1 Categories and functors

Much of the so called 'conceptual mathematics' can be phrased short and precise in terms of categories and functors. This is more an efficient language than a theory in its own right en the categorial notions are justified largely by the number of concrete examples we find in all areas of math. The mathematical content of this section therefor lies mostly in the numerous examples.

Definition 1.1. *A category $\mathcal{C}$ consists of a collection of object and for each pair A, B of objects in $\mathcal{C}$ a set $Hom_{\mathcal{C}}(A, B)$ of morphisms from A to B. The sets of morphisms are pairwise disjoint an for every three objects A, B and C in $\mathcal{C}$ there is a compostion of morphisms*

$$\circ : Hom_{\mathcal{C}}(A, B) \times Hom_{\mathcal{C}}(B, C) \longrightarrow Hom_{\mathcal{C}}(A, C).$$

The following criteria are satisfied.

1. *For every $A \in \mathcal{C}$ the set $Hom_{\mathcal{C}}(A, A)$ contains an indentity id_A which acts as a neutral element with respect to composition.*

2. *For morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ it holds that $(h \circ g) \circ f = h \circ (g \circ f)$.*

The *morphisms* in $\mathcal{C}$ are also refered to as the *arrows* or *maps* in $\mathcal{C}$. One usually writes $\mathrm{Hom}(A, B)$ for $\mathrm{Hom}_{\mathcal{C}}(A, B)$ if it is clear which category is meant.

To avoid certain set theoretic paradoxes like the 'set of all sets' one does not demand that the objects of $\mathcal{C}$ form a set. They are in fact a *class* in the sense of set theory. We shall not deal with such logical finesses, which one usually avoids by working with *small categories* inside a suitable *universe*.

The existence of an identity morphism for every object enables us to talk about inverses of morphisms and with that of *isomorphisms*, morphisms with a two-sided inverse. Morphisms in $\mathrm{Hom}(A, A)$ are called *endomorphisms* of A and isomorphisms in this set are called *automorphisms*. The automorphisms form a group $Aut(A)$ under composition.

Note that there is no mention of elements anywhere in the definition of a category; we don't even assume the objects consist of elements. Also it is possible for the set $\mathrm{Hom}(A, B)$ to be empty for some A and B.

Categories are often refered to in terms of their objects, for example, the category $\mathfrak{Ab}$ of abelian groups or the category $\mathfrak{Mod}_R$ of R-modules. The reader is expected to understand that the morphisms in the category are the 'obvious' ones. In the case of $\mathfrak{Ab}$ this are the group homomorphisms and in the case of $\mathfrak{Mod}_R$ the R-module homomorphisms.

The category $\mathfrak{Ab}$ is in a natural way a *subcategory* of the category $\mathfrak{Grp}$ of all groups. In general, one calles a category $\mathcal{C}$ a subcategory of $\mathcal{D}$ if the objects in $\mathcal{C}$ are also objects in $\mathcal{D}$ and for every two objects A, B in $\mathcal{C}$ there is an inclusion $\mathrm{Hom}_{\mathcal{C}}(A, B) \subset \mathrm{Hom}_{\mathcal{D}}(A, B)$. If in fact $\mathrm{Hom}_{\mathcal{C}}(A, B) = \mathrm{Hom}_{\mathcal{D}}(A, B)$ for all A, B in $\mathcal{C}$ then $\mathcal{C}$ is called a *full subcategory* of $\mathcal{D}$.

Examples 1.2.

Before we proceed we give some of the numerous examples. Every reader can extend the following list in her favourite direction.

1. The category $\mathfrak{Sets}$ of sets with 'ordinary' maps as morphisms is a standard example of a category. The subcategory $\mathfrak{FSets}$ of finite sets forms a full subcategory f $\mathfrak{Sets}$. For every group G there is a category of G-$\mathfrak{sets}$ of sets with a G action. The morphisms are the G-equivariant maps.

2. The category $\mathfrak{Grp}$ of groups with group homomorphisms contains the category $\mathfrak{Ab}$ of abelian groups as a full subcategory. Similarly the category $\mathfrak{Rng}$ of rings with ring homomorphisms contains the full subcategory $\mathfrak{CRng}$ of commutative rings. These are all 'large' categories and one often works with smaller subcategories such as *finite* abelian groups or *Noetherian* rings.

3. The category $\mathfrak{Vec}_K$ of vector spaces over a field K with K-linear maps as morphisms had a full subcategory $\mathfrak{FVec}_K$ of finite dimensional K-vector spaces.

4. The modules over a ring R together with R-homomorphisms form a category $\mathfrak{Mod}_R$. For commutative R once can do just about every universal construction (fibered sums and products, quotients) in $\mathfrak{Mod}_R$. This makes $\mathfrak{Mod}_R$ into the typical example of an *abelian category.*

If R is taken to be the group ring $K[G]$ of a group G over a field K then $\mathfrak{Mod}_R = \mathfrak{Rep}_K(G)$, the category of K-representations of G.

5. The category $\mathfrak{Top}$ of topological spaces has continuous maps as its morphisms. One often works in full categories of topological spaces which have one or more extra properties (connected, Hausdorff, metric, compact, ...). The topology $\mathfrak{T}_X$ of a topological space X is itself a category. The objects are the open subsets of X and the morphisms are inclusions between these open subsets.

6. From every category $\mathcal{C}$ we can construct the *opposite category* $\mathcal{C}^{\text{opp}}$ by 'inverting all the arrows'. More precisely: the objects of $\mathcal{C}^{\text{opp}}$ are the same as those of $\mathcal{C}$ and the sets of morphisms $\text{Hom}_{\mathcal{C}^{\text{opp}}}(A, B)$ are in bijection with $\text{Hom}_{\mathcal{C}}(B, A)$, say, $f^{\text{opp}} \leftrightarrow f$. The composition of morphisms in $\mathcal{C}^opp$ is defined by $f^{\text{opp}} \circ g^{\text{opp}} = (g \circ f)^{\text{opp}}$.

As we can see in some of the examples above the sets $\text{Hom}_{\mathcal{C}}(A, B)$ sometimes inherit some extra structure from $\mathcal{C}$. In $\mathcal{C} = \mathfrak{Ab}$ we get abelian groups and in $\mathfrak{Mod}_R$ for commutative R we get R-modules.

The morphisms in a category $\mathcal{C}$ form another category $\mathfrak{Mor}(\mathcal{C})$. A morphism $\phi : f \to g$ in $\mathfrak{Mor}(\mathcal{C})$ from $f \in \text{Hom}_{\mathcal{C}}(A, B)$ to $g \in \text{Hom}_{\mathcal{C}}(C, D)$ is an ordered pair $\phi = (\phi_1, \phi_2)$ of morphisms in $\mathcal{C}$ that makes the following diagram commutes.

One often gets interesting subcategories of $\mathfrak{Mor}(\mathcal{C})$ by considering morphisms to or from a fixed object in $\mathcal{C}$. In the first case we take a fixed $A = C$ in the diagram above and only look at the morphisms $\phi = (\phi_1, \phi_2)$ with $\phi_1 = \text{id}_A$. In the second case we fix $B = D$ and look at morphisms with $\phi_2 = \text{id}_B$. We refed to either case as a category of objects over a fixed base object.

Examples 1.3.

For every commutative ring R one can view the category $\mathfrak{CAlg}_R$ of commutative R-algebras as the category of rings over R. After all, a morphisms of R-algebras $A-1 \to A-2$

respects the R-algebra structure and with that is a morphism between the structure maps $f_i : R \to A_i$ in $\mathfrak{CRng}$ which is the identity on R.

An interesting example of topological spaces over a fixed base object is given by the category $\mathfrak{Cov}_X$ of *coverings* of a topological space X. A map $f : Y \to X$ of topological spaces is called a *covering* if every point $x \in X$ has a neighbourhood $U_x \subset X$ such that $f^{-1}[U_x] \xrightarrow{f} U - x$ is the *trivial covering*. This means that the *fibre* $f^{-1}(x)$ above x is discrete in Y and that there is a homeomorphism $f^{-1}(x) \times U_x \to f^{-1}[U_x]$ which composed with f gives the projection on the second co-ordinate (make a picture!). A morphism ϕ from a covering $f_1 : Y_1 \to X$ to $f_2 : Y_2 \to X$ (a so-called *decking transform*) is a continuous map $\phi : Y_1 \to Y_2$ with $f_2 \circ \phi = f_1$.

Definition 1.4. *A (co-variant) functor $F : \mathcal{C} \to \mathcal{D}$ is a map that assigns to every object A of $\mathcal{C}$ an object $F(A)$ of $\mathcal{D}$ and to every morphism $f \in Hom_{\mathcal{C}}(A, B)$ a morphism $f_* = F(f) \in Hom_{\mathcal{D}}(F(A), F(B))$. Moreover, $(id_A)_* = id_{F(A)}$ and $(f \circ g)_* = f_* \circ g_*$.*

One often simply says that the construction of $F(A) \in \mathcal{D}$ from $A \in \mathcal{C}$ is 'functorial'. Such a construction has all sorts of nice 'stability properties' that make functorial notions much more managable than non-functorial ones.

Examples 1.5.

1. The forming of the commutator subgroup $[G, G]$ from a group G is a functor from $\mathfrak{Grp}$ to itself. The functor $G \mapsto G^{\mathrm{ab}} = G/[G, G]$ that assigns to every group its largest abelian quotient is a functor $\mathfrak{Grp} \to \mathfrak{Ab}$. Forming the center $Z(G)$ of a group G is *not* a functor $\mathfrak{Grp} \to \mathfrak{Ab}$ since in general a group homomorphism $f : G_1 \to G_2$ does not induce a group homomorphism between the centers.

2. Taking the unit group R^* of a ring R is a functor $U : \mathfrak{Rng} \to \mathfrak{Grp}$. For every positive integer n there is a functor $\mathrm{GL}_n : \mathfrak{CRng} \to \mathfrak{Grp}$ that assigns to a commutative ring R the group $\mathrm{GL}_n(R)$ of invertible $n \times n$-matrices with coefficients from R. Note that U and GL_1 are 'the same' functor.

3. The map $\mathfrak{CRng} \to \mathfrak{CRng}$ that assigns to every commutative ring R its *reduced ring* R/N_R, with N_R the nil-radical of R, is a functor. On the subcategory of reduced rings it is the identity.

4. A *forget functor* is a functor that forgets part of the structure of an object. There are, for example, forget functors from most of the categories metioned in 1.2 to $\mathfrak{Sets}$ that assign to a group (ring, vector space, etc.) the underlying set. Of the same nature are the functors from $\mathfrak{Rng}$ and $\mathfrak{Vec}_K$ to $\mathfrak{Ab}$ that assign to a ring or a vector space the underlying abelian addition group or the functors $\mathfrak{Rep}_K(G) \to \mathfrak{Vec}_K$ and G-$\mathfrak{sets} \to \mathfrak{Sets}$ that forget the G-action.