COURSE

MATHEMATICAL METHODS

OF PHYSICS.

R.J.Kooman

University of Leiden

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MATHEMATICAL METHODS

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PROBLEM SET.

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Chapter I: Ordinary linear differential equations.

1a. Use Frobenius' method to find two linearly independent solutions of the differential equation $y''(z) + y(z) = 0$.

b. Show that the power series converge for all $z \in \mathbf{C}$.

c. Why are the two solutions you found linearly independent?

2a. Find two linearly independent solutions of the Airy equation $y''(z) + zy(z) = 0$.

b. Show that the power series converge for all $z \in \mathbf{C}$.

3. Show that $y(z)$ is a solution of the Hermite equation $y''(z) - 2zy'(z) + \lambda y(z) = 0$ if and only if $w(z) = y(z)e^{-z^2/2}$ is a solution of the equation
$w''(z) + (\lambda + 1 - z^2)w(z) = 0$.

4a. Solve the differential equation using the substitution $z = e^s$:

$$4y''(z) + \frac{1}{z}y'(z) - \frac{1}{z^2}y(z) \ = \ 0.$$

b. Give the singular points in $\mathbf{C} \cup \{\infty\}$. Which singular points are regular?

5a. Give two linearly independent solutions of Laguerre's equation about $z = 0$:

$$zy''(z) + (1 - z)y'(z) + \lambda y(z) = 0.$$

b. For which values of λ is there a polynomial solution?

c. Show that the power series converge for all $z \in \mathbf{C}$.

d. Is $z = \infty$ an ordinary or a singular point of the DE? If singular, is it regular or irregular?

6a. Show that substitution of $x = \cos\theta$ transforms Chebyshev's equation

$$(1 - x^2)y''(x) - xy'(x) + \lambda y(x) = 0$$

into the constant coefficient equation $w'' + \lambda w = 0$.

b. Show that for $\lambda = N^2$ the DE has a polynomial solution of degree N.

c. Show that the general solution of the Chebyshev equation for $\lambda = N^2$ is $y(x) = A\cos(N \arccos x) + B\sin(N \arccos x)$. For which A, B is the solution a polynomial?

7a. Give two linearly independent power series solutions about $z = 0$ of the DE

$$y''(z) - \frac{2z}{1-z^2}\, y'(z) - \frac{n(n+1)}{(1-z^2)z^2}\, y(z) = 0$$

where $n \in \mathbf{Z}_{\geq 0}$.

b. For what $z \in \mathbf{C}$ do the power series converge?

c. Give the singular points of the DE in $\mathbf{C} \cup \{\infty\}$. Are they regular or not?

8. Give the singular points of the Bessel equation $x^2y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0$ and find out if they are regular or not.

9. Consider the DE

$$x^2y''(x) - 3xy'(x) + 4y(x) = 0.$$

a. Use Frobenius' method to find a solution of the DE. Call it $y_1(x)$

b. Give a solution y_2 that is linearly independent from y_1. Use reduction of the order.

10. Prove the following properties of the Bessel functions J_ν (using the power series representation):

a. $\dfrac{d}{dx}(x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x)$.

b. $\dfrac{d}{dx}(x^{-\nu} J_\nu(x)) = -x^{-\nu} J_{\nu+1}(x)$.

c. $J_{-n}(x) = (-1)^n J_n(x)$ for $n \in \mathbf{Z}$.

11. Let $y''(z) + P(z)y'(z) + Q(z)y(z) = 0$ be a differential equation with three regular singular points in $z = 0, 1$ en $z = \infty$ (and no other singular points).

a. Show that $P(z) = \dfrac{p(z)}{z(z-1)}$, $Q(z) = \dfrac{q(z)}{z^2(z-1)^2}$ where $p(z), q(z)$ are polynomials of degree 1 and 2, respectively.

b. Show that the are numbers $\alpha, \beta \in \mathbf{C}$ such that $u(z) = z^\alpha (z-1)^\beta y(z)$ is a solution of the hypergeometric differential equation

$$z(1-z)u''(z) + (b - (a+c+1)z)u'(z) - acu(z) = 0. \qquad (\dagger)$$

c. Show that (†) has, besides $F(a, c; b; z)$, also a solution $z^{1-b}F(a-b+1, c-b+1; 2-b; z)$. Why are these solutions linearly independent if $b \neq 1$? What happens if $b = 1$?

d. Show that the elliptic function of the first kind $K(z) = \displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - z^2 \sin^2\theta}}$ is equal to $\dfrac{\pi}{2}F(\dfrac{1}{2}, \dfrac{1}{2}; 1; z^2)$.

12a. Show that $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi}}\dfrac{\sin x}{\sqrt{x}}$ en $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi}}\dfrac{\cos x}{\sqrt{x}}$.

b. Write $J_{3/2}(x)$ and $J_{-3/2}(x)$ in terms of $x, \sin x$ and $\cos x$. (Use problem 10).

c. Show that for $n = 0, 1, 2, \ldots$ there exist polynomials P_n and Q_n or degrees n and $n-1$ resp. such that

$$J_{n+1/2}(x) = x^{-n-1/2}(P_n(x)\cos x + Q_n(x)\sin x), \quad J_{-n-1/2}(x) = x^{-n-1/2}(P_n(x)\sin x - Q_n(x)\cos x).$$

The *spherical Bessel functions* $m = 0, 1, 2, \ldots$ are defined as $j_m(x) = \sqrt{\frac{\pi}{2x}} J_{m+1/2}(x)$ and $n_m(x) = \sqrt{\frac{\pi}{2x}} J_{-m-1/2}(x)$.

d. Show that $j_m(x)$ and $n_m(x)$ are solutions of the DE

$$x^2 y''(x) + 2xy'(x) + (x^2 - m(m+1))y(x) = 0.$$

13a. Show that $\displaystyle \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^n \theta d\theta = \begin{cases} \frac{1}{2^n} \cdot \binom{n}{n/2} & \text{for } n \in \mathbf{Z},\ n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases}$

b. Prove that $\displaystyle \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\cos\theta} d\theta \ = \ J_0(z).$
(Hint: give a power series for the integrand and use a.)

14. (*zeroes of the Bessel function.*) Let $y_m(x) = \sqrt{x} J_m(x)$ for $m \in \mathbf{R}$.

a. Show that

$$y_m''(x) + \left(1 + \frac{1/4 - m^2}{x^2}\right) y_m(x).$$

b. Show that $y_{1/2}(x) = a \sin x$ for some $a > 0$.

c. Use (b) and theorem 1.5 to prove that $J_m(x)$ has for $|m| \le 1/2$ infinitely many positive (and also infinitely many negative) zeroes.

d. Use problem 10 to show that $J_m(x)$ has infinitely many zeroes for all real values of m. (use induction to $[m]$.)

15. (*multipole expansion.*) Consider the function $F(z, x) = \dfrac{1}{\sqrt{1 - 2zx + z^2}}$.

a. Fix $x \in \mathbf{R}$, $|x| \le 1$. Show that $F(z, x)$ is a (complex) analytic function is for $|z| < 1$. Let the power series be $\displaystyle\sum_{n=0}^{\infty} A_n(x) z^n$.

b. Show that $(1 - x^2)F_{xx} - 2xF_x + z(zF)_{zz} = 0$ for $x \in \mathbf{R}$, $|x|, |z| < 1$.

c. Prove that it follows from (b) that $A_n(x)$ is a solution of the n-th order Legendre equation $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$.

d. Prove that $A_n(x) = P_n(x)$.

Let $\mathbf{x}, \mathbf{y}$ be vectors in $\mathbf{R}^N$ such that $\|\mathbf{y}\| < \|\mathbf{x}\|$. Let θ be the angle between $\mathbf{x}$ and $\mathbf{y}$.

e. Show that

$$\frac{1}{\|\mathbf{x}+\mathbf{y}\|} = \frac{1}{\|\mathbf{x}\|} \sum_{n=0}^{\infty} \left(\frac{\|\mathbf{y}\|}{\|\mathbf{x}\|} \right)^n P_n(\cos\theta).$$

16. The modified Bessel function of the first kind is defined as

$$I_\nu(z) = e^{-i\pi\nu/2} J_\nu(iz) = \sum_{m=0}^{\infty} \frac{(z/2)^{2m+\nu}}{m!\Gamma(m+\nu+1)}.$$

An integral expression for $I_\nu(z)$ is

$$I_\nu(x) = \frac{1}{2\pi i} \int_C e^{(x/2)(z+1/z)} \frac{dz}{z^{\nu+1}}$$

where the curve $C \subset \mathbf{C}$ starts in $-\infty$, approaches the origin $z = 0$, circles it counterclockwise and goes back to $-\infty$.

a. Give (the first term of) an asymptotic expression for $I_\nu(x)$ as $x \in \mathbf{R}$, $x \to \infty$.

17. The Bessel equation is given by

$$z^2 y''(z) + zy'(z) + (z^2 - \nu^2) y(z) = 0. \qquad (**)$$

a. Give the Bessel equation in self-adjoint form.

b. What is the adjoint equation of (**)?

c. Solve the adjoint equation. Express the solution in terms of the solutions of the Bessel equation. (Hint: use the Lagrange identity.)

18. Express the zeroth order Bessel function as an integral $\int_C f(t) e^{ixt} dt$. Use the method described in §1.9 (or see Chapter 15 of Hassani).

19. Give integral expressions of two linearly independent solutions of Airy's equation $y''(z) + zy(z) = 0$.

20. Show that

$$\int_x^\infty e^{x^2-t^2} dt \sim \frac{1}{2x} - \frac{1}{2^2 x^3} + \frac{1 \cdot 3}{2^3 x^5} - \frac{1 \cdot 3 \cdot 5}{2^4 x^7} + \ldots$$

Chapter II. Hilbert spaces.

1. The functions $e^{2\pi nix/\ell}$ ($n \in \mathbf{Z}$) form an orthogonal basis of the Hilbert space $H = L_2(-\ell, \ell)$.

a. Explain that the functions 1 and $\sin 2\pi nx/\ell, \cos 2\pi nx/\ell$ for $n \in \mathbf{Z}, n > 0$ also form an orthogonal basis of H.

b. Give the Fourier series of the function x with respect to the latter basis.

2. Let $H = L_2(-\infty, \infty)_w$ with weight function e^{-x^2}. An orthogonal basis is given by the Hermite polynomials $\{H_0(x), H_1(x), \ldots\}$ where the degree of H_n is n. Give the Fourier series of x^2 with respect to this basis. (Do not look up the form of the Hermite polynomials; the information given here should be sufficient to give the answer.)

3. Let H be the Hilbert space $\ell_2(\mathbf{C})$. The map $T : H \to \mathbf{C}$ is given by $T(x) = \sum_{n=1}^{\infty} x_n/n$.

a. Show that T is a well-defined linear operator.

b. Is T bounded? If so, give its norm $\|T\|$.

c. Illustrate the Riesz representation theorem for the case of T.

4. The evaluation operator $E : D \subset L_2(-\pi, \pi) \to \mathbf{C}$ is given by $E(f) = f(0)$. Its domain $D = D(E)$ is the linear subspace of continuous functions in $L_2(-\pi, \pi)$ (more precisely, functions having a continuous representant).

Show that E is not a bounded operator.

5a. Show that the spectrum of the differentiation operator $D = \frac{d}{dx}$ on $L_2(-\pi, \pi)$ is $\mathbf{C}$.

b. Use the functions $\cos nx$ to show that D is not bounded.

6. Let $H = \ell_2(\mathbf{C})$ and let L, R be the left- and right-shift operators

$$L(x_1, x_2, \ldots) = (x_2, x_3, \ldots), \qquad R(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots).$$

Give the adjoint operators $L^\dagger$ and $R^\dagger$.

7. Give an example of a Hilbert space and a linear operator $T : H \to H$ such that $\mathrm{im}(T^\dagger)$ is not equal to $(\ker(T))^\perp$.

8. Let $T : H \to H$ be a hermitian operator with domain H. Suppose that $\langle x, T(x)\rangle = 0$ for all $x \in H$. Show that $\langle x, T(y)\rangle = 0$ for all $x, y \in H$ and hence, that $T = 0$.

9. Let H be the Hilbert space $L_2(-1, 1)$ and let C be the subset of continuous functions in H.

a. Give an example to show that C is not a closed subset.

b. Show that the closure of C is H. (The closure of C is the smallest closed subspace that contains U.) Hint: show that the orthogonal complement of C is the zero set $\{0\}$. Use that the Legendre polynomials form an orthogonal basis of H.

10. Let $H = \ell_2(\mathbf{C})$ and let the operator $C : H \to H$ be given by

$$C(x_1, x_2, x_3, \ldots) = (x_1, x_2/2, x_3/3, \ldots).$$

a. Show that C is a bounded hermitian operator. What is $\|C\|$?

b. Show that C is a compact operator.

The operator C' is given by

$$C'(x_1, x_2, \ldots) = (x_2/2, x_3/3, \ldots).$$

c. Why is C' compact? You may use (b).

d. Give the spectra of C and C'. Is zero an eigenvalue?

11. Let H be the Hilbert space $L_2(0,1)$. Let $X(x) = x$. The operator $T : H \to H$ is given by $T(f) = Xf$.

a. Show that T is bounded and give the value of $\|T\|$.

b. Show that T is hermitian.

c. Give the eigenvalues of T.

d. Show that $\sigma(T) = [0,1]$.

e. Is T compact?

12. Let $H = \ell_2(\mathbf{C})$. The operator $T : H \to H$ is given by

$$T(x_1, x_2, x_3, \ldots) = (x_2 - x_1, x_3 - x_2, x_4 - x_3, \ldots).$$

a. Show that $\lambda \in \mathbf{C}$ is an eigenvalue of T if and only if $|\lambda + 1| < 1$.

b. Show that T is bounded and give the value of $\|T\|$.

c. Show that $\sigma(T) = \{\lambda \in \mathbf{C} : |\lambda + 1| \leq 1\}$.

d. Is T compact?

13. Let $R : \ell_2(\mathbf{C}) \to \ell_2(\mathbf{C})$ be the right-shift operator. Prove that $\sigma(R) = \{\lambda \in \mathbf{C} : |\lambda| \leq 1\}$.

14. Calculate the distribution derivative $\dfrac{d}{dx}|x|$.

15. Prove that $x\delta(x) = -\delta'(x)$.

16. Show that the following sequences $\{\delta_n\}_{n=1}^{\infty}$ are delta-sequences:

a. $\delta_n = \dfrac{n}{\sqrt{\pi}} e^{-n^2x^2}$.

b. $\delta_n = \dfrac{\sin nx}{\pi x}$.

17a. Let $f : \mathbf{R} \to \mathbf{R}$ be a differentiable function with zeroes $x_1, x_2, \ldots$ such that $f'(x_i) \neq 0$ in $i = 1, 2, \ldots$. Show that

$$\delta(f(x)) = \sum_{n=1}^{\infty} \frac{1}{|f'(x_n)|} \delta(x - x_n).$$

b. Give the value of the integral $\int_{-\infty}^{\infty} \delta(x^2 - \pi^2) \cos x dx$.

Chapter III. Integral equations.

1. For what values of λ has the equation

$$f(x) = x + \lambda \int_0^\pi f(t) \sin(x+t) dt$$

a solution?

2. Consider the integral equation

$$f(x) = x^2 + \lambda \int_0^1 (1+xt) f(t) dt.$$

Give the characteristic values and the eigenfunctions. Solve the equation. For what values of λ does the series converge?

3. Solve $f(x) = x^a + \lambda \int_0^\infty e^{-(x+t)} f(t) dt$. where $a \geq 0$. Are there any values of λ for which there is no solution?

4. Solve the following equations:

a. $f(x) = x + \frac{1}{2} \int_{-1}^1 (x+t) f(t) dt.$

b. $f(x) = x + \int_0^x f(t) dt.$

c $f(x) = \lambda \int_0^\pi f(t) \sin(x-t) dt.$

5. Transform the differential equation

$$y''(x) + xy'(x) + y(x) = 0, \quad y(0) = 1,\ y'(0) = 0$$

into a Volterra integral equation of the second type. Use partial integration to remove derivatives from within the integral. Solve the integral equation.

Chapter IV. Sturm-Liouville systems.

1. Consider the inhomogeneous Sturm-Liouville system

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0,\ y'(\pi) = 0.$$

a. Give the eigenvalues and eigenfunctions and state the orthogonality relation for the eigenfunctions.

b. Give the Fourier series of the function $f(x) = 1$ (with respect to the eigenfunctions).

c. Apply Parseval's theorem to the function $f(x) = 1$.

d. Give the Green's function $G(x,t)$ for the operator $Ly = y''$ on $[0,\pi]$ with boundary values $y(0) = y'(\pi) = 0$. Give both an explicit form and the Fourier series.

e. Solve the inhomogeneous boundary value problem

$$y''(x) = f(x), \quad y(0) = y'(\pi) = 0.$$

Give the solution in the form of an integral.

f. Consider the inhomogeneous boundary value problem

$$y''(x) + y(x) = f(x), \quad y(0) = y'(\pi) = 0.$$

For what $f(x)$ is there a solution? Give the solution in the case that it exists, in whatever form you like.

2. Consider the inhomogeneous S.L. problem

$$y''(x) + n^2 y(x) = \sin mx, \qquad y(0) = y(\pi) = 0,$$

where m, n are positive integers.

a. Fix n. For what values of m is there a solution? (Use the Fredholm alternative.)

b. Use the theory of Fredholm integral equations to solve the system, in the case that a solution exists.

3. Consider the Sturm-Liouville system $y'' + \lambda y = 0$, $y'(0) = 0$, $y'(1) = 0$ op $[0,1]$.

a. Give the eigenvalues and the corresponding eigenfunctions. State the orthogonality relation for the eigenfunctions.

b. Give the Fourier series of the function $f(x) = x$ (with respect to the eigenfunctions).

c. Apply Parseval's theorem to the function $f(x) = x$.

d. Give the Green's function $G(x,t)$ for the operator $Ly = y'' + (\pi^2/4)y$ on $[0,1]$ with the boundary values $y'(0) = y'(1) = 0$. What is the Fourier series of $G(x,t)$?

e. Solve the inhomogeneous boundary value problem

$$y''(x) + (\pi^2/4)y(x) = f(x), \quad y'(0) = y'(1) = 0.$$

Give the solution in the form of an integral.

f. Consider the inhomogeneous boundary value problem

$$y''(x) = f(x), \quad y'(0) = y'(1) = 0.$$

Give a condition on $f(x)$ such that there is a solution.

g. Give an integral form of the solution in the case that it exists. (Express the solution as a single integral.)

4. Consider the singular Sturm-Liouville system on the interval $[-1, 1]$ given by the Legendre equation

$$(1-x^2)y''(x) - 2xy'(x) + \lambda y(x) = 0, \quad \text{where } y(x),\ (1-x^2)y'(x) \quad \text{are bounded in } (-1, 1).$$

a. Show that for all $n = 0, 1, \ldots$ there is a polynomial eigenfunction of degree n.

b. Show that $P_n(x) = \dfrac{d^n}{dx^n}(x^2-1)^{2n}$ is a solution of the Legendre equation. Show that it is a polynomial of degree n and that $P_n(1) = 1$. ($P_n(x)$ is called the n-th Legendre polynomial).

c. Argue that $\int_{-1}^{1} P_n(x)P_m(x)dx = 0$ if $m \neq n$ so that the Legendre polynomials form a system of orthogonal polynomials.

Remark: By the Stone-Weierstrasz theorem mentioned in chapter 2, the Legendre polynomials form an orthogonal basis of $L_2(-1, 1)$.

5. Consider the boundary value problem

$$r^2R''(r) + rR(r) + (\lambda r^2 - n^2)R(r)$$

for $n = 0, 1, \ldots$, with $R(1) = 0$, and $R(r)$ continu in $r = 0$.

a. Write the differential equation in self-adjoint form and show that we obtain a singular Sturm-Liouville problem.

b. Show that the eigenvalues are α_{nj}^2 $(j = 1, 2, \ldots)$ where $0 < \alpha_{n1} < \alpha_{n2} \ldots$ are the positive zeroes of the Bessel function J_n and that the eigenfunctions are $y_n(r) = J_n(\alpha_{nj}r)$.

c. Give the orthogonality relation for the eigenfunctions.

6. Consider the Sturm-Liouville system $y'' + \lambda y = 0$ with boundary conditions $y(0) = 0, y'(1) - 2y(1) = 0$.

a. Find the eigenvalues and the eigenfunctions. Show explicitly that there are infinitely many eigenvalues $\lambda_1 < \lambda_2 < \ldots$ and that $\lambda_n/n^2 \to C$ as $n \to \infty$ (with $C \neq 0$ a constant.

b. Give the Green's function for the system.

7. Apply a Liouville substitution to Bessel's equation

$$(xy')'(x) + \left(x - \frac{\nu^2}{x}\right) y(x) = 0$$

to bring in into the form

$$v''(t) + \left(1 - \frac{\nu^2 - 1/4}{t^2}\right) v(t) = 0.$$

Let $A = A(t), \phi = \phi(t)$ be functions such that

$$v(t) = A \sin \phi, \quad v'(t) = A\sqrt{S} \cos \phi$$

where $S = S(t) = 1 - \frac{\nu^2 - 1/4}{t^2}$. (See also §4.4 of the lecture notes.)

a. Show that

$$\phi'(t) = 1 - \frac{\nu^2 - 1/4}{2t^2} + O(\frac{1}{t^3}), \qquad \frac{A'(t)}{A(t)} = O(\frac{1}{t^3}).$$

b. Integrate the above equations the show that

$$\phi(t) = t - \phi_\infty + \frac{\nu^2 - 1/4}{2t} + O(\frac{1}{t^2}), \qquad A(t) = A_\infty + O(\frac{1}{t^2})$$

where $A_\infty \neq 0$.

c. Conclude that $v(t) = A_\infty \sin(t - \phi_\infty + \frac{\nu^2 - 1/4}{2t}) + O(\frac{1}{t^2})$ as $t \to \infty$.

d. Give the asymptotic behaviour of the solutions of the Bessel equation as $x \to \infty$. (Do not bother which values of ϕ_∞, A_∞ belong to J_ν and $J_{-\nu}$ (or Y_ν).)

8. Consider the wave equation $u_{tt} = \Delta u$ for $t > 0$ on the square $G = \{(x, y) \in \mathbf{R}^2 : 0 < x, y < 1\}$ in $\mathbf{R}^2$ with homogeneous boundary conditions $u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$. Use separation of variables to find de frequencies of the eigenmodes.

9. Consider the one-dimensional heat equation $u_t = ku_{xx}$ where $u(x, t)$ is the temperature of a bar $0 \leq x \leq 1$. At time $t = 0$ the temperature is given by $u(x, 0) = f(x)$, the left end of the bar is kept at a constant temperature $u(0, t) = 0$ and the right end is isolated, so $u_x(1, t) = 0$ (there is no heat current).

Solve this initial- and boundary values problem by separation of variables.

10. Solve Laplace's equation $\Delta u = 0$ on the unit disk $\{x^2 + y^2 < 1\}$ in $\mathbf{R}^2$ with boundary condition $u(x, y) = \begin{cases} 1 & \text{if } x^2 + y^2 = 1, y > 0 \\ -1 & \text{if } x^2 + y^2 = 1, y < 0 \end{cases}$ by separation of variables. Use polar coordinates (the Laplacian in polar coordinates is $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\phi\phi}$).

Chapter V. Partial differential equations.

1. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a differentiable function that is invariant under the dilatation group, i.e. $f(x,y) = f(ax, ay)$ for $x, y, a \in \mathbf{R}$ and $a \neq 0$.

a. Show, by considering an infinitesimal transformation, that f satisfies the first order PDE $xf_x + yf_y = 0$.

b. Solve the PDE and show that f is a function of y/x (or x/y) only.

2. Consider the PDE $xu_x + yu_y = u$ where $u = u(x,y)$ is a real-valued function on $\mathbf{R}^2$.

a. Give the characteristics.

b. Impose on u the condition $u(x,0) = \phi(x)$ for some function ϕ. Is this boundary value problem well-posed?

c. Solve the boundary value problem

$$xu_x + yu_y = u, \qquad u(x,1) = \phi(x)$$

where ψ is some differentiable function on $\mathbf{R}$.

3. Consider the PDE $u_x + 2xu_y = C$ where $u = u(x,y)$ is a real-valued function on $\mathbf{R}^2$ and C is a real constant.

a. Give the characteristics.

b. Solve the boundary value problem with boundary condition $u(0,y) = \psi(y)$ where ψ is some differentiable function on $\mathbf{R}$.

c. Now impose instead of (b) the boundary condition $u(x,0) = \psi(x)$. What condition must be imposed on ψ in order that there is a solution?

4. Consider the second-order PDE

$$u_{xx} + 4u_{xy} + u_{yy} + 2u_x + 4u_y + 2u = 0.$$

Transform it into a PDE in standard form (5.13') for some function w and express w as a function of u.

5. Consider the diffusion equation on the half line $x > 0$

$$\begin{cases} u_t - ku_{xx} = 0 & \text{for } x > 0, t > 0 \\ u(x,0) = \phi(x) & \text{for } x > 0 \\ u(0,t) = 0 & \text{voor } t > 0 \end{cases} . \qquad (\dagger)$$

We can use the solution formula for the diffusion equation on $\mathbf{R}$ by defining $\phi(x)$ properly on the negative x-axis: let $\phi(-x) = -\phi(x)$.

a. Why is this a good choice? In what way would you extend ψ if the boundary condition on $t = 0$ were $u_x(0,t) = 0$?

b. Give a formula for the solution of † as an integral from $x = 0$ to ∞. What is the fundamental solution for the half-line?

6. Let A and B be two points in $\mathbf{R}^2$ and let ℓ be the (closed) segment between A and B. Let $H = \mathbf{R}^2 \backslash \ell$. For $X \in H$ let $u(X)$ be the directed angle between the half-lines XA and XB, $-\pi < u(X) < \pi$. Show that u is a harmonic function on H.

7. Show that if $u(r, \phi)$ is harmonic on the disk $\{r < R\}$ in $\mathbf{R}^2$, then the function $v(r, \phi) := u(R^2/r, \phi)$ is harmonic on the exterior $\{r > R\}$.

b. Prove (5.23).

8a. Show that for $0 \le r < 1$:

$$1 + 2\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}.$$

b. Use Poisson's formula to solve the following Dirichlet problem on the unit disk $\{r < 1\}$ in $\mathbf{R}^2$:

$$\begin{cases} \Delta u(x, y) = 0 & \text{voor } x^2 + y^2 < 1 \\ u(x, y) = 1 & \text{als } x^2 + y^2 = 1, y > 0 \\ u(x, y) = 0 & \text{als } x^2 + y^2 = 1, y < 0. \end{cases}$$

Use (a) and write the solution in the form of a Fourier cosine series.

c. Derive the following closed form for the solution:

$$u(x, y) = \frac{1}{2} + \frac{1}{\pi}\text{Arg}\left(\frac{1 + x + iy}{1 - x - iy}\right).$$

9. *The mean value theorem for harmonic functions in* $\mathbf{R}^n$. Let $\mathbf{a} \in \mathbf{R}^n$ $(n > 2)$ and $0 < \epsilon < R$ and let u be a harmonic function in $B(\mathbf{a}, R) = \{\mathbf{x} \in \mathbf{R}^n : \|\mathbf{x} - \mathbf{a}\| = R\}$.

Apply Green's second identity (5.15) for $G = B(\mathbf{a}, \epsilon)$ and $v = u_f + c$ where u_f is the fundamental solution of the Laplace equation and c is some real constant. Conclude that

$$\oint_{\|\mathbf{x}-\mathbf{a}\|=\epsilon} \frac{\partial u}{\partial n} d^{n-1}A = 0, \qquad u(\mathbf{a}) = \frac{1}{\Omega_n \epsilon^{n-1}} \oint_{\|\mathbf{x}-\mathbf{a}\|=\epsilon} u(\mathbf{x}) d^{n-1}A,$$

where Ω_n is the surface area of the unit ball $B(0, 1)$ in $\mathbf{R}^n$.

10. Show that $(\Delta + k^2)\dfrac{e^{ikr}}{r} = -4\pi\delta(\mathbf{x})$ for $\mathbf{x} \in \mathbf{R}^3$, $r = \|\mathbf{x}\|$ and $k^2 \in \mathbf{R}$.

11. Suppose that the function $u = u(r)$ satisfies $u_{rr} + \dfrac{n-1}{r}u_r + k^2 u = 0$. Let $w = r^{-1}u_r$. Prove that $w_{rr} + \dfrac{n+1}{r}w_r + k^2 w = 0$.

12. *A movie problem.* Let $u(x, t)$ be a solution of the one-dimensional wave equation $u_{tt} - c^2 u_{xx} = 0$ for $t > 0, x \in \mathbf{R}$ with initial conditions $u(x, 0) = \phi(x), u_t(x, 0) = \psi(x)$.

a. Let $\psi(x) = 0, \phi(x) = \begin{cases} \cos x & \text{if } |x|, \pi/2 \\ 0 & \text{otherwise} \end{cases}$. Draw the graph of $u(x, t)$ for both small and large values of t.

b. Let $\phi(x) = 0, \psi(x) = \begin{cases} \cos x & \text{if } |x|, \pi/2 \\ 0 & \text{otherwise} \end{cases}$. Draw the graph of $u(x,t)$ for both small and large values of t.

13. *The Doppler effect.* We consider a source that moves with speed $0 < v < c$ along the x-axis and which sends a signal that is observed by some stationary observer on the x-axis. This is modelled by the boundary value problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{voor } x \in \mathbf{R}, x \neq vt, t > 0 \\ u(vt,t) = \sin \omega t & t > 0 \\ u(x,0) = u_t(x,0) = 0 & x \neq 0 \end{cases}$$

a. Give the solution $u(x,t)$. Distinguish between the cases $x < vt$, $vt < x < ct$ and $x > ct$. (Hint: the (x,t)-plane is divided into two parts by the straight line $x = vt$ along which the source is moving. On $x = vt$ the solution is continuous but not differentiable. On each of the parts $x < vt$, $x > vt$ the solution is differentiable and satisfies the wave equation so that d'Alembert's formula holds for suitable functions ϕ, ψ. We must extend these functions to the whole of $\mathbf{R}$ in order to find a solution. We can use the value $u(vt,t)$ together with continuity of the solution. Compare the boundary value problem (5.28) of the lecture notes.

14. *Refraction of a one dimensional wave at the boundary of two media with different propagation speeds.*

Consider the initial value problem

$$\left\{ \begin{array}{c} u_{tt}(x,t) = c(x)^2 u_{xx}(x,t) \\ u(x,0) = f(x), \; u_t(x,0) = 0 \end{array} \right\} \quad (x \in \mathbf{R}, \; t > 0)$$

met $c(x) = \begin{cases} c_1 & \text{for } x > 0 \\ c_2 & \text{for } x < 0 \end{cases}$ for certain $c_1, c_2 > 0$, and $f(x) = \begin{cases} \sin x & \text{for } -2\pi \leq x \leq -\pi \\ 0 & \text{otherwise} \end{cases}$.

a. Solve the initial value problem. Express the solution $u(x,t)$ in terms of the function f. Assume that u and u_x are continuous at the boundary $x = 0$.

b. Draw the region in the (x,t)-plane where $u(x,t) \neq 0$ in the case that $c_1 > c_2$.

c. Discuss reflection and transmission/refraction of the wave at the boundary $x = 0$. Does the the sign of the solution change?

d. What happens if $c_1 < c_2$?

15. Does Huygens' principle hold in 1 dimension? Explain your answer.

16a. Derive d'Alembert's formula for the one-dimensional wave equation from Poisson's formula for the three-dimensional wave equation (5.32) by the method of descent.

b. Use Duhamel's principle to find a solution $u = u(x,t)$ of the inhomogeneous one-dimensional wave equation (with source term) with homogeneous boundary conditions

$$\Delta u(x,t) = f(x,t), \qquad u(x,0) = u_t(x,0) = 0.$$

c. Take $f(x,t) = \delta(x - x_0)\delta(t - t_0)$ (for $x_0, t_0 \in \mathbf{R}$ fixed) and give the Green's function for the one-dimensional wave equation on $\mathbf{R}$.

Chapter VI. Tensor algebra.

1. Let V be a (real or complex) vector space with basis $\{e_1,\ldots,e_n\}$. Let A be an invertible (real or complex) $n\times n$-matrix . Set $f_j = A^i_j e_i$ for $j=1,\ldots,n$.

a. Why is $\{f_1,\ldots,f_n\}$ a basis of V?

Let $\{e^1,\ldots,e^n\}$ and $\{f^1,\ldots,f^n\}$ be the dual bases in V^* of $\{e_1,\ldots,e_n\}$ and $\{f_1,\ldots,f_n\}$, respectively.

b. Show that $f^j = (A^{-1})^j_i e^i$ for $j=1,\ldots,n$.

2. Let T be a tensor of rank (r,s) with components $T^{i_1\ldots i_r}_{j_1\ldots j_s}$ with respect to some coordinate basis of the (finite-dimensional) vector space V and let T' be the contraction of T with respect to the k-th upper (contravariant) index and the ℓ-th lower (covariant) index

$$(T')^{i_1\ldots\hat{p}\ldots i_r}_{j_1\ldots\hat{p}\ldots j_s} = T^{i_1\ldots p\ldots i_r}_{j_1\ldots p\ldots j_s} \qquad (\dagger)$$

(where the hat means that the corresponding index is omitted and where the Einstein summation convention has been used).

Show that after transformation to a different basis of V (and the corresponding dual basis of V^*) T' transforms like a tensor of rank $(r-1,s-1)$. In other words, contraction of a tensor yields indeed a tensor.

3. Let $v_1,\ldots,v_n$ be vectors in some vector space V. Prove that

$$v_1\wedge\ldots\wedge v_n = \sum_{i_1=1}^{n}\ldots\sum_{i_n=1}^{n}\epsilon^{i_1\ldots i_n} v_{i_1}\otimes\ldots\otimes v_{i_n}.$$

(Note that this justifies in some sense the choice of coefficients in the definitions of the antisymmetriser and the wedge product of two tensors in §6.3).

4. Prove that the Levi-Cività-(pseudo)tensor ϵ with components ϵ_{ijk} is the only Cartesian pseudotensor of rank 3 in $\mathbf{R}^3$ and that there are no other Cartesian (pseudo)tensors of rank 3 in any R^n for $n>1$.

5. The tensor $\epsilon\otimes\epsilon$ (with components $\epsilon_{ijk}\epsilon_{\ell mn}$) is a Cartesian tensor of rank 6 in $\mathbf{R}^3$ (why?). We know that all tensors of even rank are tensor products of the Kronecker-deltatensor. Show that in fact

$$\epsilon_{ijk}\epsilon_{\ell mn} = \begin{vmatrix} \delta_{i\ell} & \delta_{im} & \delta_{in} \\ \delta_{j\ell} & \delta_{jm} & \delta_{jn} \\ \delta_{k\ell} & \delta_{km} & \delta_{kn} \end{vmatrix}.$$

6. Let u,v be given Cartesian vector fields in $\mathbf{R}^n$. Assume that there exists a linear connection between v and the tensor of second partial derivatives of u:

$$v_i = C_{ijk\ell}\frac{\partial^2 u_j}{\partial x^k\partial x^\ell} \qquad (*)$$

where $x^1,\ldots,x^n$ are Cartesian coordinates. Assume moreover that the tensor of coefficients C is isotropic, i.e.the values of the components $C_{ijk\ell}$ does not depend on the choice of the Cartesian coordinates (it remains the same whenever the coordinate axes are translated or rotated). Show that (∗) can be written in the form

$$v = A\Delta u + B\nabla(\nabla \cdot u)$$

where $\Delta u = \nabla \cdot \nabla u$ is the Laplacian of u.

7. Consider in $\mathbf{R}^3$ the Cartesian rank-2-tensor $\mathbf{I}$. With respect to a certain Cartesian coordinate system x^1, x^2, x^3 the tensor $\mathbf{I}$ has components $I_{11} = I_1$, $I_{22} = I_{33} = I_2$ and $I_{ij} = 0$ als $i \neq j$. (The matrix is then $\begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{pmatrix}$.) Determine how the components transform under a coordinate transformation $(x^1, x^2, x^3) \to (x'^1, x'^2, x'^3)$ in the following cases:

a. The coordinate axes are rotated about the x^1-axis about an angle θ.
b. The coordinate axes are rotated about the x^3-axis about an angle θ.

8. Let T be a tensor of rank (r, s). Any component of T has r contravariant and s covariant indices. Take any subset of either contravariant or covariant indices and symmetrize the components with respect to the chosen set of indices. This yields an object T' which is symmetric in the chosen set of indices. Is T' again a tensor? (In other words, is the concept of a tensor that is symmetric with respect to a given set of indices (either contravariant or covariant) a meaningful concept?) And how about antisymmetry? And what happens if we do not separate contravariant and covariant indices?

9. let V be a vector space with an inner product. Show that the definition of the Hodge star operator is independent of the chosen orthonormal basis, provided that the two bases have the same orientation. What happens if the orientation is different?

Chapter VII. Differential Geometry.

1a. Show that both cilindrical and spherical coordinates are regular coordinates on the subset $U \subset \mathbf{R}^3$ that one gets by omitting some (closed) half-plane that has the x_3-axis as its boundary.

b. Give the components of the metric tensor for cylindrical and for spherical coordinates in $\mathbf{R}^3$.

c. Let f be a differentiable function on U. Give the components of the (contravariant) gradient ∇f of f both in cylindrical and in spherical coordinates.

2. Let $x^1, \ldots, x^n$ be Cartesian coordinates on $\mathbf{R}^n$ and let $y^1, \ldots, y^n$ be regular coordinates on $U \subset \mathbf{R}^n$. Let $P \in U$. On the cotangent space $(T_P\mathbf{R}^n)^*$ we define an inner product by $(dx^i, dx^j) = \delta^{ij}$. Let $g = g_{ij} dy^i \otimes dy^j$ be the metric tensor on U. Show that $g^{ij} = (dy^i, dy^j)$.

3. Prove that the covariant derivative of the metric tensor is zero, i.e. $\nabla_i g_{jk} = 0$.

4. Give the values of the Christoffel symbols Γ^k_{ij} for polar coordinates in $\mathbf{R}^2$.

5. Give the expression of the Laplacian Δf of a function f in cilindrical and spherical coordinates.

6. Let $B = \{\mathbf{x} \in \mathbf{R}^3 : \|\mathbf{x}\| = 1\}$ be the unit sphere in $\mathbf{R}^3$. Let N, S be the points $(0,0,1)$ and $(0,0,-1)$ respectively, and $U_1 = B\backslash\{N\}$, $U_2 = B\backslash\{S\}$. The maps $\phi_1 : U_1 \to \mathbf{R}^2$ and $\phi_2 : U_2 \to \mathbf{R}^2$ that project a point $P \in B$ onto the intersection point of the line through P and N (and the line through P and S, respectively) with the plane $x_3 = 0$ are homeomorphisms between U_1 and $\mathbf{R}^2$ (U_2 and $\mathbf{R}^2$ resp.). Show that the transition function $\phi_2 \circ \phi_1^{-1} : \mathbf{R}^2 \to \mathbf{R}^2$ maps the point (x_1, x_2) onto $(\frac{x_1}{x_1^2 + x_2^2}, \frac{x_2}{x_1^2 + x_2^2})$ and argue that it is a diffeomorphism. This shows that the sphere is a differentiable manifold.

7. Let $M = \mathbf{R}^n$ and let $S = \{\mathbf{x} \in M : \|\mathbf{x}\| = 1\}$ be the unit sphere. $S = f^{-1}(0)$ where $f(\mathbf{x}) = \|\mathbf{x}\|^2 - 1$. Show that $df(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in S$. Conclude that S is a subvariety of M.

8. Let M, N be differentiable manifolds with dimensions m and n respectively. Let $x^1, \ldots, x^m$ and $y^1, \ldots, y^n$ be local coordinates about P and $f(P)$ on M and N respectively. Show that, for any tangent vector $X_P = X^i \frac{\partial}{\partial x^i}$ in $T_P M$,

$$f_*(X)_{f(P)} = X^i \frac{\partial f^j}{\partial x^i} \frac{\partial}{\partial y^j} = X(f^j) \frac{\partial}{\partial y^j}.$$

Furthermore, if $m = n$ and $\omega = g(\mathbf{y}) dy^1 \wedge \ldots \wedge dy^n$ is an n-form in some neighbourhood of $f(P)$, then show that

$$f^*\omega(\mathbf{x}) = (g \circ f)(\mathbf{x}) \left| \frac{\partial f}{\partial x} \right| dx^1 \wedge \ldots \wedge dx^n. \qquad (7.13')$$

9. Let $d : \bigwedge^p M \to \bigwedge^{p+1} M$ be the exterior differentiation operator on M. Prove that $d^2 = 0$.

10. Let ω be a 1-form on a differentiable manifold M and X, Y vector fields on M. Prove that

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

11. x, y are Cartesian coordinates on $\mathbf{R}^2$. Let the vector field X on $\mathbf{R}^2$ be given by $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. Show that the flow of X through the point (p, q) is given by

$$x(t) = p\cos t - q\sin t, \qquad y(t) = p\sin t + q\cos t.$$

(Thus the integral curves of X are circles).

12. Give the flow of the vector field $X = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$ through the point (p, q).

13a. Let M be a differentiable manifold and X a vector field on M. Show that for a 1-form ω

$$(L_X\omega)_i = X^j\partial_j\omega_i + \omega_j\partial_i X^j.$$

b. Show that $L_X dx^i = dX^i$.

c. Let $\omega = a_1dx^1 + a_2dx^2 + a_3dx^3 =: \mathbf{a}\cdot ds$ (where $ds = (dx^1, dx^2, dx^3)^T$) be a 1-form on $\mathbf{R}^3$ and let $\mathbf{v}$ be a vector field. Prove that

$$L_{\mathbf{v}}\omega = ((\nabla\times\mathbf{a})\times\mathbf{v} + \nabla(\mathbf{a}\cdot\mathbf{v}))\cdot ds$$

d. Let $\omega = b_1dx^2\wedge dx^3 + b_2dx^3\wedge dx^1 + b_3dx^1\wedge dx^2 =: \mathbf{b}\cdot d\sigma$ (where $d\sigma = (dx^2\wedge dx^3, dx^3\wedge dx^1, dx^1\wedge dx^2)^T$) be a 2-form on $\mathbf{R}^3$ and let $\mathbf{v}$ be a vector field. Prove that

$$L_{\mathbf{v}}\omega = (\nabla\times(\mathbf{b}\times\mathbf{v}) + \mathbf{v}\nabla\cdot\mathbf{b})\cdot d\sigma.$$

e. Suppose that M is Riemannian with metric tensor g. Calculate the components $(L_Xg)_{ij}$.

14 Let M be a differentiable manifold, $P \in M$, and $X, Y \in T_PM$. The commutator $[X, Y]$ is defined by $[X, Y](f) = X(Y(f)) - Y(X(f))$ where $f : M \to \mathbf{R}$ is a differentiable function.

a. Show that $[X, Y] \in T_PM$ (use the definition of a tangent vector given in §7.4)

b. Let N be another manifold and $\phi : M \to N$ be a differentiable map. Show that

$$\phi_*[X, Y] = [\phi_*X, \phi_*Y].$$

c. Let X be a vector field on M with flow f_t. Show that, for $g : M \to \mathbf{R}$ a differentiable function, and $P \in M$,

$$\lim_{t\to 0}\frac{(g\circ f_t)(P) - g(P)}{t} = X_P(g).$$

15. Let V be a vector space. Show that the inner product i_X with respect to a vector X is an antiderivation, i.e.

$$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^p\alpha\wedge d\beta.$$

16. Show that the covariant derivative of the metric tensor g on a Riemannian manifold is zero. (Note that this result holds for the affine connection, not in general).

17. Let $C = \{x_1^2 + x_2^2 = r^2\}$ be a cylinder in $\mathbf{R}^3$.

a. Show that C is a submanifold of $\mathbf{R}^3$.

b. Give the geodesic equation for C in terms of the cylindrical coordinates ϕ, z.
c. What are the geodesics on C?
d. What is the result of parallel displacement of a vector from a point on C along a circle $x_3=$ constant?

18. Let $K = \{x_1^2 + x_2^2 = x_3^2, x_3 > 0\}$ be a cone in $\mathbf{R}^3$.
a. Is K a submanifold of $\mathbf{R}^3$?
b. Give the geodesic equations for K. Choose suitable coordinates.
c. What are the geodesics on K?
d. What is the result of parallel displacement of a vector from a point on K along a circle $x_3=$ constant?

19. Consider the curve γ on the cylinder $C = \{x_1^2 + x_2^2 = 1\}$ in $\mathbf{R}^3$ with the parametric equations

$$x_1 = \cos\phi, x_2 = \sin\phi, x_3 = a\phi \text{ for } 0 \leq \phi \leq 2\phi$$

where ϕ, z are cylindrical coordinates.
a. What is the length of C?
b. Show that the angle between the curve and the curves $\phi = \phi_0$ is constant.
c. Displace the vector ∂_z parallel along C from the point $\phi = 0, z = 0$. What is the result?

20. On the unit sphere $S^2 = \{\|\mathbf{x}\| = 1\}$ in $\mathbf{R}^3$ the metric tensor in spherical coordinates is given by $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$.
a. Give all Christoffel symbols Γ^i_{jk}.
b. Show that for a point on the equator $\theta = \pi/2$ the coordinates θ, ϕ are normal coordinates.
c. Give the geodesic equation for S^2.
d. Explain why the equation of a great circle (i.e. a circle which has its center in the center of the sphere) is given by $A\cos\phi + B\sin\phi + C\cot\theta = 0$ where A, B, C are not all zero.
d. Show that the great circles are exactly the geodesics on on S^2.
e. What is the result of parallel displacement of the vector ∂_ϕ along the circle $\theta = \pi/4$?
f. What are the Killing fields on S^2?

21a. Give all Killing fields on Euclidian space E_3.
b. Give all Killing fields on Minkowski space M_4.

22. Show that $L_{[X,Y]} = L_X \circ L_Y - L_Y \circ L_X$ if X, Y are vector fields on some manifold M. Conclude that, if M is Riemannian and X and Y are Killing fields on M, then $[X, Y]$ is a Killing field.

23. Let M be an n-dimensional Riemannian manifold with metric tensor $\mathbf{g}$. Let $g = \det(g_{ij})$ and let $x^1, \ldots, x^n$ be a set of local coordinates. The n-form $\omega = \sqrt{g}dx^1 \wedge \ldots \wedge dx^n$ is a volume form on M. For a vector field X on M the divergence is (as in §7.7) defined by $\text{div}(X)\omega = d(i_X\omega)$.
a. Give an expression for the $(n-1)$-form $i_X\omega$ in terms of the local coordinates.
b. Let ∇ the metric connection. Show that $\text{div}(X) = \nabla_i X^i$.

24. Let $T^{\mu\nu}$ be a (contravariant) Lorentz tensor van rank 2.

a. Fix $\nu = \alpha$ and let $v^\mu = T^{\mu\alpha}$. Is v^μ a Lorentz vector?

b. Show that $\ell^i = T^{0i}$ $(i = 1, 2, 3)$ are the components of a Cartesian vector.

25. The energy momentum tensor for a perfect fluid $T^{\mu\nu}$ has with respect to a certain coordinate system (called its rest system) components $\begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$. How does $T^{\mu\nu}$ transform under a Lorentz boost

$$x'^0 = \gamma(x^0 + \mathbf{v}\cdot\mathbf{x}),\ \mathbf{x}' = \mathbf{x} + \frac{\gamma^2}{1+\gamma}(\mathbf{v}\cdot\mathbf{x})\mathbf{v} + \gamma x^0\mathbf{v}?$$

Here $\mathbf{x} = (x^1, x^2, x^3)^T$ is the spatial part of the 4-vector x^μ, $\mathbf{v} \in \mathbf{R}^3$ is the velocity vector and $\gamma = (1 - v^2)^{-1/2}$. Express the components of $T^{\mu\nu}$ in terms of the 4-velocity $u^\mu = (\gamma, \gamma\mathbf{v})$ and show that $T^{\mu\nu} = (\rho + p)u^\mu u^\nu - p\eta^{\mu\nu}$. (Hint: write the transformation matrices $\Lambda^\mu{}_\nu$ in terms of u^μ.)

26. Vector fields and orthogonal surfaces.

Let $\mathbf{v}(\mathbf{x}) = (v^1, v^2, v^3)$ be some vector field in $\Omega \subset E_3$. If $\mathbf{v}$ is continuous on Ω and nowhere zero, then the flow of $\mathbf{v}$ determines a set of integral curves, i.e. curves that are tangent to $\mathbf{v}$ in every point of Ω. These integral curves are solutions of the system of DE $x'^1(t) = v^1, x'^2(t) = v^2, x'^3(t) = v^3$, or $\dfrac{dx^1}{v^1} = \dfrac{dx^2}{v^2} = \dfrac{dx^3}{v^3}$. We ask ourselves if there also exists (locally) a family of surfaces $F(x^1, x^2, x^3) = c$ such that the vector field is everywhere orthogonal to the surfaces $F = c$. Such surfaces are called *orthogonal surfaces* of the vector field.

a. Express the condition that $F = c$ are orthogonal surfaces of $\mathbf{v}$ in terms of F and $\mathbf{v}$. Why is F a differentiable function of x^1, x^2, x^3?

b. Show that a necessary condition for the existence of a family of orthogonal surfaces is that $\mathbf{v}\cdot$ curl$(\mathbf{v})$= 0. (In fact it can be shown that this condition is also sufficient.)

We now consider the case that the vector field $\mathbf{v}(\mathbf{x})$ is nowhere zero and that the integral curves of $\mathbf{v}$ are geodesics with respect to some metric (not necessarily the standard Euclidian metric) on Ω. Assume that there exists some surface $F(x^1, x^2, x^3) = 0$ that is orthogonal to the vector field, so that the geodesics intersect the surface orthogonally. The surface $F = 0$ is a 2-dimensional submanifold of $\mathbf{R}^3$ and so there exists a local parametrisation $\mathbf{x}(t, u)$ of the surface. De geodesics can then also be parametrized by t en u: the geodesic $\gamma_{t,u}$ intersects $F = 0$ in $\mathbf{x}(t, u)$; if s is the arc length of the geodesic and we choose $s = 0$ on the surface $F = 0$, then s, t, u are regular coordinates.

c. Why is the surface $F = 0$ a submanifold of $\mathbf{R}^3$?

d. Show that $g_{st} = g_{su} = 0$ en show that curl$(\partial_s) = 0$.

e. Show that the surfaces $s = s_0$ are orthogonal to the bundle of geodesics and show that the distance between the planes $s = s_0$ en $s = s_1$ is everywhere the same.

Remark: Light rays in some medium $M \subset E_3$ with isotropic index of refraction $n(\mathbf{x})$ (i.e. the index of refraction is a scalar field - there is no dependence on the direction) are geodesics with respect to the metric $ds^2 = n(\mathbf{x})^2(dx^2 + dy^2 + dz^2)$. This is a result of Fermat's principle (light rays follow the path of shortest time; if c is the velocity of light in a vacuum, then s/c is a measure of the time) and the fact that geodesics are (locally) the paths of shortest length, a fact that can be shown with the aid of the theory of calculus of variations (for which see chapter 10). A bundle of light rays originating in a point P has an orthogonal surface (an infinitesimally small sphere with center P). As we showed above it follows that they then have a bundle of orthogonal surfaces $s = s_0$. If we choose $s = 0$ for the time that the bundle leaves P, then s/c measures the time that has elapsed since leaving P. The surfaces $s =$ constant are the *wave fronts*. Notice that this result is a result of geometrical optics and does not use the wave theory of light. It is known as *Malus' law*.

f. Show that the wave fronts are solutions of the scalar equation $(\nabla s)^2 = 1$.

Chapter VIII. Groups and representations.

1. Let G be a group with subgroups H, H'. Show that the intersection $H \cap H'$ is also a subgroup.

2. $g \in S_7$ is a permutations that maps

$$1 \to 2, 2 \to 5, 3 \to 4, 4 \to 7, 5 \to 6, 6 \to 1, 7 \to 3$$

Write g as a product of cycles. What is the smallest positive n such that g^n is the identity? How can you see this immediately from the cycle structure of g?

3. Show that the groups D_3 and S_3 are isomorphic.

4. Let n be a positive integer. $\phi : \mathbf{Z} \to \mathbf{C}^*$ maps $m \in \mathbf{Z}$ to $e^{2\pi i m/n}$.

a. Show that ϕ is a homomorphism.

b. Prove that the image of ϕ is isomorphic to $\mathbf{Z}_n$.

c. Use the homomorphism theorem to show that $\mathbf{Z}/n\mathbf{Z} \cong \mathbf{Z}_n$.

5. Is $SO(2) \times \mathbf{Z}_2 \cong O(2)$? Is $O(2)/\mathbf{Z}_2 \cong SO(2)$?

6. Is $\mathbf{Z}_n \times \mathbf{Z}_2 \cong D_n$? Is $D_n/\mathbf{Z}_2 \cong \mathbf{Z}_n$?

7. *Matrix groups.* Any bilinear form $(\ , \ )$ on $\mathbf{R}^n$ is given by $(x, y) = x^T K y$ for some $n \times n$-matrix K.

a. Show this and show that the form is non-degenerate if and only if K is invertible (the form is non-degenerate if the only $x \in \mathbf{R}^n$ such that $(x, y) = 0$ for all y, is $x = 0$).

b. Suppose that the form is non-degenerate. Let G be the subset of $n \times n$-matrices A such that $(x, y) = (Ax, Ay)$ for all $x, y \in \mathbf{R}^n$. Show that G is a subgroup of $GL(n, \mathbf{R})$. What might go wrong if the form is degenerate?

c. Suppose that the form is non-degenerate and symmetric, i.e. K is symmetric. Show that there exists some invertible matrix B such that $B^T K B = \text{diag}(I_p, -I_q)$ where $p + q = n$ (you can use that K is orthogonally diagonalizable).

From a theorem by Sylvester it follows that p and q are uniquely determined for a given K. If $K = \text{diag}(I_p, -I_q)$ and $p, q \neq 0$, the group G is called the pseudo-orthogonal group and is denoted by $O(p, q)$. If $q = 0$ then $G = O(n)$, the orthogonal group.

d. Argue that if the bilinear form is non-degenerate and symmetric, then G is isomorphic to one of the groups $O(p, q)$ (or $O(n)$).

e. Show that all matrices in $O(p, q)$ have determinant ± 1.

The subgroup of matrices with determinant 1 is denoted by $SO(p, q)$.

f. Show that the general form of a matrix in $SO(1, 1)$ is $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ with $t \in \mathbf{R}$. What is the general form of a matrix in $O(1, 1)$?

g. Suppose that the form is non-degenerate and antisymmetric, i.e. K is antisymmetric. Show that there exists some invertible matrix B such that $B^T K B = J_m = \begin{pmatrix} O & -I_m \\ I_m & O \end{pmatrix}$ where $n = 2m$. (This one may be hard. One of the ways to proceed is to show that K is (complex) unitarily diagonalizable, and subsequently show that $\mathbf{R}^n$ is the direct sum of linear subspaces which have an orthonormal basis $\{e, f\}$ such that $Ke = -af, Kf = ae$ for some real a. Then proceed as in the symmetric case.

In the case that $K = J_m$, G is called the real symplectic group $Sp(m, \mathbf{R})$. As in (d), one can see that if the form is non-degenerate and antisymmetric, then G is isomorphic to $Sp(m, \mathbf{R})$.

h. Show that the groups $Sp(1, \mathbf{R})$ and $SL(2, \mathbf{R})$ are isomorphic.

8. Prove that $U(2) \cong SU(2) \times U(1)/\{\pm I\}$.

9. (*the Lorentz group*). The Lorentz group $O(3, 1)$ consists of the real 4×4-matrices Λ such that $\Lambda^T H \Lambda = H$ where $H = diag(1, -1, -1, -1)$.

a. Show that $\Lambda \in O(3, 1) \Rightarrow \Lambda^T \in O(3, 1)$.

b. The orthogonal group $O(3)$ can be embedded as a subgroup of $O(3, 1)$. Prove that if $\Lambda \in O(3, 1) \cap O(4)$ (i.e. if Λ is an orthogonal Lorentz transformation) then $\Lambda = \begin{pmatrix} \pm 1 & \mathbf{0} \\ \mathbf{0}^T & R \end{pmatrix}$ where R is an orthogonal 3×3-matrix.

c. Give a basis of the vector space of infinitesimal generators of the Lorentz group.

d. Since an element $\Lambda \in O(3, 1)$ is invertible, Λ can be uniquely decomposed as SO, where S is a symmetric positive definite matrix and O is an orthogonal matrix (this is the *polar decomposition* of a matrix). S and O lie themselves in $O(3, 1)$. Show this. (You can use (a).)

O is a spatial rotation possibly combined with a spatial reflection ($\mathbf{x} \to -\mathbf{x}, x^0 \to x^0$) and/or time inversion ($x^0 \to -x^0, x^i \to x^i$) and S is a Lorentz boost.

e. Prove this by showing that $S = \begin{pmatrix} \cosh\theta & \mathbf{b}^T \\ \mathbf{b} & R \end{pmatrix}$ with $\theta \in \mathbf{R}$, $\mathbf{b} = \sinh\theta \mathbf{n}$, $\mathbf{n}$ is a unit vector in $\mathbf{R}^3$, and $R = I_3 + \mathbf{n}\mathbf{n}^T(\cosh\theta - 1)$. Subsequently show that S is the matrix for a Lorentz boost in the direction of $\mathbf{n}$. If you like, you can first consider the case that $\mathbf{n} = e_1$).

f. The group $O(3, 1)$ has 4 connected components, two of these have determinant 1 and two have determinant -1; two have $\Lambda_{00} > 0$ and two have $\Lambda_{00} < 0$ (Λ_{00} is the element in the first row and column of the matrix Λ; the zeroth component is the time component in special relativity.) The orthochronous Lorentz transformations are those which have $\Lambda_{00} > 0$ en $\det(O) = 1$. Show that these form a group (denoted by $SO^+(3, 1)$); (note that the orthogonal Lorentz transformations act almost trivially on the zeroth component and that is sufficient to consider only the action of the matrix S of part (d). It can be shown that this component is the connected component of the identity I_4 (in other words, there is a path $\Lambda(t)$ in $O(3, 1)$ from I_4 to every element in $SO^+(3, 1)$, but not to any other element of $O(3, 1)$).

g. Minkowski space M_4 can be divided into six parts: we denote a point $x \in M_4$ by $(x^0, \mathbf{x})$ where $\mathbf{x} \in \mathbf{R}^3$ is the spatial component of x and $(\mathbf{x}, \mathbf{x}) = \mathbf{x}^2$): 1. timelike vectors ($(x^0)^2 - \mathbf{x}^2 > 0$) with $x^0 > 0$. 2. timelike vectors with $x^0 < 0$. 3. spacelike vectors ($(x^0)^2 - \mathbf{x}^2 < 0$). 4. Lightlike vectors (which have $(x^0)^2 - \mathbf{x}^2 = 0$) with $x^0 > 0$. 5. Lightlike vectors with $x^0 < 0$. 6. $x^0 = 0, \mathbf{x} = \mathbf{0}$.

h. Show that the orthochronous Lorentz transformations map each of these six regions onto itself.

Representations.

10. Consider the symmetric group S_3. S_3 acts as a permutation group on the set $\{1, 2, 3\}$. For $g \in S_3$, let $T(g)$ be the matrix $(e_{g(1)}\ e_{g(2)}\ e_{g(3)})$ (i.e. with columns $e_{g(1)}, \ldots$).

a. Show that $T : S_3 \to GL(3, \mathbf{R})$ is a representation of S_3 (it is called the fundamental representation of S_3).

b. Show that $U = \text{span}(1, 1, 1)^T$ and $W = U^\perp$ are invariant subspaces of T.

c. Show that the restrictions T_U and T_W of T to U and W are irreducible. Give matrix representations of T_U and T_W.

11. Consider S_3 with the fundamental representation T given in problem 10. S_3 acts as a permutation group on the space $\mathcal{F}$ of functions from $\mathbf{R}^3$ to $\mathbf{R}$ as follows: for $g \in S_3$ and F such a function, let $S_g(F)$ be the function $S_g(F)(\mathbf{x}) = F(T_g^{-1}\mathbf{x})$ where $\mathbf{x} \in \mathbf{R}^3$.

a. Show that S is a representation of S_3.

b. Show that every symmetric function (like $x_1+x_2+x_3, x_1x_2x_3$) determines a 1-dimensional invariant subspace of $\mathcal{F}$.

c. What is the smallest invariant subspace of $\mathcal{F}$ that contains the function x_1x_2?

d. What is the smallest invariant subspace of $\mathcal{F}$ that contains the function $(x_1)^2$?

12. Let G be a group and T a finite-dimensional representation with character χ. Explain why $\chi(g) = \chi(g')$ if $g' = hgh^{-1}$ for $g, h \in G$. (A character has the same value for all elements in the same conjugacy class).

13. Prove that the regular representation of a finite group is indeed a representation.

14. (*Dual and conjugate representation.* Let $T : G \to GL(n, K)$ be a finite-dimensional representation of the group G. The conjugate and dual representations $\bar{T}$ and T^* have matrices

$$\bar{T}(g) = \overline{T(g)} \quad \text{and} \quad T^*(g) = (T(g)^{-1})^T$$

respectively. Show that $\bar{T}$ and T^* are indeed representations of G.

15a. Give the conjugation classes of S_4.

b. Give the characters of S_4 and classify the irreducible representations.

The fundamental representation T of S_4 is given by $T(g) = (e_{g(1)}\ e_{g(2)}\ e_{g(3)}\ e_{g(4)})$ (compare problem 9).

c. Decompose T as a direct sum of irreducible representations.

16. The dihedral group D_3, the symmetry group of the equilateral triangle, is generated by a rotation R over 120 degrees and a reflection S in one of the axes of symmetry. Then $R^3 = S^2 = I$, the identity element, and $RS = SR^2$.

a. Prove that D_3 is isomorphic to S_3.

Consider the representation $T : D_3 \to GL(2, \mathbf{C})$ given by

$$T(R) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}, \quad T(S) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

b. Show that T defines a representation of D_3 (and thus, of S_3).

c. Show that T is equivalent to the two-dimensional representation $T^{(3)}$ of S_3 that is given in the lecture notes.

17a. D_4 is the complete symmetry group of the square. It consists of all rotations and reflections that transform the square into itself. By numbering the vertices of the square (1,2,3,4) we see that D_4 acts as a permutation group on the vertices. As such, it is a subgroup of S_4.

a. Give the conjugation classes of D_4.

b. Give the characters of D_4 and classify the irreducible representations. Give matrix representations of each of them.

c. Give the Clebsch-Gordan decompositions of all tensor products of the irreducible representations.

18. Let S, T be finite-dimensional representations with representation spaces V and W, and with characters χ^S and χ^T respectively.

a. Show that the tensor product representation $S \otimes T$ is a representation with representation space $V \otimes W$ and character $\chi^{S \otimes T}$ such that

$$\chi^{S \otimes T}(g) = \chi^S(g)\chi^T(g).$$

The group S_3 has three irreducible representations $T^{(1)}, T^{(2)}, T^{(3)}$. Give the Clebsch-Gordan decompositions of each of the tensor product representations $T^{(\alpha)} \otimes T^{(\beta)} = m_\gamma^{\alpha\beta} T^{(\gamma)}$.

19. Consider a two-dimensional system of three masses positioned in an equilateral triangle and connected by springs with equal strength. Perform an analysis as in §8.3 to find the normal (vibrational) modes. Which modes are degenerate? (Hint: take as generalized coordinates the deviations (x_i, y_i) from the equilibrium position for each of the three masses ($i = 1, 2, 3$). How many zero modes do you expect?)

Chapter IX. Lie groups and Lie algebras.

1a. Show that every matrix in $SU(2)$ has the form $\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$ with $|a|^2 + |b|^2 = 1$ and argue that $SU(2)$ is a 3-parameter group.

b. Show that every matrix in $SU(2)$ is equal to $e^{i(\sigma\cdot\mathbf{n})\phi}$ with $\sigma = (\sigma_1\sigma_2\sigma_3)^T$ the column vector of Pauli matrices, $\mathbf{n}$ is a unit vector in $\mathbf{R}^3$ and $\phi \in \mathbf{R}$; show furthermore that

$$e^{i(\sigma\cdot\mathbf{n})\phi} = I\cos\phi + i(\sigma\cdot\mathbf{n})\sin\phi.$$

2. Let $M_{ij} = E_{ij} - E_{ji}$ and $J_1 = -M_{23}, J_2 = -M_{31}, J_3 = -M_{12}$. J_1, J_2, J_3 generate the Lie-algebra $so(3)$.

a. Show that $e^{\mathbf{x}\cdot J} \in SO(3)$ where $\mathbf{x}\cdot J = x^1 J_1 + x^2 J_2 + x^3 J_3$.

b. Conversely, show that every matrix in $SO(3)$ is of the form e^A with A antisymmetric (and thus, a linear combination of the J_k).

c. Prove that the linear map $\phi : so(3) \to su(2)$ given by $\phi(J_k) = -i\sigma_k/2$ is a Lie algebra isomorphism.

d. Prove that the map $\psi : SU(2) \to SO(3)$ given by $\psi(e^{-i\mathbf{x}\cdot\sigma/2}) = e^{\mathbf{x}\cdot J}$ is a Lie group homomorphism. Show moreover that for every $A \in SO(3)$ there are exactly two matrices $B \in SU(2)$ such that $\psi(B) = A$.

3. Show that $\mathbf{R}^3$ with the vector product $\mathbf{a}\times\mathbf{b}$ is a Lie algebra isomorphic to $so(3)$.

4a. Show that the vector space generated by the differential operators

$$y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

on $\mathbf{R}^3$ has a Lie-algebra structure (with commutation as the Lie bracket operation) and is isomorphic to $so(3)$.

b. Show that the Killing fields on $\mathbf{R}^3$ form a Lie algebra isomorphic to the Lie algebra $p(3)$ of the Poincaré group $P(3)$.

5a. Give a basis of the Lie algebra $u(n)$.

b. Show that for $n \geq 2$, the Lie algebra $u(n) \cong su(n) \oplus u(1)$.

c. Give the Killing form on $u(2)$. Show that the center of $u(2)$ is isomorphic to $u(1)$ and show that the restriction of the Killing form is non-degenerate on $su(2)$, so that $su(2)$ is semisimple.

d. Give the Cartan metric tensor on $su(2)$.

6. Give a condition for a $2m \times 2m$-matrix A such that $A \in sp(m, \mathbf{R})$ if and only if the condition is fulfilled.

7. Let $G = \mathbf{R}^n$ be Euclidian space with vector addition as the group operation. G is a Lie group.

a. What is the Lie algebra g of G?

b. What is the exponential map $\exp : g \to G$?

8. The group $A(1)$ is the group of dilatations and translations in one-dimensional Euclidian space. It consists of the maps $f : \mathbf{R} \to \mathbf{R}$ given by $f_{ab} : x \to ax + b$ with $a, b \in \mathbf{R}$, $a \neq 0$.

a. Is the group connected? If not, what are the connected components?

b. Show that $T : A(1) \to GL(2, \mathbf{R})$ given $T : f_{ab} \to \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ is a faithful representation of the group.

c. Give a matrix representation of the Lie algebra $a(1)$. Give a basis of $a(1)$ and determine the structure constants with respect to the basis.

d. Let $X \in a(1)$. Show that $\exp(tX)$ is of the form $\begin{pmatrix} e^{ct} & d(e^{ct} - 1) \\ 0 & 1 \end{pmatrix}$ for certain $c, d \in \mathbf{R}$. Use the defining property of the exponential map. What is X?

e. Is the image of $\exp : a(1) \to A(1)$ equal to the connected component of the identity?

9. Let G be the Lie group $GL(n, K)$ where $K = \mathbf{R}$ or $\mathbf{C}$. $f : G \to K$ is the map defined by $f(A) = \det(A)$.

a. Show that f is a differentiable map.

b. Show that $f_* : T_A G \to K$ is surjective for all A. Can you give an expression for $f_*(X)$ in terms of A and X?

10. Let $G = GL(n, K)$ where $K = \mathbf{R}$ or $\mathbf{C}$. Let $f : G \to Sym^n(K)$ be the differentiable map $f(A) = A^T A$.

a. Show that the set of symmetric matrices $Sym^n(K)$ is a differentiable manifold with a global Euclidian structure. What is its dimension?

b. Why is $T_B Sym^n(K) = Sym^n(K)$ for $B \in Sym^n K$?

c. Show that for all $A \in G$ the push-forward $f_* : T_A G \to Sym^n(K)$ is given by $f_*(X) = X^T A + A^T X$.

11. Let g be a Lie algebra. The *derived algebra* $[g, g]$ is the linear subspace of g that is generated by the commutators $[X, Y]$ with $X, Y \in g$.

a. Show that $[g, g]$ is an ideal of g.

b. Show that $[g, g] = g$ if g is simple.

c. Show that $[g, g] = g$ if g is semisimple.

12. Let g be a Lie algebra. $\mathrm{Der}(g)$ is the set of derivations $D \in \mathcal{L}(g)$. Show that $\mathrm{Der}(g)$ is a Lie subalgebra of $gl(g)$.

13. Consider the 3-dimensional Lie algebra ℓ generated by X, Y, Z with $[X, Y] = [X, Z] = 0$ and $[Y, Z] = X$. ℓ is called the *Heisenberg algebra*.

a. What is the center $z(\ell)$?

b. Show that ℓ is isomorphic to the Lie algebra n of strict upper triangular 3×3-matrices.

c. The Lie group N consists of all matrices e^A with $A \in n$. Give a description of N (i.e. what matrices lie in N?).

14. Consider the 3-dimensional Lie algebra p generated by X, Y, Z with $[X, Y] = Z, [X, Z] = -Y$ and $[Y, Z] = 0$.

a. What is the center $z(\ell)$?

b. Is p semisimple?

c. Show that ℓ is isomorphic to the Lie algebra $p(2)$ of the Poincaré group $P(2)$.

15. Let g be a Lie-algebra over $K = \mathbf{R}$ or $\mathbf{C}$ and let h be an ideal of g. We introduce the following equivalence relation $\sim$ on g: for $X, Y \in g$ we define $X \sim Y$ if $X - Y \in h$. The quotient g/h is the set of equivalence classes $\overline{X} = \{Y \in g : Y \sim X\}$.

a. Show that $\sim$ is an equivalence relation on g.

b. Show that g/h has the structure of a Lie-algebra with operations

$$\overline{aX + bY} = a\overline{X} + b\overline{Y}, \quad (a, b \in K), \qquad \overline{[X,Y]} = [\overline{X}, \overline{Y}].$$

16. Let g be a Lie algebra. The *lower central series* is $C^0 g = g \supset C^1 g \supset C^2 g \ldots$ where $C^{n+1} g = [g, C^n g]$. g is called *nilpotent* if $C^n g = \{0\}$ for some n.

a. Let g be a nilpotent Lie algebra. Show that ad_X is nilpotent for all $X \in g$.

b. Show that the Heisenberg algebra of problem 13 is nilpotent.

c. Let N be the subset of the Lie-algebra of (real or complex) strictly upper diagonal $n \times n$-matrices (with $[A, B] = AB - BA$), i.e. $A \in N$ if $A_{ij} = 0$ for $i > j$. Show that N is nilpotent.

17. Let g be a Lie algebra. The *derived series* $D^0 g, D^1 g, \ldots$ is defined as follows: $D^0 g = g$, $D^{n+1} g = [D^n g, D^n g]$ for $n \geq 0$. g is called *solvable* if $D^n g = \{0\}$ for some n.

a. Show that the Lie algebra p of problem 14 is solvable.

b. Show that a nilpotent Lie algebra is solvable.

Let g be a finite-dimensional Lie-algebra with solvable ideals a, b.

c. Show that the sum $a + b = \{X \in g : X = Y + Z \text{ with } Y \in a, Z \in b\}$ is a solvable ideal of g.

d. Conclude from (c) that g has a unique maximal solvable ideal. This ideal is called the *radical* $\mathrm{Rad}(g)$ of g.

e. Prove that g is semisimple if and only if its radical is zero.

f. Prove that the quotient Lie algebra $g/\mathrm{Rad}(g)$ is semisimple (see problem 15 for the concept of a quotient Lie algebra).

g. Let B be the subset of upper triangular $n \times n$-matrices (with $[A, B] = AB - BA$), i.e. a $n \times n$-matrix A lies in B if $A_{ij} = 0$ for $i > j$. Show that B is a solvable Lie algebra.

18. Let g be a Lie algebra and let $X, Y, Z \in g$. .

a. Show that ad_X is a derivation.

b. Show that $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$.

c. Let $(\, , \,)$ be the Killing form on g. Show that $(X, [Y, Z]) = ([X, Y], Z)$.

19. Let g be a semisimple Lie algebra with Cartan metric tensor g_{ij}. Let $X_1, \ldots, X_n$ be a (vector space) basis of g. The operator C is given by $C = g^{ij} X_i X_j$.

a. Show that the definition of C is in fact independent of the basis.

b. Show that C commutes with every $X \in g$.

20. Let X be a vector field on the Lie group G with flow f_t. Let $\phi : G \to G'$ a Lie group homomorphism. Show that $\phi \circ f_t$ is the flow of the vector field $\phi_* X$ on G'. Conclude that $\exp(t\phi_* X) = \phi(\exp(tX))$.

21. Let G be a Lie (matrix) group with Lie algebra g.

a. Let $h \in G$. Show that, if $X \in g$, then $hXh^{-1} \in g$.

b. Show that the map $\mathrm{Ad}(h) : g \to g$ defined by $\mathrm{Ad}(h)(X) = hXh^{-1}$ is a Lie algebra homomorphism.

c. Show that the *adjoint representation* Ad:$G \to GL(g)$ is indeed a representation of G.

d. Prove that the push-forward Ad_* is the adjoint representation ad: $g \to gl(g)$ of the Lie algebra g.

22. Show that the representation $T^{(1)}$ of $SU(2)$ is equivalent to the adjoint representation.

23. Consider the fundamental representation $T^{(1/2)}$ of dimension 2 of the group $SU(2)$. For $g \in SU(2)$ the matrix of $T^{(1/2)}(g)$ with respect to an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of the representation space V is given by $T^{(1/2)}(g) = \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix}$ for $a, b \in \mathbf{C}$ with $|a|^2 + |b|^2 = 1$. Now consider the tensor product representation $T^{(1/2)} \otimes T^{(1/2)}$.

a. Show that the matrix van g with respect to the basis $\{\mathbf{e}_1 \otimes \mathbf{e}_1, (\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)/\sqrt{2}, \mathbf{e}_2 \otimes \mathbf{e}_2, (\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1)/\sqrt{2}\}$ of $V \otimes V$ is given by $\begin{pmatrix} a^2 & -\sqrt{2}a\bar{b} & \bar{b}^2 & 0 \\ \sqrt{2}ab & |a|^2 - |b|^2 & -\sqrt{2}\bar{a}\bar{b} & 0 \\ b^2 & \sqrt{2}a\bar{b} & \bar{a}^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Use that if $A, B \in \mathcal{L}(V)$ are linear maps then the linear map $A \otimes B \in \mathcal{L}(V \otimes V)$ is defined by $(A \otimes B)(v \otimes w) = A(v) \otimes B(w)$ for $v, w \in V$.

b. Show that $T^{(1/2 \otimes 1/2)} = T^{(0)} \oplus T^{(1)}$ where $T^{(0)}$ is the trivial representation and $T^{(1)}$ is the three-dimensional irreducible representation of $SU(2)$.

By considering the infinitesimal transformations, we can find the corresponding representation (which we also call $T^{(j)}$) of the Lie algebra $su(2)$. Let $X_j = i\sigma_j$ ($j = 1, 2, 3$) (with σ_j the Pauli spin matrices). (Strictly speaking, the matrices $i\sigma_j$ represent X_j in the fundamental representation, so the matrix of $T^{(1/2)}(X_j)$ with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of V is $i\sigma_j$.)

c. Give the matrices of $T^{(1)}(X_j)$ with respect to the basis $\{\mathbf{e}_1 \otimes \mathbf{e}_1, (\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)/\sqrt{2}, \mathbf{e}_2 \otimes \mathbf{e}_2\}$ of $\mathrm{Sym}^2(V)$.

d. Show explicitly that $T^{(1)}$ is equivalent to the fundamental representation of $so(3)$ (which is given by the matrices J_1, J_2, J_3 in §9.1 of the lecture notes).

24. Left translation on the Lie group can be used to define a metric on A, making A into a Riemannian manifold: let $g(\, , \,)$ be an inner product on the tangent space T_eA at the identity e. Then we define the metric tensor g on A by requiring that left translations are isometries, i.e. $(L_h)^*g = g$ for $h \in A$ (where $L_h : A \to A$ is the left translation on A defined by $L_h(k) = hk$).

a. Explain that this means that $g_h(X_h, Y_h) = g(X_e, Y_e)$ for $h \in A$ and for $X, Y \in \mathcal{A}$ (i.e. X, Y are left-invariant vector fields on A), where g_h, X_h are the values of g resp. X in h.

It follows that parallel displacement on A is also defined by left translation: $(L_h)_*X_k$ is the parallel displacement of the vector X_k in T_kA from k to $hk = L_h(k)$. Left invariant vector fields are thus parallel everywhere on A.

b. Show that this implies that the curves $\gamma(t) = \exp(tX)$ are geodesics on A.

Chapter X. Calculus of variations.

1. Prove the chain rule (10.2) for the functional derivative.

2. Show that a (smooth) curve $\gamma(t) = (x(t), y(t))$ in $\mathbf{R}^2$ which has curvature zero in each of its points, is (part of) a straight line.

3. Let $z = z(x, y)$ be the equation of a surface W in $\mathbf{R}^3$ where (x, y) lie in a bounded domain $G \subset \mathbf{R}^2$. Assume that $z(x, y)$ is differentiable everywhere in G (so W is a smooth surface).

a. Show that the area of W is given by $\displaystyle\int_G \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy.$

Let ∂W be the boundary of W. The projection of ∂W onto the plane $z = 0$ is a closed curve. Consider all surfaces $z = z(x, y)$ that have boundaries ∂W. From this family of surfaces we want to find the surface that has smallest area. Such a surface is called a *minimal surface*.

b. Give the Euler-Lagrange equation for a minimal surface.

4a. Let $A(0, a)$ and $B(b, c)$ be two points in the plane with $a, b, c > 0$. Find a function $y = y(x)$ on $[0, b]$ whose graph passes through A and B such that the area of the surface of revolution that is obtained by rotating the graph of $y(x)$ about the x-axis is minimal. (Answer: $y = A \cosh B(x - C)$ for suitable A, B, C; the surface is called a *catenoid*).

b. Is the catenoid a minimal surface (see problem 3)?

5. A chain of a homogeneous material is suspended on two points with equal height. Find the shape that the chain assumes under the influence of gravity. Hint: you may assume for simplicity that the chain is a curve in a plane; let $(-a, 0)$ and $(a, 0)$ be the coordinates of the points of suspension. (The second coordinate is for the vertical direction). The length of the curve is fixed. The shape is determined by the fact that the center of gravity is at the lowest possible point.

6. Consider the cone $K \subset \mathbf{R}^3$ with equation $z = a\rho$ for $a > 0$ (in cylindrical coordinates ρ, ϕ, θ).

a. Show that the metric tensor is given by $ds^2 = (1 + a^2)d\rho^2 + \rho^2 d\phi^2$.

b. Derive the geodesic equations for K (in terms of ρ, ϕ) by using that the geodesics are stationary solutions of a suitable functional L.

7. Consider the Sturm-Liouville problem on $[0, 1]$

$$y''(x) + \lambda x y(x), \qquad y(0) = y(1) = 0.$$

Derive an upper bound for the smallest eigenvalue λ by the Rayleigh-Ritz method, using a suitable polynomial of degree two as a trial function.

8. Answer the same question as in (7) for the Sturm-Liouville problem

$$y''(x) - x^2 y(x) + \lambda y(x) = 0, \qquad y(-1) = y(1) = 0.$$

9. Determine the minimal value of the functional $J(y) = \displaystyle\int_0^1 x^4 y''(x)^2 + 4x^2 y'(x)^2\, dx$ under the condition that y is not singular in $x = 0$, and $y(1) = y'(1) = 1$.

10. Consider the two-dimensional boundary value problem on the disk $\{(r, \phi) : r \leq a\} \subset \mathbf{R}^2$

$$\Delta u + k^2 u = 0 \quad \text{for } r < a, \qquad u(a, \phi) = 0.$$

a. Find an upper bound for the smallest eigenvalue k^2. Use as a trial function a suitable linear function of r.

b. Use the method of separation of variables to show that the smallest eigenvalue is in fact $k^2 = \alpha^2/a^2$ where α is the smallest positive zero of the Bessel function $J_0(x)$. (Problem 10 of chapter I may be useful here.)