Endomorphism Rings of Abelian Varieties and their Representations

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1 Introduction

This lecture aims to give a basic understanding of the structure of the endomorphism ring on an abelian variety A and its representations on certain linear objects associated to A. I will first give the reader a reminder of the definitions that will be particularly helpful for this lecture.

Definition 1.1. Let k be a field. An *Abelian variety (AV) A over k* is a connected, complete (proper) algebraic variety together with a point $\mathcal{O} \in A(k)$ and morphisms of algebraic varieties

$$m : A \times A \to A, \quad s : A \to A$$

such that A forms a group with multiplication m and inversion s.

Definition 1.2. An Abelian variety is *simple* if it has no proper Abelian subvarieties.

Definition 1.3. An *isogeny* of Abelian varieties is a surjective morphism with finite kernel.

Definition 1.4. A *Lie group* is a smooth manifold M together with smooth maps

$$m : M \times M \to M, \quad s : M \to M$$

which define a group structure on M with multiplication m and inversion s.

Fact 1.5. A an Abelian variety over $\mathbb{C}$. Then A can be viewed as a compact, connected, complex Lie group.

2 The Category $\mathbb{Q} \otimes \mathcal{A}(k)$

Let $\mathcal{A}(k)$ be the category of Abelian varieties over the field k. This is an additive category, i.e.

$-\mathrm{Hom}_{\mathcal{A}(k)}(A, B)$ has the structure of an Abelian group
$-$composition is bilinear
$-\mathcal{A}(k)$ has finite direct products which also function as finite direct sums.

Definition 2.1. The *category* $\mathbb{Q} \otimes \mathcal{A}(k)$ is the same category as $\mathcal{A}(k)$, but for all objects A, B of $\mathcal{A}(k)$, $\mathrm{Hom}(A, B)$ is replaced by $\mathbb{Q} \otimes \mathrm{Hom}(A, B)$ (and the $\mathbb{Z}$-bilinear composition maps

$$\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \xrightarrow{\circ} \mathrm{Hom}(A, C)$$

are extended to $\mathbb{Q}$-bilinear maps).

Definition 2.2. The *universal functor of $\mathcal{A}(k)$ into a $\mathbb{Q}$-linear category* is the canonical functor

$$\begin{array}{rcl} \mathcal{A}(k) & \longrightarrow & \mathbb{Q} \otimes \mathcal{A}(k) \\ A & \mapsto & \mathbb{Q} \otimes A \\ \mathrm{Hom}(A, B) & \mapsto & \mathbb{Q} \otimes \mathrm{Hom}(A, B). \end{array}$$

We will write $\mathbb{Q} \otimes \mathrm{End}(-)$ for endomorphism rings in $\mathbb{Q} \otimes \mathcal{A}(k)$.

Remark 2.3. This functor sends isogenies to isomorphisms. To see this, consider an isogeny of Abelian varieties

$$g : A \longrightarrow B.$$

Then there exists $N \in \mathbb{Z}_{>0}$ and an isogeny

$$h : B \longrightarrow A$$

such that

$$g \circ h = h \circ g = [N].$$

Hence you can define an isomorphism

$$\begin{array}{rcl} \mathbb{Q} \otimes \mathrm{End}(A) & \xrightarrow{\sim} & \mathbb{Q} \otimes \mathrm{End}(B) \\ f & \mapsto & \frac{1}{N} \otimes (g \circ f \circ h). \end{array}$$

Fact 2.4. Every Abelian variety is isogenous to a direct product of simple Abelian varieties, and $\mathbb{Q} \otimes \mathcal{A}(k)$ is a semi-simple Abelian category, i.e. every object is the direct sum of simple objects. This will be discussed in more detail in Remark 3.8.

3 Linear objects associated to an Abelian variety

Consider first complex Abelian varieties. By Fact 1.5, a complex Abelian variety A can be viewed as a complex Lie group. Then it is natural to consider the following:

- $T_0 A$ - the tangent space at the identity
- $T_0^*(A)$ - the cotangent space at the identity
- $H_1(A, \mathbb{Z})$ - the first homology group
- $H_1(A, \mathbb{Z})$ - the first cohomology group.

Here we are considering the homology and cohomology as topological groups, so intuitively $H_1(A, \mathbb{Z})$ can be thought of as loops on A modulo homotopy. In fact there is a more explicit description for $H_1(A, \mathbb{Z})$. Lie group theory gives us a surjective map

$$\begin{array}{rcl} \exp : T_0(A) & \to & A \\ v & \mapsto & [\gamma_v(1)], \end{array}$$

where $\gamma_v(1)$ is a smooth curve on A (for those comfortable with Lie groups, this is the flowline of v through 1), and [ ] denotes homotopy class. In particular, the kernel can be identified with smooth closed curves on A modulo homotopy, so

$$H_1(A, \mathbb{Z}) = \ker(\exp : T_0(A) \to A).$$

We also have the following properties of our canonical linear objects:

$$\dim_{\mathbb{C}}(T_0(A)) = \dim_{\mathbb{C}}(T_0^*(A)) = \dim(A)$$

$$\text{rank}(H_1(A, \mathbb{Z})) = \text{rank}(H^1(A, \mathbb{Z})) = 2\dim(A).$$

We have a functor

$$H_1(-, \mathbb{Z}) : \mathcal{A}(\mathbb{C}) \longrightarrow \{\text{finite free Abelian groups}\}.$$

We could also take the homology with rational coefficients:

$$H_1(-, \mathbb{Q}) : \mathcal{A}(\mathbb{C}) \longrightarrow \{\text{finite dimensional } \mathbb{Q}\text{-vector spaces}\},$$

which extends uniquely to the $\mathbb{Q}$-linear functor

$$H_1(-, \mathbb{Q}) : \mathbb{Q} \otimes \mathcal{A}(k) \longrightarrow \{\text{finite dimensional } \mathbb{Q}\text{-vector spaces}\}.$$

Now consider Abelian varieties over a general field k. $T_0(A)$ and $T_0^*(A)$ are still defined, they are now k-vector spaces of dimension $\dim(k)$. However, $H_1(A, \mathbb{Z})$ and $H^1(A, \mathbb{Z})$ are no longer defined, so we attempt to define an analogue. There is not a unique analogue, but we will consider the Tate module.

Definition 3.1. The *Tate module* of an Abelian variety A over a field k is defined to be

$$T_l(A) = \varprojlim_n A[l^n](\overline{k}),$$

where $\overline{k}$ is some fixed algebraic closure, and the projective limit is taken with respect to the maps

$$l : A[l^{n+1}](\overline{k}) \to A[l^n](\overline{k}).$$

If char(k)=0, then the functor defined by T_0 extends uniquely to a $\mathbb{Q}$-linear functor

$$T_0 : \mathbb{Q} \otimes \mathcal{A}(k) \longrightarrow \{\text{finite dimensional } k\text{-vector spaces}\}.$$

Note that this gives

$$T_0 : \mathbb{Q} \otimes \mathrm{End}(A) \to \mathrm{End}_k(T_0 A).$$

Now consider $l > 0$, l prime, prime to the characteristic of k. Compose the functor

$$T_l : \mathcal{A}(k) \longrightarrow \{\text{finite free } \mathbb{Z}_l\text{-modules}\}$$

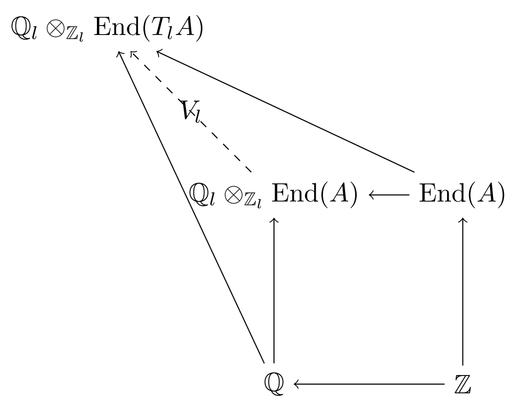
with the canonical functor

$$\begin{array}{ccc} \{\text{finite free } \mathbb{Z}_l\text{-modules}\} & \longrightarrow & \{\text{finite dimensional } \mathbb{Q}_l\text{-vector spaces}\} \\ M & \mapsto & \mathbb{Q}_l \otimes_{\mathbb{Z}_l} M \end{array}$$

to get the functor

$$\begin{array}{cccc} \phi : & \mathcal{A}(k) & \longrightarrow & \{\text{finite dimensional } \mathbb{Q}_l\text{-vector spaces}\} \\ & \mathrm{End}(A) & \mapsto & \mathbb{Q}_l \otimes_{\mathbb{Z}_l} \mathrm{End}(T_l A) \end{array}$$

Now we have the following commutative diagram



where

$$V_l : \mathbb{Q}_l \otimes_{\mathbb{Z}_l} \mathrm{End}(A) \longrightarrow \mathbb{Q}_l \otimes_{\mathbb{Z}_l} \mathrm{End}(T_l A)$$

is the unique $\mathbb{Q}_l$-algebra homomorphism induced by the universal property of the tensor product in the category of finite dimensional $\mathbb{Q}_l$-vector spaces.

Definition 3.2. For $a \in \mathbb{Q} \otimes \mathrm{End}(A)$, let $\chi(a)$ be the *characteristic polynomial* of the endomorphism $T_l a$ of $V_l A$.

Fact 3.3. χ has coefficients in $\mathbb{Z}$ and is independent of the choice of l.

Definition 3.4. Let K be a field. An *algebra* over K is a ring R with a homomorphism
$$K \to Z(R),$$
where $Z(R)$ denotes the centre of R. From now on we will be considering algebras which are finite dimensional over K. A K-algebra is *simple* if it has exactly 2 2-sided ideals, and *semi-simple* if it is the product of simple K-algebras. A K-algebra R is *central* if the map

$$K \to Z(R)$$

is an isomorphism.

For example, $\mathrm{Mat}_n(K)$ is a central simple K-algebra. An important example is the following:
If R a simple algebra over K. Then R is a central simple $Z(R)$-algebra.

Fact 3.5. If R is a semi-simple K-algebra and L is a separable extension of K, then $L \otimes_K R$ is a semi-simple L-algebra.

Fact 3.6. If R is a central simple K-algebra and K^{sep} a separable closure of K, then there exists an isomorphism

$$\iota : K^{\mathrm{sep}} \otimes_K R \xrightarrow{\sim} \mathrm{Mat}_n(K^{\mathrm{sep}})$$

of K^{sep}-algebras for some $n \in \mathbb{Z}_{>0}$. In particular,

$$n^2 = [\mathrm{Mat}_n(K^{\mathrm{sep}}) : K^{\mathrm{sep}}] = [K^{\mathrm{sep}} \otimes_K R : K^{\mathrm{sep}}] = [R : K].$$

Further,

$$\begin{array}{rcl} K^{\mathrm{sep}} \otimes_K R & \longrightarrow & \{\text{monic polynomials of degree} = n \text{ over } K^{\mathrm{sep}}\} \\ r & \mapsto & \chi(\iota(r)) \end{array}$$

is independent of the choice of ι and induces a function

$$\chi^{\mathrm{red}}_{R/K} : R \longrightarrow \{\text{monic polynomials of degree} = n \text{ over } K\}.$$

We will take this to be the definition of $\chi^{\mathrm{red}}_{R/K}$ for R a central simple K-algebra.

We now extend this definition.
If R is a simple algebra over K, define

$$[R : K]^{\mathrm{red}} := [R : Z(R)]^{1/2}[Z(R) : K]$$

and for $r \in R$,
$$\chi^{\mathrm{red}}_{R/K}(r) := N_{Z(R)[X]/K[X]}(\chi^{\mathrm{red}}_{R/Z(R)}(r)).$$

If R is a semi-simple algebra over K, say

$$R \cong R_1 \times \ldots \times R_s,$$

where the R_i are simple K-algebras, define

$$[R:K]^{\mathrm{red}} := \sum_i [R_i : K]^{\mathrm{red}}$$

and for $r \in R$, $r = (r_1, ..., r_s)$,

$$\chi_{R/K}^{\mathrm{red}}(r) = \prod_i \chi_{R/K}^{\mathrm{red}}(r_i).$$

Definition 3.7. $[R:K]^{\mathrm{red}}$ is the *reduced degree of* R, and for all $r \in R$, $\chi_{R/K}^{\mathrm{red}}(r)$ is the *reduced characteristic polynomial of* r.

Note that if R is commutative, then

$$[R:K]^{\mathrm{red}} = [R:K],$$

$$\chi_{R/K}^{\mathrm{red}}(r) = \chi_{R/K}(r).$$

Remark 3.8. If A/k an Abelian variety, then there exists a decomposition (up to isogeny) of A into simple Abelian varieties:

$$A \sim A_1^{h_1} \times \ldots \times A_s^{h_s},$$

where the A_i are non-pairwise isogenous. Now by using Remark 2.3 and that there do not exist any non-trivial homomorphisms between non-isogenous simple Abelian varieties, it can be seen that

$$\mathbb{Q} \otimes \mathrm{End}_k A \cong \mathrm{Mat}_{h_1}(\mathbb{Q} \otimes \mathrm{End} A_1) \times \ldots \times \mathrm{Mat}_{h_s}(\mathbb{Q} \otimes \mathrm{End} A_s).$$

Further, each $\mathrm{Mat}_{h_i}(\mathbb{Q} \otimes \mathrm{End} A_i)$ is in fact a simple $\mathbb{Q}$-algebra, since $\mathbb{Q} \otimes \mathrm{End}(A_i)$ is a division algebra over $\mathbb{Q}$. (Exercise: verify this!)
Therefore, $\mathbb{Q} \otimes \mathrm{End}_k A$ is a semi-simple $\mathbb{Q}$-algebra. In what follows we will look at commutative semi-simple subalgebras of semi-simple $\mathbb{Q}$-algebras. This set is partially ordered under inclusion, and contains maximal elements.

Fact 3.9. Let K be a field, R a semi-simple K-algebra and E a commutative semi-simple subalgebra of R. Then

$$[E:K] \leq [R:K]^{\mathrm{red}},$$

with equality iff E is maximal.

4 Representations of Simple Algebras

Let $\mathbb{Q}$ be the base field. Let R be a simple $\mathbb{Q}$-algebra, K its centre, $[R:K]=n^2$. Let F be a field with characteristic 0.
Consider an F-linear representation of R, i.e. a finite dimensional F-vector space V together with a $\mathbb{Q}$-algebra homomorphism

$$R \longrightarrow \mathrm{End}_F V.$$

Choose some algebraically closed field $\overline{F} \supseteq F$. Define

$$V_{\overline{F}} := \overline{F} \otimes_F V.$$

$V_{\overline{F}}$ can be considered as an $\overline{F}$-linear representation of the $\overline{F}$-algebra

$$\begin{aligned} \overline{F} \otimes_{\mathbb{Q}} R &\cong \overline{F} \otimes_{\mathbb{Q}} K \otimes KR \\ &\cong \left(\prod_{j:K\to\overline{F}} \overline{F}\right) \otimes_K R \\ &\cong \prod_{j:K\to\overline{F}} (\overline{F}\,{}_j \otimes_K R) \\ &\cong \prod_{j:K\to\overline{F}} \mathrm{Mat}_n(\overline{F}), \end{aligned}$$

where the product is taken over the embeddings of K into $\overline{F}$, and ${}_j\otimes_K$ denotes that the tensor product should be taken with $\overline{F}$ as a K-algebra by $j: K \to \overline{F}$. Further, the only finite dimensional $\overline{F}$-linear representations of $\mathrm{Mat}_n(\overline{F})$ are finite direct sums of the standard representation $\overline{F}_n$, hence

$$V_{\overline{F}} \cong \bigoplus_{j:K\to\overline{F}} (\overline{F}^n)^{m_j}.$$

V is a vector space, so it is commutative, hence

$$\chi_V(r) = \chi_V^{\mathrm{red}}(r) = \prod_{j:K\to\overline{F}} \chi^{\mathrm{red}}_{(\overline{F}^n)^{m_j}/K}(r_j) = \prod_{j:K\to\overline{F}} j(\chi^{\mathrm{red}}_{R/K}(r))^{m_j}.$$

I will now state some useful lemmas and theorems for deducing representations. Proofs can be found in [1].

Lemma 4.1. *Let R be a semi-simple $\mathbb{Q}$-algebra, V a finite dimensional faithful representation of R over a field F of characteristic 0. Then*

$$dim_F V \geq [R:\mathbb{Q}]^{red}.$$

If equality holds, then $\forall r \in R$,

$$\chi_V(r) = \chi^{red}_{R/\mathbb{Q}}(r).$$

Lemma 4.2. *Let R, V, F be as above, and E a commitative semi-simple subalgebra of R. Then*

$$[E : \mathbb{Q}] \leq [R : \mathbb{Q}]^{red} \leq dim_F V.$$

If equality holds, $\forall r \in E$,

$$\chi_V(r) = \chi_{E/\mathbb{Q}}(r)$$

and the commutant of E inside R is equal to E.

Theorem 4.3. *A/k an Abelian variety, $char(k) = 0$. Then the following are equivalent:*

(1) *$\mathbb{Q} \otimes EndA$ containes a commutative semi-simple $\mathbb{Q}$-algebra of degree $2dimA$.*
(2) *For each i, the division algebra $\mathbb{Q} \otimes EndA_i$ is a field of degree $2dimA_i$ over $\mathbb{Q}$.*

Finally, I will state a proposition that is intended to motivate the study of CM-fields, by showing a situation in which they appear. A proof can be found in [2].

Proposition 1. *Let K be an algebraic number field of finite degree, $\sigma \in Aut(K)$, $\sigma^2 = 1$. Let K_0 be a subfield of K, defined by*

$$K_0 := \{x \in K | \, \sigma(x) = x\}.$$

Suppose that for all non-zero x in K

$$\chi^{red}_{K/\mathbb{Q}}(x\sigma(x)) > 0.$$

Then K_0 is totally real*, and if $\sigma \neq id$ on K, then K is* totally imaginary*, and*

$$\forall \tau : K \xrightarrow{\sim} \mathbb{C}, \, \tau(\sigma(x)) = \overline{\tau(x)}.$$

References

[1] P. Bruin, *Endomorhpism rings of Abelian varieties and their representations*, (http://homepages.warwick.ac.uk/staff/P.Bruin/endomorphisms.pdf).

[2] Shimura G., Taniyama Y. *Complex multiplication of abelian varieties and applications to number theory*, (Math.Soc.Japan, 1961).