The material in this lecture is based on Van Haandel's PhD thesis.

Riesz completion of Archimedean POVS

Def 1. A partially ordered vector space (POVS) (X, K) is called pre-Riesz if for every $x, y, z \in X$, the inclusion $\{x+y, z+y\}^u \subseteq \{x, z\}^u$ implies $y \in K$. ($y \geq 0$)

Lemma 2 Let X be POVS If for every $a, y \in X$, $\{a+y, y\}^u \subseteq \{a, 0\}^u$ implies $y \in K$. then X is pre-Riesz.

Proposition 3 Every pre-Riesz space is directed, every directed Archimedean POVS is pre-Riesz

Proof. <i> Suppose X is a POVS but not directed.

Then $x, y \in X$. $\{x, y\}$ has no upper bounds. $\forall z \in X$, $\{x+z, y+z\}$ has no upper bounds neither. $\{x+z, y+z\}^u \subseteq \{x, y\}^u$ trivially holds, but z could not be in K. Infact, take $z = y - x$, then $z \ngeq 0$ otherwise $y \geq x$. y will be an upper bound of $\{x, y\}$, so $z \notin K$. Hence X not pre-Riesz, contradic

<ii> Let X be an Archimedean POVS, and $\forall x, y, z \in X$ be s.t. $\{x+z, y+z\}^u \subseteq \{x, y\}^u$. Since X directed, so $\exists u \in \{x, y\}^u$, then $u+z \in \{x+z, y+z\}^u \subseteq \{x, y\}^u$

Inductively. $u+nz \in \{x,y\}^u$. $\forall n \in \mathbb{N}$.

Hence. $u+nz \geq x$, $-nz \leq u-x$

Since X Archimedean. $-z \leq 0$

so $z \geq 0$. Hence X pre-Riesz. $\square$

<u>Proposition 4</u> Let X be POVS, Y pre-Riesz and $i: X \to Y$ be bipositive linear map. If $i[X]$ is order dense in Y. then X pre-Riesz.

<u>Proof</u>: Let $x, y, z \in X$ be s.t. $\{x+z, y+z\}^u \subseteq \{x, y\}^u$. Let $a \in \{i(x)+i(z), i(y)+i(z)\}^u$

As $i[X]$ order dense in Y. we have.

$a = \inf\{i(u) : u \in X,\ i(u) \geq a\}$. For $u \in X$ with $i(u) \geq a$. we have $i(u) \geq i(x+z)$ $i(u) \geq i(y+z)$. so $u \in \{x+z, y+z\}^u$

so $u \in \{x, y\}^u$, therefore $i(u) \in \{i(x), i(y)\}^u$

Hence $\{i(x)+i(z), i(y)+i(z)\}^u \subseteq \{i(x), i(y)\}^u$

As Y pre-Riesz. $i(z) \geq 0$, $z \geq 0$ $\square$

Thus X pre-Riesz

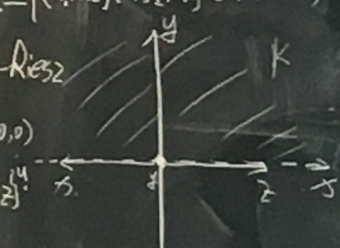
<u>Lemma 5</u> Let X be a pre-Riesz space.

i) For every nonempty finite $A \subseteq X$. $\forall x \in X$ the inclusion $(A+x)^u \subseteq A^u$ implies $x \geq 0$

ii) For every nonempty finite $A, B, C \subseteq X$. $(A+C)^u \subseteq (B+C)^u$ implies $A^u \subseteq B^u$

<u>Example 6</u>. $X = \mathbb{R}^2$. $K = \{(x_1, x_2) : x_2 > 0\} \cup \{(0,0)\}$

X directed. not pre-Riesz



Indeed. $x = (-1, 0)$, $y = (0,0)$ $z = (1,0)$. If $u \in \{x+z, y+z\}^u$.

$\underline{u \geq (0,0)}$, $\underline{u \geq (1,0)}$
$u_2 > 0$ $u \neq (0,0)$. so $u \in \{x, y\}^u$

So $\{x+z, y+z\}^u \subseteq \{x, y\}^u$, but $z \notin K$ $\square$

$\boxed{\text{Riesz completion of Archimedean POVS}}$

Def 7 Let X, Y be POVSs. A linear map $h: X \to Y$ is called Riesz* homomorphism (R^*h) if for every nonempty finite $F \subseteq X$, one has: $h[F^{ul}] \subseteq h[F]^{ul}$

Lemma 8 Let X, Y be POVSs. $h: X \to Y$ linear

i) If h R^*h, then h positive

ii) h R^*h iff for every nonempty finite $F \subseteq X$

$$h[F^{ul}]^{ul} = h[F]^{ul}$$

iii) If X, Y are Riesz spaces, then h is R^*h iff h is Riesz homomorphism (Rh)

Proof: i) Take $F = \{0\}$, $\forall x \in X$, $x \geq 0$, we have $-x \in F^{ul}$

So $h(-x) \in h[F^{ul}] \subseteq h[F]^{ul} = \{0\}^l$, $h(x) \geq 0$

ii) "$\Leftarrow$" $\emptyset \neq F \subseteq X$ finite, $h[F^{ul}]^{ul} = h[F]^{ul}$

then $h[F^{ul}] \subseteq h[F^{ul}]^{ul} = h[F]^{ul}$. $h: R^*h$

"$\Rightarrow$" $\forall\, \emptyset \neq F \subseteq X$ finite, $F^{ul} \supseteq F$. So $h[F^{ul}] \supseteq h[F]$

hence $h[F^{ul}]^{ul} \supseteq h[F]^{ul}$.

If h R^*h, then $\forall\, \emptyset \neq F \subseteq X$ finite

$h[F^{ul}] \subseteq h[F]^{ul}$, hence

$$h[F^{ul}]^{ul} \subseteq h[F]^{ul}$$

iii)

iii> X, Y be Riesz space, then let $F=\{a_1, \dots, a_n\}$, $a_i \in X$, $i=1, \dots, n$

$h[F^{ul}] \subseteq h[F]^{ul}$ means

$h[\{a_1 \vee \cdots \vee a_n\}^l] \subseteq \{h(a_1) \vee \cdots \vee h(a_n)\}^l$

If h is Rh, obviously h is R^*h

If h is R^*h, we have

$$h(a_1 \vee \cdots \vee a_n) \leq h(a_1) \vee \cdots \vee h(a_n)$$

As h positive, $h(a_1 \vee \cdots \vee a_n) \geq h(a_1) \vee \cdots \vee h(a_n)$

Thus h is Rh □

Lemma 9 Let Y be a pre-Riesz, X POVS, $h: X \to Y$ linear bipositive. Then h is R^*h in both of the following cases.

i> $h[X]$ is order dense in Y

ii> Y is a Riesz space, $h[X]$ is a Riesz subspace of Y

Lemma 10 Let X, Y be directed POVSs, $h: X \to Y$ linear bipositive. If Y is pre-Riesz, and h is R^*h, then X is pre-Riesz □

Proof: By Proposition 4

Proposition 11 Let X_1, X_2 POVSs, $h: X_1 \to X_2$ linear. Assume there exist Riesz spaces Y_1, Y_2 and bipositive linear maps $i_1: X_1 \to Y_1$, $i_2: X_2 \to Y_2$ s.t. $i_1[X_1]$ is order dense in Y. If $\exists$ Rh $\hat{h}: Y_1 \to Y_2$ s.t. $i_2 \circ h = \hat{h} \circ i_1$, then h is R^*h.

Proof. Let $F=\{a_1, \dots, a_n\}$, $a_i \in X$, we want is $h[F^{ul}] \subseteq h[F]^{ul}$. Let $x \in F^{ul}$, assume $h[F] \neq \phi$. Let $y \in h[F]^u$. We have

$$i_1(a_1) \vee \cdots \vee i_1(a_n) = \inf\{i_1(u) : u \in \{a_1, \dots, a_n\}^u\} \geq i_1(x)$$

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Then $i_2(h(x)) = \hat{h}(i_1(x)) \leq \hat{h}(i_1(a_1)) \vee \cdots \vee i_1(a_n))$

$= \hat{h}(i_1(a_1)) \vee \cdots \vee \hat{h}(i_1(a_n))$

$= i_2(h(a_1)) \vee \cdots \vee i_2(h(a_n))$

$\leq i_2(y)$

Hence $h(x) \leq y$, thus $h(x) \in h[F]^{ul}$ $\square$

Theorem 12 Let X_1, X_2 be POVSs. $h: X_1 \to X_2$ linear. Assume $\exists$ Y_1, Y_2 Riesz spaces, and bipositive linear $i_1: X_1 \to Y_1$, $i_2: X_2 \to Y_2$ s. t. $i_1[X_1]$ order dense in Y_1, and generates Y_1 as a Riesz space, $i_2[X_2]$ order dense in Y_2.

TFSE

i) h is $R^* h$

ii) $\phi \neq F, G \subseteq X$. finite. one has $F^u = G^u \Rightarrow h[F]^u = h[G]^u$

iii) $\exists$ $\hat{h}: Y_1 \to Y_2$. Rh. with $i_2 \circ h = \hat{h} \circ i_1$

Proof: i) $\Rightarrow$ ii) Use Lemma 8, we have

$$h[F]^{ul} = h[F^{ul}]^{ul} = h[G^{ul}]^{ul} = h[G]^{ul}$$

So $h[F]^u = h[F]^{ulu} = h[G]^{ulu} = h[G]^u$

ii) $\Rightarrow$ iii) $\forall y \in Y_1$, $y = \bigvee_{i=1}^{m} i_1(a_i) - \bigvee_{j=1}^{n} i_1(b_j)$, $a_i, b_j \in X$,

define $\hat{h}: Y_1 \to Y_2$ by

$$\hat{h}(y) = \bigvee_{i=1}^{m} i_2(h(a_i)) - \bigvee_{j=1}^{n} i_2(h(b_j))$$

$\hat{h}$: well defined, linear, Rh, range $= Z \subseteq Y_2$, generated by $i_2[h[X_1]]$

$G]^u$

nerated

$x_i]]$

iii) $\Rightarrow$ i) Proposition 11. $\square$

<u>Corollary 13</u> Let (X, K) be directed Archimedean POVS with <u>Riesz completion (X^ρ, i)</u>. For a linear functional $\phi : X \to \mathbb{R}$, TFSE

i) ϕ is $\mathbb{R}^\ast h$.

ii) There exists a unique Rh $\psi : X^\rho \to \mathbb{R}$ with

$$\psi \circ i = \phi$$

<u>Proposition 14</u> Let X be a directed POVS. For every $a_1, \dots, a_m \in X$ and $A = \bigvee_{i=1}^{m} \{a_i\}^l$, the identity

$$A \oplus \ominus A = \{0\}^l$$

holds iff X is a pre-Riesz space.

- $A \oplus B = (A+B)^{ul}$

$\ominus A = -A^u$

$$\boxed{A^{ul} = \bigvee_{i=1}^{m} \{a_i\}^{lul} = \bigvee_{i=1}^{m} \{a_i\}^l = A}$$

Proof: "$\Leftarrow$" Assume X pre-Riesz space, $a_1, \dots, a_m \in X$, $A = \bigvee_{i=1}^{m} \{a_i\}^l$. Let $x \in (A \oplus \ominus A)^u = (A + (-A^u))^{ulu}$

$$= (A - A^u)^u$$

For $a \in A$, $u \in A^u$, we have $x \geq a - u$

So $a - x \leq u$. So $a - x \in A^{ul} = A$

So $A - x \subseteq A$. Then $A^u \subseteq (A - x)^u$

So $(A+x)^u \subseteq A^u$. Since X pre-Riesz, $x \geq 0$

Hence $(A \oplus \ominus A)^u \subseteq \{0\}^u$ thus

$$A \oplus \ominus A = (A \oplus \ominus A)^{ul} \supseteq \{0\}^{ul} = \{0\}^l$$

- $A \oplus \ominus A \subseteq \{0\}^l$ by previous lecture

$$A \oplus \ominus A = \{0\}^l$$

"$\Rightarrow$" Let $F = \{a_1, \dots, a_m\}$, $a_i \in X$, and let $x \in X$ be s.t. $(F + x)^u \subseteq F^u$

we want to get $x \geq 0$

Let $A = \bigvee_{i=1} \{a_i\}^l$. By assumption

$$\{0\}^l = A \oplus \ominus A = (A + (-A^u))^{ul} = (A^{ul} - A^u)^{ul}$$

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Note: $A=\bigvee_{i=1}^{m}\{a_i\}^l=\left(\bigcup_{i=1}^{m}\{a_i\}^l\right)^{ul}$

$\cdot\ A^u=\left(\bigcup_{i=1}^{m}\{a_i\}^l\right)^u=\{a_1,\cdots,a_m\}^u$

Let $B=A^{ul}-A^u$. $\quad \{0\}^l=B^{ul}$

So $\{0\}^{lu}=B^{ulu}=B^u$

$\{0\}^u=B^u=\left(\{a_1,\cdots,a_m\}^{ul}-\{a_1,\cdots,a_m\}^u\right)^u$

For every $u\in\{a_1,\cdots,a_m\}^u$, $v\in\{a_1,\cdots,a_m\}^{ul}$

we have $u+x\geqslant a_i+x$ $\quad i=1,\cdots,m$

So $u+x\in(F+x)^u\subseteq F^u$.

So $u+x\geqslant a_i$

Hence $u+x\in\{a_1,\cdots,a_m\}^u$. so $u+x\geqslant v$. $\quad x\geqslant v-u$

So $x\in\left(\{a_1,\cdots,a_m\}^{ul}-\{a_1\cdots a_m\}^u\right)^u$ $\quad$ implies $x\geqslant 0$ □

Theorem 15 Let X pre-Riesz. X^ρ be subset of X^δ given by

$$X^\rho=\left\{\bigvee_{i=1}^{m}\{a_i\}^l\oplus\ominus\bigvee_{j=1}^{n}\{b_j\}^l,\ a_i, b_j\in X,\ m,n\in\mathbb{N}_{\geq 1}\right\}$$

Then X^ρ is Riesz space. $J: X\to X^\rho$, $x\mapsto\{x\}^l$, $x\in X$ is a bipositive linear map. $J[X]$ order dense in X^ρ. $J[X]$ generates X^ρ as a Riesz space, J is R^*h

Proof: Let $\mathcal{V}=\left\{\bigvee_{i=1}^{m}\{a_i\}^l,\ a_i\in X,\ i\in\mathbb{N}_{\geq 1}\right\}$

Then $X^\rho=\{V\oplus\ominus W,\ V,W\in\mathcal{V}\}$

$\cdot\ X^\rho$

- X^ρ with addition $\oplus$, scalar multiplication $*$ and zero element $\{0\}^\rho$ be a vector space
- J is linear, bipositive, $J[X]$ order dense in X^ρ
- X^ρ closed under supremum. $\square$

<u>Proposition 16</u> Let X be pre-Riesz, X^ρ, J defined in Th 15. If Y is Riesz, $i: X \to Y$ bipositive linear s.t. $i[X]$ order dense in Y, and generates Y as a Riesz space, then Y and X^ρ are isomorphic Riesz space.

<u>Theorem 17</u> Let X be POVS. TFRE

i) X is pre-Riesz

ii) $\exists$ Riesz space Y, and bipositive linear map $i: X \to Y$ s.t. $i[X]$ order dense in Y

iii) $i[X]$ generates Y as a Riesz space

Moreove, all Riesz space Y are isomorphisc

<u>Def 18</u> A pair (Y, i) as in Th 17 is called a vector lattice cover of X. a pair (Y, i) at (ii) $\underline{\underline{a}}$ (the) Riesz completion of X

<u>Th 19</u> Let X, Y be pre-Riesz spares. (X^ρ, J_X), (Y^ρ, J_Y) be their Riesz comptions. $h: X \to Y$ bipostive linear. There exists a Riesz homomorphism $\hat{h}: X^\rho \to Y^\rho$ s.t. $\hat{h} \circ J_X = J_Y \circ h$ iff h is $R^* h$. Moreaer if h is $R^* h$ then $\hat{h}$ a uique and for every $a_1 \ldots a_m, b_1 \ldots b_n \in X$, $\hat{h}$ satisfies

$$\hat{h}\left(\bigvee_{i=1}^{m} J_X(a_i) - \bigvee_{j=1}^{n} J_X(b_j)\right) = \bigvee_{i=1}^{m} J_Y(h(a_i)) - \bigvee_{j=1}^{n} J_Y(h(b_j))$$