of part (a) shows that for all $\lambda > 1$ and $\delta > 0$, there exists $v_{\lambda,\delta} \in X \hat{\otimes}_\varepsilon Z$ with $I \hat{\otimes}_\varepsilon Q(v_{\lambda,\delta}) = u$ and $\varepsilon(v_{\lambda,\delta}) \leq \lambda(1+\delta)\,\varepsilon(u)$.

If $\varepsilon > 0$ is given, selecting $\lambda > 1$ and $\delta > 0$ such that $\lambda/(1-\delta) < 1+\varepsilon$ shows that the condition in Lemma 5(a) is satisfied

□

<u>Corollary 6</u>: Let S be a loc. cpt. Hd space. If $Q: X \to Y$ is a quotient operator, then so is $I \otimes_\varepsilon Q: C_0(S) \hat{\otimes}_\varepsilon X \to C_0(S) \hat{\otimes}_\varepsilon Y$.

<u>Proof</u>: Combine Thm 2 and Thm 4

□

<u>Now</u>: (Ω, Σ, μ) finite measure space. Assume μ to be complete (because results from Section 2.2 are used, where this is assumed – possibly this is not necessary).

<u>Goal</u>: find model for $L_1(\mu) \hat{\otimes}_\varepsilon X$.

<u>Answer</u>: completion of μ-measurable Pettis integrable functions $f: \Omega \to X$ in the Pettis norm (or actually: equivalence classes of such functions).

Suppose $f: \Omega \to X$ is weakly integrable, i.e., $\varphi \circ f$ is integrable for all $\varphi \in X^*$. (Bochner integrable f would do, for example). Then we have a linear map

$$\begin{aligned} \int_f : X^* &\to L_1(\mu) \\ \varphi &\mapsto \varphi \circ f \end{aligned}$$

<u>Claim</u> S_f is bounded.

⌈ Suppose $\varphi_n \to \varphi$ in X^* and $S_f \varphi_n = \varphi_n \circ f \to g$ in $L_1(\mu)$. Then there is a subsequence (φ_{n_k}) such that $\varphi_{n_k} \circ f \to g$ a.e. Hence, for almost $\omega \in \Omega$, $g(\omega) = \lim_{k\to\infty} (\varphi_{n_k} \circ f)(\omega) = \lim_{k\to\infty} \varphi_{n_k}(f(\omega))$
$= \varphi(f(\omega)) = (\varphi \circ f)(\omega) = (S_f \varphi)(\omega)$.

Now apply the Closed Graph Theorem ⌋

Consider the adjoint $S_f^* : L_\infty(\mu) \to X^{**}$, so

$$\langle \varphi, S_f^* g \rangle = \langle S_f \varphi, g \rangle \qquad (\varphi \in X^*, g \in L_\infty(\mu))$$

$$= \int_\Omega g \cdot (\varphi \circ f)\, d\mu.$$

Take $g = \chi_E$ and write $\int_E f\, d\mu$ for $S_f^* \chi_E$.
$(E \in \Lambda)$

Hence

$$\left\langle \varphi, \int_E f\, d\mu \right\rangle = \int_E \varphi \circ f\, d\mu \qquad \left(\begin{array}{l}\varphi \in X^* \\ E \in \Lambda\end{array}\right)$$

$\int_E f\, d\mu$ is the <u>Dunford integral of f over E</u>.

It is in X^{**}.

<u>Def.</u> f (weakly int.) is <u>Pettis integrable</u> if

$$\int_f^* \chi_E = \int_E f\,d\mu \in X \quad (\subset X^{**}) \text{ for all } E \in \Lambda$$

<u>Note</u>: If f is Bochner integrable, then f is Pettis integrable, and $(B)-\int_\Omega f\chi_E\,d\mu = (P)-\int_E f\,d\mu$

because $(B)-\int_\Omega f\chi_E$ is in X and it satisfies the defining equation for $(P)-\int_E f\,d\mu$:

$$\langle \varphi, (B)-\int_\Omega f\chi_E\,d\mu \rangle = \int_\Omega \varphi \circ (f\chi_E)\,d\mu$$

$$= \int_\Omega \chi_E \cdot (\varphi \circ f)\,d\mu = \int_E \varphi \circ f\,d\mu$$

<u>Example</u> ~~that~~ where f is weakly L_1, but not Pettis integrable. Take $(\Omega, \Lambda, \mu) = [0,1]$ with Lebesgue measure, and $X = c_0$. (so $X^* = \ell_1$)

Choose $[0,1] = \bigcup_{n=1}^{\infty} E_n$ with $\mu(E_n) > 0$ for all n and define $f : [0,1] \to c_0$ by

$f(t) = \sum_{n=1}^{\infty} \frac{\chi_{E_n}(t)}{\mu(E_n)} \cdot e_n$ (one non-zero term for each t)

If $\varphi = \sum_{1}^{\infty} \varphi_n e_n \in \ell_1$, then

$(\varphi \circ f)(t) = \sum_{1}^{\infty} \frac{\varphi_n}{\mu(E_n)} \chi_{E_n}(t)$, which implies that

f is weakly integrable since $\varphi \in \ell_1$, and

$\int_{[0,1]} \varphi \circ f \, d\mu = \sum_{1}^{\infty} \varphi_n = \langle \varphi, e \rangle$ with

$e = (1,1,1,\ldots) \in \ell_\infty = c_0^{**}$. Since $e \notin c_0$,

the Dunford integral $\int_{[0,1]} f \, d\mu$ is not in c_0.

<u>Note</u>: $\int_E f \, d\mu = S_f^* \chi_E \in X$

$\iff S_f^* g \in X$ for all $g \in L_\infty(\mu)$,

since X is closed in X^{**} and the simple functions are dense in $L_\infty(\mu)$.

Lemma 7 If $f : \Omega \to X$ is Pettis integrable, and $g \in L_\infty(\mu)$, then $g \cdot f$ is Pettis integrable and

$$\int_E g \cdot f \, d\mu = \mathcal{J}_f^{*}(g \chi_E) \qquad (E \in \Lambda)$$

In particular: $\displaystyle\int_E \chi_F f \, d\mu = \int_{E \cap F} f \, d\mu \quad (g = \chi_F)$

$$\int_\Omega g \cdot f \, d\mu = \mathcal{J}_f^{*} g \qquad (E = \Omega)$$

Proof: Certainly $g \cdot f$ is weakly integrable since g is measurable and essentially bounded. If $\varphi \in X^*$ and $E \in \Lambda$ we compute:

$$\begin{aligned}
\langle \varphi, \mathcal{J}_{gf}^{*} \chi_E \rangle &= \langle \mathcal{J}_{gf} \varphi, \chi_E \rangle \\
&= \int_\Omega (\varphi \circ (gf)) \chi_E \, d\mu \\
&= \int_{\Omega} g \cdot (\varphi \circ f) \, \cancel{d\mu} \, \chi_E \, d\mu \\
&= \langle \mathcal{J}_f \varphi, g \chi_E \rangle \\
&= \langle \varphi, \mathcal{J}_f^{*}(g \chi_E) \rangle
\end{aligned}$$

We know that $\mathcal{J}_f^{*}(g \chi_E) \in X$ since f

is Pettis integrable and $g\chi_E \in L_\infty(\mu)$. Hence
$\int^*_{gf} \chi_E = \int^*_f (g\chi_E)$ is in X, and
gf is Pettis integrable. Choosing $g = \chi_F$ or
~~g=~~ $E = \Omega$ yields the special cases.

□

<u>Prop 8</u> Let $f : \Omega \to X$ be Pettis integrable and
$R : X \to Y$ be bounded. Then $R \circ f$ is Pettis
integrable and, for $E \in \Lambda$,
$$\int_E R \circ f \, d\mu = R\left(\int_E f \, d\mu \right).$$

<u>Proof</u>: Let $y \in Y^*$. Then $y \circ (R \circ f) = (y \circ R) \circ f$
$= \underbrace{(R^* y)}_{\in X^*} \circ f$, so $R \circ f$ is weakly integrable.

Furthermore, for $E \in \Lambda$:
$$\left\langle y, \int^*_{R \circ f} \chi_E \right\rangle = \left\langle \int_{R \circ f} y, \chi_E \right\rangle$$
$$= \left\langle y \circ (R \circ f), \chi_E \right\rangle = \left\langle \underbrace{(R^* y)}_{\text{in } X^*} \circ f, \chi_E \right\rangle$$

$= \langle \oint_f (R^* y), \chi_E \rangle = \langle R^* y, \int_f^* \chi_E \rangle$

$\overset{f \text{ Pettis}}{=} \langle R^* y, (P\text{-})\int_E f \, d\mu \rangle = \langle y, \underbrace{R\,(P\text{-})\int_E f \, d\mu}_{\in Y} \rangle$

so $\oint_E R \circ f$ is Pettis integrable and

$\int_E R \circ f \, d\mu = R \left(\int_E f \, d\mu \right)$.

<u>Remark</u>: Hence $S^*_{R \circ f}(\chi_E) = (R \circ S_f^*)(\chi_E)$.

By continuity: $S^*_{R \circ f}(g) = (R \circ S_f^*)(g)$

$g \in L_\infty(\mu)$!

<u>Def / notation</u>: Let $f : \Omega \to X$ be Pettis integrable.

Write $T_f : L_\infty(\mu) \to X$ for $S_f^* : L_\infty(\mu) \to X \subset X^{**}$

and define

$|f|_1 = \|S_f\| = \|S_f^*\| = \|T_f\|$: the

<u>Pettis semi norm</u> of f.

<u>Note</u>: · Pettis int. functions form vector space

and $f \mapsto T_f$ is linear, so

$|.|_1$ is a seminorm.

- If $R: X \to Y$ is bounded, then

$$|R \circ f|_1 = \|T_{R\circ f}\|_? = \|R \circ T_f\| \leq \|R\|\|T_f\| = \|R\| \, |f|_1 .$$

Hence two ways to compute $|f|_1$:

$$|f|_1 = \|S_f\| = \sup_{\|\varphi\|\leq 1} \|S_f \varphi\| = \sup \left\{ \int_\Omega |\varphi \circ f| \, d\mu ; \ \varphi \in B_{X^*} \right\} \quad (*)$$

and

$$|f|_1 = \|S_f^*\| = \sup_{\|g\|_\infty \leq 1} \|S_f^* g\| = \sup_{\|g\|_\infty \leq 1} \|\int_\Omega^* fg\| = \sup_{\|g\|_\infty\le 1} \left\{ \| \int_\Omega g \cdot f \, d\mu \| ; \ \|g\|_\infty \leq 1 \right\}$$

Remark

1° f Bochner integrable $\Rightarrow$ f Pettis integrable and $(*)$ shows that $|f|_1 \leq \|f\|_1$.

2° Example of Pettis integrable function which is μ-measurable, but not Bochner-integrable.

$X = c_0$, (Ω, Σ, μ) is Lebesgue measure on $[0,1]$.
Let $I_n = (\frac{1}{n+1}, \frac{1}{n}]$ (so that $\mu(I_n) = \frac{1}{n(n+1)}$)
and put $f : [0,1] \to c_0$

$$t \mapsto \sum_{n=1}^{\infty} n \chi_{I_n} e_n.$$

For $\varphi \in c_0^* = \ell_1$, say $\varphi = \sum_1^\infty \varphi_n e_n$, we have
$(\varphi \circ f)(t) = \sum_1^\infty n \chi_{I_n}(t) \varphi_n$. Hence f is
weakly measurable. By the Pettis Measurability Theorem
and the separability of c_0, f is μ-measurable.
For $\varphi \in \ell_1$ we have

$$\int_{[0,1]} |\varphi \circ f| \, d\mu = \int_{[0,1]} \sum_1^\infty n \chi_{I_n}(t) |\varphi_n| \, dt$$

$$= \sum_1^\infty n |\varphi_n| \cdot \frac{1}{n(n+1)} < \infty, \text{ Hence } f \text{ is}$$

weakly integrable and for E measurable,

$$\int_E \varphi \circ f \, d\mu = \sum_{n=1}^\infty n \varphi_n \mu(I_n \cap E)$$

$$= \langle \varphi, (1\mu(I_1 \cap E), 2\mu(I_2 \cap E), \dots) \rangle$$

Since $n\mu(I_n \cap E) \leq n \cdot \frac{1}{n(n+1)} = \frac{1}{n+1}$, the sequence above is in c_0. We conclude that f is Pettis integrable, and that

$$\int_E f \, d\mu = (1\mu(I_1 \cap E), 2\mu(I_2 \cap E), \ldots). \quad (E \in \mathcal{A})$$

However: $\|f(t)\| = \sum_{n=1}^{\infty} n \chi_{I_n}(t)$, so

$$\int_{[0,1]} \|f\| \, d\mu = \sum_1^{\infty} n \cdot \frac{1}{n(n+1)} = \infty.$$ Hence f is not Bochner integrable.

Note: If $f = 0$ a.e., then $T_f = 0$, so $A(f) = \|T_f\| < \infty$. The converse is not true!

Example: $[0,1]$ with Lebesgue measure.

$$X = c_0(I) \quad (\text{so } X^* = \ell_1(I)).$$

Let $f : \Omega \to X$, $t \mapsto e_t$. Then for $\varphi \in X^*$,

$\varphi \circ f = 0$ a.e., so f weakly integrable, and

$\int_f = 0$. Then $\int_f^{\varphi} = 0 \Longrightarrow f$ is Pettis integrable, and $|f|_1 = \|T_f\| = 0$. Yet f is nowhere zero!

Note: If f is Pettis integrable and μ-measurable then $|f|_1 = 0 \iff f = 0$ a.e.

Proof. $\Leftarrow$ ok.

$\Rightarrow$ May assume: f is separably valued. Take $(\varphi_n) \subset X^*$ countable norming for $f(\Omega)$. If $|f|_1 = 0$ then from ⊛ p. 20 certainly $|\varphi_n(f(\omega))| = 0$ for a.a. ω. Since there are countably many φ_n, this implies that $\|f(\omega)\| = 0$ for a.a. ω. ☐

Def: $\mathcal{P}_1(\mu, X)$: eq. classes (a.e. equal) of μ-measurable Pettis integrable functions, with norm $|.|_1$.

Let $\hat{P}_1(\mu, X)$ be its completion.

<u>Thm</u>: Let (Ω, Λ, μ) be a finite <u>complete</u> measure space. Then the map

$$J : L_1(\mu) \otimes X \longrightarrow P_1(\mu, X)$$

$$f \otimes x \longmapsto \{\omega \mapsto f(\omega) x\}$$

yields an isometric isomorphism between $L_1(\mu) \hat{\otimes}_\varepsilon X$ and $\hat{P}_1(\mu, X)$.

<u>Proof</u>: $J(L_1(\mu) \otimes X)$ consists of Bochner integrable functions, which are Pettis integrable, so J is well-defined.

Let $u = \sum_i f_i \otimes x_i \in L_1(\mu) \otimes X$. Then:

$$\cancel{\| Ju \| = \sup_{\varphi \in B_{X^*}} \|}$$

$$| Ju |_1 = \| T_{Ju} \| = \sup_{\varphi \in B_{X^*}} \| T_{Ju} \varphi \|_1$$

$$= \sup_{\varphi \in B_{X^*}} \| \varphi \circ J u \|_1$$

$$= \sup_{\varphi \in B_{X^*}} \| \underbrace{\sum \varphi(x_i) f_i}_{\in L_1(\mu)} \|_1$$

$$\overset{\langle L_1^*, L_\infty \rangle}{=} \sup_{\varphi \in B_{X^*}, g \in L_\infty(\mu)} \left| \langle \sum \varphi(x_i) f_i , g \rangle \right|$$

$$= \sup_{\varphi \in B_{X^*}, g \in L_\infty(\mu)} \left| \sum \cancel{\varphi(x_i)} \langle \varphi, x_i \rangle \langle g, f_i \rangle \right|$$

$$= \varepsilon(u).$$

Since the image is dense ("to be proved shortly"), we are done.

$\Box$

<u>Aim</u>: the μ-measurable simple functions $\sum \chi_{E_i} x_i$ are dense in $P_1(\mu, X)$

(certainly in image of J).

Proof in the book is based on Prop. 2.9, which

is false (counterexample due to [Karbus Hoodé]).
We give a proof based on vector measures.

Def: (Ω, Λ, μ) given. (μ σ-add finite measure).

- $F : \Lambda \to X$ is vector measure if it is finitely additive.
- F is μ-ct if for all $\varepsilon > 0$ there exists $\delta > 0$ s.t. $\|F(E)\| < \varepsilon$ for all $E \in \Lambda$ with $\mu(E) < \delta$.

Fact (Diestel and Uhl (cor.) p. 9): a σ-additive vector measure is bounded

Thm 10 (Pettis, see DU p. 10) If F is σ-additive, then TFAE: (μ finite)

1° F is μ-ct.

2° $F(E) = 0$ whenever $\mu(E) = 0$ $(E \in \Lambda)$.

($1° \Rightarrow 2°$ is trivial)

Let F be a σ-add. vector measure (can be relaxed). Define

$\|F\| : \Lambda \to \mathbb{R} \cup \{\infty\}$ by

$\|F\|(E) = \sup \{ |x^* \circ F|(E) : x^* \in B_{X^*} \}$

where $|x^* \circ F|$ is the variation of the complex σ-add. measure $x^* \circ F$.

$\|F\|$ is the <u>semivariation of F</u>: it is monotone and subadditive. By Diestel & Uhl Prop 11, p. 9, $\|F\|(\Omega) < \infty$.

Basic inequality (D & U, p. 4):

F σ-add vector measure. Then for $E \in \Lambda$,

$\sup \{ \|F(H)\| : H \subset E, H \in \Lambda \}$

$\leq \|F\|(E)$

$\leq 4 \sup \{ \|F(H)\| : H \subset E, H \in \Lambda \}$.

Relates semivariation and norm.

Consequently, can add in Thm 10.

3° For all $\varepsilon > 0$ there exists $\delta > 0$ such that

$\| F(E) \| < \varepsilon$ whenever $\mu(E) < \delta$.

Furthermore we need:

Thm 11 (Pettis, see Dunford & Schwartz I, p 318).

If F is a vector measure and $x^* \circ F$ is σ-add. for all $x^* \in X^*$, then F is σ-additive.

Relation with our material:

Suppose $f : \Omega \to X$ is Pettis integrable.

Let $F_f : \Lambda \to X$ be defined by

$$F_f(E) = T_f \chi_E \in X \left(= \int_E f \, d\mu \right)$$

Then F_f is finitely additive $\left(\chi_{\dot\cup E_i} = \sum \chi_{E_i} \right)$

and $\mu(E) = 0$ implies $\overset{F_f(E)}{\overset{\shortparallel}{T_f \chi_E}} = \int_E f \, d\mu = \int_\Omega \chi_E f \, d\mu = 0$.

Claim: $x^* \circ F_f$ is represented by $x^* \circ f$

Namely:

$$(x^* \circ F_f)(\chi_E) = \langle x^*, T_f \chi_E \rangle$$
$$= \langle x^*, \int_E f \, d\mu \rangle$$
$$= \int_E x^* \circ f \, d\mu.$$

Since $x^* \circ f \in L_1(\mu)$, $x^* \circ f$ is σ-additive.

By Theorem 11, F_f is σ-additive.

Then 3° in Theorem 10 shows: if $\varepsilon > 0$, then there exists $\delta > 0$ such that $\|F_f\|(E) < \varepsilon$ whenever $\mu(E) < \delta$. This is what we need.

Just see: $x^* \circ F_f$ is represented by $x^* \circ f$

$\overset{PdI}{\Longrightarrow}$ $|x^* \circ F_f|$ is represented by $|x^* \circ f|$.

So:

$$|x^* \circ F_f|(E) = \int_E |x^* \circ f| \, d\mu$$
$$= \int_\Omega |x^* \circ (\chi_E f)| \, d\mu$$

and we conclude: $\|F_f(E)\| = \#|\chi_E f|_1$. (p.20)

All in all: If $\varepsilon > 0$, then there exists $\delta > 0$ s.t.
$\#|\chi_E f|_1 < \varepsilon$ whenever $\mu(E) < \delta$.

Can now prove:

Thm 12: Let (Ω, Λ, μ) be a finite complete measure space. If f is μ-measurable and Pettis integrable, then for every $\varepsilon > 0$ there exists a simple function $s = \sum_{\text{finite}} \chi_{E_i} x_i$ $(E_i \in \Lambda)$ with $|f - s|_1 < \varepsilon$.

Proof: From the above (where μ-measurability has not yet been used):

$$\lim_{\mu(E) \to 0} |\chi_E f|_1 = 0.$$

For $n = 0, 1, 2, \ldots$, let $E_n = \{\omega \in \Omega : \|f(\omega)\| \geq n\}$.
Then $\chi_{E_n} \downarrow 0$ everywhere. Since μ is finite,
Note: $\|f\|$ is measurable!

$\mu(E_n) \downarrow 0$. Let $\varepsilon > 0$ be given. Pick $\delta > 0$ such that $|\chi_E f|_1 < \frac{\varepsilon}{2}$ whenever $\mu(E) < \delta$. and next select n_0 such that $\mu(E_{n_0}) < \delta$. Then $f = f\chi_{E_{n_0}} + f\chi_{E^c_{n_0}}$. Now $f\chi_{E^c_{n_0}}$ is μ-measurable since f is. It is bounded, hence Bochner integrable, so we can pick simple function s s.t. $\|f\chi_{E^c_{n_0}} - s\|_1 < \frac{\varepsilon}{2}$. Then

$$|f - s|_1 \leq |f\chi_{E_{n_0}}|_1 + |f\chi_{E^c_{n_0}} - s|_1$$

$$< \frac{\varepsilon}{2} + \|f\chi_{E^c_{n_0}} - s\|_1 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\square$

We can use this in compactness properties of T_f:

For f Pettis integrable, have $T_f : L_\infty(\mu) \to X$
$$g \to T_f(g) = \int_\Omega fg\, d\mu.$$

<u>Thm 12</u> (μ finite <u>complete</u>)

1° If f is Pettis integrable, then T_f is weakly cpt.

2° If f is Pettis integrable and μ-measurable, then T_f is compact (even a limit of finite rank operators)

3° If f is Bochner integrable then T_f is nuclear.

<u>Proof.</u>

1° For $g \in L_\infty(\mu)$, $\varphi \in X^*$:

$T_f g = T_{gf} \chi_\Omega$, so

$$\langle \varphi, T_f g\rangle = \langle \varphi, T_{gf}\chi_\Omega\rangle = \langle \varphi\circ (gf), \chi_\Omega\rangle$$

$$= \int_\Omega \varphi\circ(gf)\, d\mu = \int_\Omega g\cdot \underbrace{(\varphi\circ f)}_{\text{in } L_1(\mu)}\, d\mu = \langle g, \varphi\circ f\rangle$$

This implies that $T_f g_\alpha \xrightarrow{w} T_f g$ in X whenever $g_\alpha \xrightarrow{w^*} g$ in $L_\infty(\mu)$. Hence $T_f : L_\infty(\mu) \to X$ is weak* to weak continuous. The Banach-Dieudonné theorem implies that T_f is weakly compact.

2° Let $s = \sum \chi_{E_i} x_i$ be a μ-measurable simple function. Then, for $g \in L_\infty(\mu)$

$$T_s g = T_{sg} \chi_\Omega = (B)\int_\Omega sg\, d\mu = (B)-\int_\Omega \sum g \chi_{E_i} x_i\, d\mu$$

$$= \sum_i \left(\int_\Omega g\, \chi_{E_i} d\mu \right) x_i .$$ Hence T_s is a finite rank operator. Picking a sequence $s_n \to f$ in $|\cdot|_1$; we see that

$$\| T_f - T_{s_n} \| = \| T_{f-s_n} \| = | f - s_n |_1 \to 0,$$

hence T_f is compact.

3° If $f \in L_1(\mu, X)_z \cong L_1(\mu) \hat{\otimes}_\pi X$, write

$f = \sum_n f_n \otimes x_n$ with $\sum_n \|f_n\|_1 \|x_n\| < \infty$. Then

$$T_f g = T_{fg} \chi_\Omega = (P)\text{-int} \int_\Omega fg \, d\mu$$

$$\overset{f) \text{ B-int}}{=} (B)\text{-int} \int_\Omega fg \, d\mu.$$

Since $\|\sum_{n \geq N} (f_n \otimes x_n) g\|_1 \leq \sum_{n \geq N} \|(f_n \otimes x_n) g\|_1$

$\leq \|g\|_\infty \sum_{n \geq N} \|f_n\|_1 \|x_n\| \to 0$ for $N \to \infty$,

we see that

$$T_f g = \sum_n \int_\Omega (f_n \otimes x_n) g \, d\mu$$

$$= \sum_n \langle f_n, g \rangle x_n$$

Hence T_f comes from $\sum_n f_n \otimes x_n \in L_\infty(\mu)' \hat{\otimes} X$

where we view $L_1(\mu) \hookrightarrow L_\infty(\mu)^*$ isometrically.

Therefore T_f is nuclear.

$\Box$

Application Orlicz-Pettis theorem.

Say $(x_n) \subset X$ is weakly subseries summable if $\sum_{k=1}^{\infty} x_{n_k}$ is weakly convergent for all subsequences (x_{n_k}). A sequence $(x_n) \subset X$ is unconditionally summable if $\sum_1^{\infty} x_{\sigma(n)}$ converges for all permutations $\sigma: \mathbb{N} \to \mathbb{N}$.

Known: uncond. summability ($\overset{B_1}{\Longleftrightarrow}$ subseries summability) $\Rightarrow$ weakly subseries summability.

Converse is also true:

Thm 13 (Orlicz-Pettis theorem). Weak subseries summability and unconditional summability are equivalent.

Proof: Suppose (x_n) is weakly subseries summable but not unconditionally summable. By Thm B, iii there exists signs (ε_n) such that $\sum \varepsilon_n x_n$

diverges. Since (x_n) is weakly subseries summable, (x_n) is weakly sign summable by Prop. 8.2.(b) and since $(\varepsilon_n x_n)$ is then also weakly sign summable, Prop 8.2.(b) again implies that $(\varepsilon_n x_n)$ is weakly subseries summable. Hence we may assume:

(x_n) is weakly subseries summable, but $\sum_1^\infty x_n$ is divergent. Then $\left(\sum_1^N x_n\right)_{N=1}^\infty$ is not Cauchy, so there exist $\varepsilon > 0$ and increasing sequences (m_k) and (n_k) with $m_k < n_k < m_{k+1}$ such that

$\|\sum_{j=m_k}^{n_k} x_j\| \geq \varepsilon$ for all k. Put $z_k = \sum_{j=m_k}^{n_k} x_k$.

Then (z_j) is weakly subseries summable (since (x_n) is), and $\|z_j\| \geq \varepsilon$ for all j.

Since $\sum_{j=1}^{\infty} z_j$ is, in particular, weakly convergent, $z_j \xrightarrow{w} 0$.

By Prop. B.2.(b), the series $\sum_{j=1}^{\infty} \varepsilon_j z_j$ is weakly convergent for all signs (ε_j). Consequently $\sum_{j=1}^{\infty} \varepsilon_j \varphi(z_j)$ is convergent for all $\varphi \in X^*$ and all signs (ε_j). Thus the sequence $(\varphi(z_j))$ is sign summable in $\mathbb{F}$, and Prop. B.1.(ii) yields that it is unconditionally convergent in $\mathbb{F}$, i.e. $\sum_j |\varphi(z_j)| < \infty$.

Hence

$$|\varphi(w - \sum \varepsilon_j z_j)| = \lim_{N\to\infty} |\varphi(\sum_{1}^{N} \varepsilon_j z_j)|$$

$$\leq \sum_{1}^{\infty} |\varphi(z_j)|.$$

By the dual form of the Uniform Boundedness Principle there exists $M > 0$ s.t.

$\|w - \sum_{1}^{\infty} \varepsilon_j z_j\| \leq M$ for all signs (ε_j).

Let $f : [0,1] \to X$ be given by

$f(t) = w\text{-}\sum_{j=1}^{\infty} r_j(t)\, z_j$

Then f is bounded. It is separably valued, since $f(t) \in \overline{\mathrm{Span}\{z_j\}}^{w} = \overline{\mathrm{Span}\{z_j\}}$.

For ~~$(x^* \circ f)$~~ $x^* \in X^*$, we have

$(x^* \circ f)(t) = \sum_{j=1}^{\infty} r_j(t)\, x^*(z_j)$, so f is weakly measurable. The Pettis Measurability Theorem implies that f is μ-measurable so by the boundedness it is Bochner integrable.

Hence $T_f : L_\infty[0,1] \to X$ is compact (even: nuclear)

We claim that $T_f r_n = z_n$ for all n.

Then done: $\left.\begin{array}{l} \|r_j\| \le 1 \\ r_j \ c.p.t. \end{array}\right\} \Rightarrow (T_f r_j) = (z_j)$

has convergent subsequence, say $z_{j_k} \to z_0$.

Then $z_{j_k} \xrightarrow{w} z_0$, but we know $z_{j_k} \xrightarrow{w} 0$.

Hence $z = 0$, contradicting $\|z_{j_k}\| \geq \varepsilon > 0$ for all k.

Remains: $T_f \eta_n = z_n$.

Recall definition:

$$\langle \varphi, T_f \eta_n \rangle = \langle S_f \varphi, \eta_n \rangle = \langle \varphi \circ f, \eta_n \rangle$$

$$= \int_{[0,1]} \varphi(f(t))\, \eta_n(t)\, dt = \int_{[0,1]} \left(\sum_{j=1}^{\infty} \eta_j(t) \varphi(z_j) \right) \cdot \eta_n(t) dt.$$

~~We already observed that~~ Note

$$\sum_1^{\infty} |\eta_j(t) \varphi(z_j)| \leq \sum_1^{\infty} |\varphi(z_j)| < \infty$$

already seen

So summation and integration can be interchanged and the orthogonality of the η_j yields that

$$\langle \varphi, T_f \eta_n \rangle = \varphi(z_n) = \langle \varphi, z_n \rangle$$

$\square$

Relation with $\ell_1 \hat{\otimes}_\varepsilon X \cong \ell_1[X]$ from Example 3.4

For finite complete μ we have seen that $L_1(\mu) \hat{\otimes}_\varepsilon X \cong \hat{P}_1(\mu, X)$. Would it be possible to interpret $\ell_1 \hat{\otimes}_\varepsilon X \cong \ell_1[X]$ in a similar context? In fact it is. We let μ be the counting measure on the power set of $\mathbb{N}$. It is not finite, but it is still σ-finite, and in particular $L_1(\mu)^* = L_\infty(\mu)$ holds. Trivially all $f: \mathbb{N} \to X$ are μ-measurable, so there are no measurability issues whatsoever.

It is ~~now~~ easy to check that the results on p. 13 - 24 are, in fact, valid for σ-finite μ, except (perhaps) for the statement that the image of the isometric embedding $L_1(\mu) \hat{\otimes}_\varepsilon X \longrightarrow \hat{P}_1(\mu, X)$ is dense?.

The finiteness of μ was only used to conclude

that $\ell_1(\mu)^* = \ell_\infty(\mu)$, and that is equally true for σ-finite μ. So, in particular, we have an isometric embedding $\ell_1 \hat{\otimes}_\varepsilon X \hookrightarrow \hat{P}_1(\mu, X)$.
The author of these notes is not aware of the (in)validity of the Theorem on p 24 for ~~σ-finite~~ general σ-finite μ, but in this particular case we can show it to hold, and at the same time identify $\hat{P}_1(\mu, X)$ with the space $\ell_1[X]$ from Example 24 in the book.
We start by showing that $\hat{P}_1(\mu, X)$ can be identified with the weakly subseries summable (eq. unconditionally summable) sequences.
Let $f: \mathbb{N} \to X$ be weakly integrable, i.e. $\sum_1^\infty |\varphi(f(n))| < \infty \ \forall \varphi \in X^*$. When is it Pettis integrable? If it is, then for all $E \subset \mathbb{N}$:

$$\langle \varphi, \overbrace{T_f \chi_E}^{\in X} \rangle = \langle S_f \varphi, \chi_E \rangle = \langle \varphi \circ f, \chi_E \rangle$$
$$= \sum_{\substack{n=1 \\ n \in E}}^{\infty} \varphi(f(n))$$

Hence, for all $E \subset \mathbb{N}$, $\sum_{\substack{n=1 \\ n \in E}} f(n)$ is weakly convergent in X (with sum $\int_E f \, d\mu$). We see that $(f(n))_{n=1}^{\infty}$ is weakly subseries summable. The Orlicz Pettis theorem implies that $(f(n))_{n=1}^{\infty}$ is unconditionally summable.

Conversely, if $(f(n))_{n=1}^{\infty}$ is unconditionally summable, then so is $(\varphi \circ f(n))_{n=1}^{\infty}$ for any fixed $\varphi \in X^*$. This implies that $\sum_{n=1}^{\infty} |\varphi \circ f(n)| < \infty$, hence f is weakly integrable. Furthermore, for $E \subset \mathbb{N}$:

$$\langle S_f \varphi, \chi_E \rangle = \langle \varphi \circ f, \chi_E \rangle$$
$$= \sum_{\substack{n=1 \\ n \in E}}^{\infty} \varphi(f(n))$$

$$= \varphi\Big(\underbrace{\sum_{\substack{n=1\\ n\in E}}^{\infty} f(n)}_{\text{exists in } X \text{ by Prop } \mathcal{D}\,i\,ii}\Big)$$

We conclude that f is Pettis integrable and that

$$\int_E f\,d\mu = \sum_{\substack{n=1\\ n\in E}}^{\infty} f(n).$$

<u>Conclusion so far</u>: the Pettis integrable functions on $\mathbb{N}$ can be identified with the unconditionally summable sequences. ~~Hence we have an isometric on~~

If $f: \mathbb{N} \to X$ is Pettis-integrable, then its norm is, according to p. 20:

$$\|f\|_1 = \sup\Big\{\int_{\mathbb{N}} |\varphi\circ f|\,d\mu : \varphi \in B_{X^*}\Big\}$$

$$= \sup\Big\{\sum_{n=1}^{\infty} |\varphi(f(n))| : \varphi \in B_{X^*}\Big\}$$

and

$$= \sup\Big\{\Big\|\int_{\mathbb{N}} g f\,d\mu\Big\| : g \in B_{L^\infty(\mu)}\Big\}$$

$$= \sup \left\{ \left\| \sum_{n=1}^{\infty} b_n f(n) \right\| : b \in B_{\ell_\infty} \right\}.$$

Hence we retrieve that $(P_1(\mathbb{N}, X), |\cdot|_1)$ can be isometrically identified with $\ell_1[X]$, and we obtain an isometric embedding

$$\ell_1 \hat{\otimes}_\varepsilon X \hookrightarrow \hat{P}_1(\mathbb{N}, X) = \hat{\ell}_1[X]$$

However, in this particular case it is shown in the book that $\ell_1[X]$ is already complete. Furthermore, since the image contains the finite sequences, which can also be shown to be dense "by hand", we see that $\ell_1 \hat{\otimes}_\varepsilon X \cong \ell_1[X]$ where we now also know that $\ell_1[X]$ can be identified with the (in this case already complete) Pettis integrable functions on $\mathbb{N}$ with values in X.

JK

Appendix: some technical results.

Proposition A.1 Let S be loc. cpt. Hd, $f_1 - f_n \in C_0(S, X)$, and $\varepsilon > 0$. Then there exist $\varphi_1 - \varphi_m \in C_c(S)$ and points $s_1 - s_m \in S$ and $t_1 - t_m \in S$ such that:

1° $0 \leq \varphi_i \leq 1$ $(i = 1, \dots, m)$

2° $\sum_{i=1}^{m} \varphi_i \leq 1$

3° $\varphi_i(s_j) = \delta_{ij}$ $(i, j = 1, \dots, m)$

4° $\| f_j - \sum_{i=1}^{m} \varphi_i f_j(t_i) \| \leq \varepsilon$ $(j = 1, \dots, n)$.

~~Proof~~: If S is cpt. Hd, then 2° can be replaced with $\sum_{i=1}^{m} \varphi_i = 1$

Proof: Assume S loc cpt Hd. Let $\hat{S} = S \cup \{\infty\}$ be the one-point compactification. Extend the f_j to $\hat{f}_j \in C(\hat{S}, X)$ by putting $\hat{f}_j(\infty) = 0$. Let $\hat{t}_0 = \infty$ and put

$$\hat{V}_0 = \{ \hat{s} \in \hat{S} : \| \hat{f}_j(\hat{s}) - \hat{f}_j(\hat{t}_0) \| < \varepsilon \text{ for all } j = 1, \dots, n \}$$

$= \{ \hat{s} \in \hat{S} : \|\hat{f}_j(s)\| < \varepsilon, j = 1, \dots, n \}$, contains ∞.

For $\hat{t} \in \hat{V}_0^c$ $(\subset S)$ let

$\hat{V}_t = \{ \hat{s} \in \hat{S} : \|\hat{f}_j(s) - \hat{f}_j(t)\| < \varepsilon, j = 1, \dots, n \}$

Then $\hat{t} \in \hat{V}_t$, but $\infty \notin \hat{V}_t$, and $\hat{S} = \hat{V}_0 \cup \bigcup_{t \in \hat{V}_0^c} \hat{V}_t$

Pick finitely many $\hat{t}_1, \dots, \hat{t}_m \in \hat{V}_0^c$ s.t.

$\hat{S} = \hat{V}_0 \cup \bigcup_{j=1}^m \hat{V}_{t_j} \overset{\text{notation}}{=} \hat{V}_0 \cup \bigcup_{j=1}^m \hat{V}_j$

We may assume that this cover is minimal:

for each $j = 0, \dots, m$ one has $\hat{V}_j \not\subset \bigcup_{\substack{k=0 \\ k \neq j}}^m \hat{V}_k$

That is: there exist points $\hat{s}_0, \dots, \hat{s}_m \in \hat{S}$ such that,

for $i = 0, \dots, m$, $\hat{s}_i \in \hat{V}_i$ but $\hat{s}_i \notin \hat{V}_j$ if

$j \in \{0, \dots, m\}$ and $j \neq i$. Note: $\hat{s}_1, \dots, \hat{s}_m \in S$.

There exist $\hat{\varphi}_0, \dots, \hat{\varphi}_m \in C(\hat{S})$ such that:

- $0 \leq \hat{\varphi}_i \leq 1$ $(i = 0, \dots, m)$
- $\sum_{i=0}^m \hat{\varphi}_i = 1$
- $\operatorname{supp} \hat{\varphi}_i \subset \hat{V}_i$ $(i = 0, \dots, m)$

The combination $\sum_{i=0}^{m} \hat{\varphi}_i = 1$, $\hat{s}_j \notin \bigcup_{\substack{i=0\\ i \neq j}}^{m} \hat{V}_i$ and $\operatorname{supp} \hat{\varphi}_i \subset \hat{V}_i$ implies that

$\hat{\varphi}_i(\hat{s}_j) = \delta_{ij}$ $(i, j = 0, \ldots, m)$.

Then, for $\hat{s} \in \hat{S}$ and $j \in \{1, \ldots, n\}$

$$\|\hat{f}_j(\hat{s}) - \sum_{i=0}^{m} \hat{\varphi}_i(\hat{s}) \hat{f}_j(\hat{t}_i)\| = \|\sum_{i=0}^{m} \hat{\varphi}_i(\hat{s})(\hat{f}_j(\hat{s}) - \hat{f}_j(\hat{t}_i))\|$$

$$\leq \sum_{i=0}^{m} \hat{\varphi}_i(\hat{s}) \|\hat{f}_j(\hat{s}) - \hat{f}_j(\hat{t}_i)\| = \sum_{\substack{i=0\\ \hat{s} \in \hat{V}_i}}^{m} \hat{\varphi}_i(\hat{s}) \|\hat{f}_j(\hat{s}) - \hat{f}_j(\hat{t}_i)\|$$

$$\leq \sum_{\substack{i=0\\ \hat{s} \in \hat{V}_i}}^{m} \hat{\varphi}_i(\hat{s}) \cdot \varepsilon \leq \varepsilon.$$

Now recall that $\hat{f}_j(\hat{t}_0) = 0$ $(j = 1, \ldots, n)$, and that $\hat{t}_1 - \hat{t}_m \in S$. Writing $t_1 - t_m$ for $\hat{t}_1 - \hat{t}_m$, we find that, for $s \in S$ and $j = 1, \ldots, m$, with ~~...~~ $\varphi_i = \hat{\varphi}_i|_S$ $(i = 1, \ldots, m)$:

$$\|f_j(s) - \sum_{i=1}^{m} \varphi_i(s) f_j(t_i)\| \leq \varepsilon.$$

With $s_i = \hat{s}_i \in S$ $(i = 1, \ldots, m)$, everything is now clear, except that $\varphi_i \in C_c(S)$

for $i = 1, \dots, m$. Fix such i. Note that

$\overline{\mathrm{supp}(\varphi_i) = \{s \in S : \varphi_i(s) \neq 0\}}^{S}$. Since $\mathrm{supp}(\hat{\varphi}_i) \subset \hat{V}_i$

and $\infty \notin \hat{V}_i$ we have $\hat{\varphi}_i(\infty) = 0$. Hence

$$\mathrm{supp}(\hat{\varphi}_i) = \overline{\{\hat{s} \in \hat{S} : \hat{\varphi}_i(\hat{s}) \neq 0\}}^{\hat{S}}$$

$$= \overline{\{s \in S : \varphi_i(s) \neq 0\}}^{\hat{S}}$$

$$= \begin{cases} \mathrm{supp}(\varphi_i) \\ \quad \text{or} \\ \mathrm{supp}(\varphi_i) \cup \{\infty\}. \end{cases}$$

Since $\infty \notin \mathrm{supp}(\hat{\varphi}_i)$, we must have $\mathrm{supp}(\hat{\varphi}_i) = \mathrm{supp}(\varphi_i)$, implying that $\mathrm{supp}(\varphi_i)$ is compact.

The proof for compact S runs along the same lines (and is easier). $\square$

<u>Proposition A2</u>: Let X be a normed space, and $\{x_1, \dots, x_n\} \subset X$ linearly independent. Then there exists $\varepsilon > 0$ such that $y_1 \dots y_n \in X$ and $\max_i \|y_i - x_i\| < \varepsilon$ imply that $\{y_1 \dots y_n\}$ are linearly independent.

<u>Proof</u> Let $x_1^* \dots x_n^*$ be extensions of the coordinate functionals on $\operatorname{Span}\{x_1 \dots x_n\}$. The function

$$(y_1, \dots, y_n) \longmapsto \begin{vmatrix} x_1^*(y_1) & \cdots & x_n^*(y_1) \\ \vdots & & \\ x_1^*(y_n) & \cdots & x_n^*(y_n) \end{vmatrix}$$

is continuous on $X \times \dots \times X$ and equal to 1 in $(x_1, \dots, x_n)$.

Thus, if $\max_i \|y_i - x_i\|$ is sufficiently small, it is non-zero, which implies that $\{y_1, \dots, y_n\}$ is linearly independent.

$\square$

Theorem A 3 Let X be a Banach space. Suppose $\{x_1 \ldots x_n\}$ is a linearly independent, and $\varepsilon > 0$. Then there exists $\delta > 0$ such that, for all $\{y_1 \ldots y_n\} \subset X$ with $\max_i \|y_i - x_i\| < \delta$, and all finite dimensional $M \supset \operatorname{span}\{y_1, \ldots, y_n\}$:

$1°$ $\{y_1 \ldots y_n\}$ is linearly independent

$2°$ $\exists N \supset \operatorname{span}\{x_1 \ldots x_n\}$ with $\dim M = \dim N$ and $d(M, N) < 1 + \varepsilon$.

Proof By Prop. A2, there exists $\delta_1 > 0$ such that
$\max_i \|y_i - x_i\| < \delta_1$ implies that $\{y_1 \ldots y_n\}$ are linearly independent.

In the notation of the Lemma in the annotation to Frijanne's lecture, pick δ_2 so small that

$$\frac{1 - (1-n)\delta_2 K}{1 - (1+n)\delta_2 K} = 1 + \frac{2n\delta_2 K}{1-(1+n)\delta_2 K} < 1 + \varepsilon.$$

and $\delta_2 (1+n) K < 1$.

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Then $\delta = \min(\delta_1, \delta_2)$ is as required.

(Note that $1 + \dfrac{2n\delta K}{1-(1+n)\delta K} \leq 1 + \dfrac{2n\delta_2 K}{1-(1+n)\delta_2 K} < 1+\varepsilon$

and $\delta(1+n)K \leq \delta_2(1+n)K < 1$)

$\square$

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Quotients II

Notation: $\overset{\circ}{B}_X(r) = \{x \in X : \|x\| < r\}$ for $r > 0$.

Lemma A4 X, Y Banach, $Q \in \mathcal{L}(X, Y)$. Then TFAE:

1° $Q(\overset{\circ}{B}_X(1)) \supset \overset{\circ}{B}_Y(1)$

2° For all $\varepsilon > 0$ and $y \in Y$ there exists $x \in X$ such that $\|y - Qx\| < \varepsilon$ and $\|x\| \leq (1+\varepsilon)\|y\|$.

3° As in 2°, for all y in a dense subset of Y.

If $\|Q\| \leq 1$, then these are also equivalent with:

4° $Q(\overset{\circ}{B}_X(1)) = \overset{\circ}{B}_Y(1)$ (i.e. Q is a quotient operator).

Proof 1° $\Rightarrow$ 2°: If $y = 0$, take $x = 0$. If $y \neq 0$, then $Q(\overset{\circ}{B}_X((1+\frac{\varepsilon}{2})\|y\|)) \supset \overset{\circ}{B}_Y((1+\frac{\varepsilon}{2})\|y\|)$ and $y \in \overset{\circ}{B}_Y((1+\frac{\varepsilon}{2})\|y\|)$.

2° $\Rightarrow$ 1° Fix $0 < \varepsilon \ll 1$, and let $y \in \overset{\circ}{B}_Y(1)$ be given. Then there exists $x_1 \in X$ with

$\|y - Qx_1\| < \varepsilon\|y\|$ and $\|x_1\| \leq (1+\varepsilon\|y\|)\|y\|$ $\leq (1+\varepsilon)\|y\|$

Then there exists $x_2 \in X$ with

$\|(y - Qx_1) - Qx_2\| < \varepsilon^2\|y\|$ and

$\|x_2\| \leq (1+\varepsilon^2\|y\|)\,\|y - Qx_1\| \leq (1+\varepsilon)\cdot\varepsilon\,\|y\|$

Assume $x_1 - x_n$ have been chosen such that

$\|y - \sum_1^n Qx_i\| < \varepsilon^n\|y\|$ and $\|x_i\| \leq (1+\varepsilon)\cdot\varepsilon^{i-1}\|y\|$,

choose x_{n+1} such that

$\|(y - \sum_1^n Qx_i) - Qx_{n+1}\| < \varepsilon^{n+1}\|y\|$ and

$\|x_{n+1}\| \leq (1+\varepsilon^{n+1}\|y\|)\,\|y - \sum_1^n Qx_i\|$.

Then $\|x_{n+1}\| \leq (1+\varepsilon)\cdot\varepsilon^n\|y\|$.

Proceeding this way, we put $x_\varepsilon = \sum_1^\infty x_i$.

Then $y = Qx_\varepsilon$ and

$\|x_\varepsilon\| \leq \frac{1+\varepsilon}{1-\varepsilon}\|y\|$.

Thus, for ε sufficiently small, $\|x_\varepsilon\| < 1$.

We conclude that $Q(B_X^0(1)) \supset S_Y^0(1)$.

$2^\circ \Rightarrow 3^\circ$ trivial

$3^\circ \Rightarrow 2^\circ$ Let $y \in Y$ and $\varepsilon > 0$ be given. If $\delta > 0$, there exists $y_\delta' \in Y$ and $x_\delta \in X$ such that

$\|y - y_\delta'\| < \delta$, $\|y_\delta' - Ax_\delta\| < \delta$ and

$\|x_\delta\| \leq (1+\delta)\|y_\delta'\|$.

Then $\|y - Ax_\delta\| \leq \|y - y_\delta'\| + \|y_\delta' - Ax_\delta\|$

$< \delta + \delta = 2\delta$, and furthermore

$\|x_\delta\| \leq (1+\delta)\|y_\delta'\| \leq (1+\delta)(\|y\| + \delta)$.

Hence chose $\delta > 0$ so small that $2\delta < \varepsilon$

and $(1+\delta)(\|y\| + \delta) \leq (1+\varepsilon)\|y\|$

(note that the latter is possible only if $y \neq 0$).

Assume now that $\|A\| \leq 1$. Then

$A(B_X^\circ(1)) \subseteq B_Y^\circ(1)$, so if 1° holds, then

$A(B_X^\circ(1)) = B_Y^\circ(1)$ which is 4°.

Certainly 4° implies 1° for arbitrary A.

$\square$

<u>Corollary A 5</u> Let $Q: X \to Y$ with $\|Q\| \leq 1$.

Then TFAE:

1° Q is a quotient operator

2° For all $y \in Y$ and all $\varepsilon > 0$ there exists $x \in X$ with $\|y - Qx\| < \varepsilon$ and $\|x\| \leq (1+\varepsilon)\|y\|$

3° For all y ~~...~~ in a dense subset of X there exists $x \in X$ with $\|y - Qx\| < \varepsilon$ and $\|x\| \leq (1+\varepsilon)\|y\|$

4° For all $y \in Y$ and all $\varepsilon > 0$ there exists $x \in X$ with $y = Qx$ and $\|x\| \leq (1+\varepsilon)\|y\|$

<u>Proof</u>: 1° $\Rightarrow$ 4° Clear if $y = 0$. If $y \neq 0$ then $\left\| \frac{y}{(1+\varepsilon)\|y\|} \right\| < 1$, hence there exists $x \in X$ with $\|x\| < 1$ and $Qx = \frac{y}{(1+\varepsilon)\|y\|}$.

Then $Q((1+\varepsilon)\|y\| x) = y$ and $\|(1+\varepsilon)\|y\| x\| < (1+\varepsilon)\|y\|$.

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$4^\circ \Rightarrow 2^\circ \Leftrightarrow 2^\circ$ is clear, and $3^\circ \Rightarrow 1^\circ$ is part of Lemma A4.

$\square$