$L_1(\mu)$ is an $\mathcal{L}_{1,1+\varepsilon}$-space for every $\varepsilon > 0$

Lemma Let E be a normed space and suppose $\{x_1 - x_n\}$ and $\{y_1 - y_n\}$ are both linearly independent subsets. Let K be such that $\sum_1^n |\alpha_i| \leq K \| \sum_1^n \alpha_i x_i \|$ for all $\alpha_1 - \alpha_n \in \mathbb{F}$. Suppose $\delta > 0$ is such that $\delta(1+n)K < 1$ and $\max_{1 \leq i \leq n} \|x_i - y_i\| < \delta$. Then, for every finite dimensional subspace Y of E containing $Y_0 \stackrel{def}{=} \operatorname{span}\{y_1 - y_n\}$, there exists a finite dimensional subspace X of E, with $\dim X = \dim Y$ and containing $X_0 \stackrel{def}{=} \operatorname{span}\{x_1 - x_n\}$, such that

$$d(X, Y) \leq \frac{1 - (1-n)\delta K}{1 - (1+n)\delta K}$$

where $d(\cdot,\cdot)$ is Banach - Mazur distance

Proof (From the proof of lemma 3.1 in J. Lindenstrauss, "Extension of linear operators", Memoirs of the AMS 48, AMS, 1964)

Let $P: E \to Y_0$ be a projection of norm $\leq \lambda$. Restriction to Y yields $Y = Y_0 \oplus \ker P|_Y$.
The aim is to show that the sum $X_0 + \ker P|_Y$ is actually a direct sum, and that is a subspace X as desired.

Let $\{z_1, \dots, z_m\}$ be a basis for $\ker P|_{Y_0}$. We start by comparing

$$\left\| \sum_1^n \alpha_i x_i + \sum_1^m \beta_j z_j \right\| \quad \text{and}$$

$$\left\| \sum_1^n \alpha_i y_i + \sum_1^m \beta_j z_j \right\|$$

for arbitrary $\alpha_i, \beta_j \in K$.

First of all

$$\Big\|\sum_1^n \alpha_i x_i\Big\| - \Big\|\sum_1^n \alpha_i y_i\Big\| \le \Big\|\sum_1^n \alpha_i x_i - \sum_1^n \alpha_i y_i\Big\| \qquad (1)$$

$$\le \delta \sum_1^n |\alpha_i| \le \delta K \Big\|\sum_1^n \alpha_i x_i\Big\|$$

Hence $\Big\|\sum_1^n \alpha_i x_i\Big\| \le \dfrac{1}{1-\delta K}\Big\|\sum_1^n \alpha_i y_i\Big\|$, and using (1) once more yields

$$\Big\|\sum_1^n \alpha_i x_i - \sum_1^n \alpha_i y_i\Big\| \le \frac{\delta K}{1-\delta K}\Big\|\sum_1^n \alpha_i y_i\Big\|$$

$$= \frac{\delta K}{1-\delta K}\Big\| P\Big(\sum_1^n \alpha_i y_i + \sum_1^m \beta_j z_j\Big)\Big\|$$

(2)
$$\le \frac{\delta n K}{1-\delta K}\Big\|\sum_1^n \alpha_i y_i + \sum_1^m \beta_j z_j\Big\|$$

Using (2) we find

$$\Big\|\sum_1^n \alpha_i y_i + \sum_1^m \beta_j z_j\Big\| \le \Big\|\sum_1^n \alpha_i x_i + \sum_1^m \beta_j z_j\Big\| + \Big\|\sum_1^n \alpha_i y_i - \sum_1^n \alpha_i x_i\Big\|$$

$$\le \Big\|\sum_1^n \alpha_i x_i + \sum_1^m \beta_j z_j\Big\| + \frac{\delta n K}{1-\delta K}\Big\|\sum_1^n \alpha_i y_i + \sum_1^m \beta_j z_j\Big\|$$

Hence, since $\frac{\delta n K}{1-\delta K} < 1$,

$$\left(1-\frac{\delta n K}{1-\delta K}\right)\left\|\sum_1^n \alpha_i y_i + \sum_1^m \beta_j z_j\right\| \leq \left\|\sum_1^n \alpha_i x_i + \sum_1^m \beta_j z_j\right\|$$

(3)

Using (2) once more, we have

$$\left\|\sum_1^n \alpha_i y_i + \sum_1^m \beta_j z_j\right\| \geq \left\|\sum_1^n \alpha_i x_i + \sum_1^m \beta_j z_j\right\| - \left\|\sum_1^n \alpha_i x_i - \sum_1^n \alpha_i y_i\right\|$$

$$\geq \left\|\sum_1^n \alpha_i x_i + \sum_1^m \beta_j z_j\right\| - \frac{\delta n K}{1-\delta K}\left\|\sum_1^n \alpha_i y_i + \sum_1^m \beta_j z_j\right\|$$

so that

$$\left\|\sum_1^n \alpha_i x_i + \sum_1^m \beta_j z_j\right\| \leq \left(1+\frac{\delta n K}{1-\delta K}\right)\left\|\sum_1^n \alpha_i y_i + \sum_1^m \beta_j z_j\right\|$$

(4)

Now (3) implies that $\{x_1 - x_n, z_1 - z_m\}$ is linearly independent, since $\{y_1 - y_n, z_1, -, z_m\}$ is. Hence we let $X = \mathrm{Ker}\, P_{T_2} \oplus X_1$

and define

$$T: X \longrightarrow Y$$

$$\sum_1^n \alpha_i x_i + \sum_1^m \beta_j z_j \longmapsto \sum_1^n \alpha_i y_i + \sum_1^m \beta_j z_j$$

Then (3) implies that

$$\|T\| \leq \frac{1}{1 - \frac{\delta n K}{1-\delta K}}$$

and (4) yields

$$\|T^{-1}\| \leq 1 + \frac{\delta n K}{1-\delta K}.$$

Therefore

$$d(X, Y) \leq \frac{1 + \frac{\delta n K}{1-\delta K}}{1 - \frac{\delta n K}{1-\delta K}} = \frac{1-(1-n)\delta K}{1-(1+n)\delta K}.$$

$\square$

Corollary: $L_1(\mu)$ is an $\mathcal{L}_{1,1+\varepsilon}$-space for every $\varepsilon > 0$.

Proof: Let $M \subset L_1(\mu)$ with $\dim M < \infty$ and $\varepsilon > 0$ be given. Choose a basis $\{f_1 \dots f_n\}$ for M and next pick K such that $\sum_1^n |\alpha_i| \le K \|\sum_1^n \alpha_i f_i\|_1$ for all $\alpha_1 \dots \alpha_n \in \mathbb{F}$. Select $\delta > 0$ so small that $\delta(n+1)K < 1$ and
$$\frac{1-(1-n)\delta K}{1-(1+n)\delta K} < 1+\varepsilon.$$
Since the simple functions are dense, there exist ~~$g_1 \dots g_n$~~ simple functions $g_1 \dots g_n$ such that

1° $\{g_1 \dots g_n\}$ is linearly independent.

2° $\max_{i=1 \dots n} \|f_i - g_i\| < \delta$.

Clearly there exists a finite dimensional subspace Y (consisting of simple functions), containing $\operatorname{span}\{g_1 \dots g_n\}$, which is isometrically isomorphic to ℓ_1^k (where $k = \dim Y$).

The lemma yields a subspace X containing M with $d(X, Y) < 1+\varepsilon$, i.e., $d(M, \ell_1^k) < 1+\varepsilon$.

$\square$