Auerbach Lemma: Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and suppose V is a finite dimensional normed space over $\mathbb{F}$. If $\dim V = n$, then there exists a basis $v_1 \dots v_n$ of V with $\|v_i\| = 1$ $(i = 1, \dots, n)$ such that the coordinate functionals v_i^*, defined by $v_i^*(v_j) = \delta_{ij}$, also have norm 1.

Proof: We may assume $V = \mathbb{F}^n$. Let $K = \{v \in V : \|v\| = 1\}$ and let $\det(\cdot, \cdot, \dots, \cdot)$ be the determinant function (w.r.t. the standard basis) on V^n. It is non-zero on K^n, hence by compactness there is a point $(v_1, \dots, v_n) \in K^n$ in which $|\det(\cdot, \cdot, \dots, \cdot)|$ attains a strictly positive maximum. Then $\{v_1, \dots v_n\}$ is a basis, and in fact a basis as required.

Indeed, since $\|v_j\| = 1$ and $|v_j^*(v_j)| = 1$, we have $\|v_j^*\| \geq 1$. If $\|v_j^*\| > 1$, then there exists $\tilde{v}_j$ with $\|\tilde{v}_j\| = 1$ and $|v_j^*(\tilde{v}_j)| > 1$. But then

$$|\det(v_1, \dots, v_{j-1}, \tilde{v}_j, v_{j+1}, \dots, v_n)| =$$

$$= |\det(v_1, \dots, v_{j-1}, \sum_{i=1}^{n} v_i^*(\tilde{v}_j) v_i, v_{j+1}, \dots, v_n)|$$

$$= |\det(v_1, \dots, v_{j-1}, v_j^*(\tilde{v}_j) v_j, v_{j+1}, \dots, v_n)|$$

$$= |v_j^*(\tilde{v}_j)| \, |\det(v_1, \dots, v_{j-1}, v_j, \dots, v_n)|$$

$$> |\det(v_1, \dots, v_n)|,$$

contradicting the maximality property of $(v_1, \dots, v_n)$.

$\square$