November 13, 2000

Dear Gan, Gross and Savin,

I looked at parts of your "Fourier … G_2"* and found the §4 painfully algebraic. Here is a geometric translation, over any base S. One has an equivalence of categories between the following three kinds of objects [morphisms := isomorphisms]

a) A vector bundle V of rank 2 and $p \in \mathrm{Sym}^3(V) \otimes (\overset{2}{\wedge} V)^{-1}$.

b) $a\colon P \to S$ proper smooth curve of genus 0, $\mathcal{O}(1)$ of relative degree 1, $p \in \Gamma(\mathcal{O}(3) \otimes a^*(\overset{2}{\wedge} a_*\mathcal{O}(1))^\vee)$.

c) A cubic algebra A (vector bundle of rank 3 with algebra structure).

The equivalence of (a) and (b) is clear: $P = \mathbb{P}(V)$ (in Grothendieck sense); $V = a_*\mathcal{O}(1)$. If $\mathcal{I} := \mathcal{O}(-3) \otimes \overset{2}{\wedge} a_*\mathcal{O}(1)$, the functor (b)→(c) is

$$A := R^0 a_*(p\colon \mathcal{I} \to \mathcal{O})$$

with $\mathcal{I}$ in cohomological degree -1. The long exact sequence of cohomology gives that $R^i a_*(p\colon \mathcal{I} \to \mathcal{O}) = 0$ for $i \neq 0$, and a short exact sequence

$$0 \to \mathcal{O}_S \to A \to R^1 a_* \mathcal{I} \to 0 .$$

If p is not a zero divisor, one has

$$Ra_*(\mathcal{I} \to \mathcal{O}) = Ra_*(\mathcal{O}/\mathcal{I}) ,$$

making the algebra structure of A clear. In general, it comes from a product on the complex $p\colon \mathcal{I} \to \mathcal{O}$, deduced from the $\mathcal{O}$-module structure of $\mathcal{I}$: $f \otimes g \mapsto$

* Duke Math J 115 (2002), 105–169.

fg, $i \otimes f \mapsto fi$, $f \otimes i \mapsto fi$, $i \otimes j \mapsto 0$. In the language a), $R^1a_*\mathcal{J} = R^1a_*(\mathcal{O}(-3) \otimes \overset{2}{\wedge} V)$ is canonically isomorphic to $V^\vee$. It could only be some $V^\vee \otimes (\overset{2}{\wedge} V)^{\otimes k}$, and that $k = 0$ is checked by looking at how $\lambda\colon V \to V$ acts.

How to go back from A is better understood in the Gorenstein case. In that case there is a curve of genus 0 $a\colon P_A \to S$ provided with $b\colon \mathrm{Spec}(A) \hookrightarrow P_A$, and it is unique up to unique isomorphism. This is a version of the fact that $PGL(2)$ is strictly 3 times transitive. The image of b is some divisor D_A, isomorphic to $\mathrm{Spec}(A)$, and one can take

$$\mathcal{O}(1) := \mathcal{O}(D_A) \otimes \Omega^2_{P/S},$$

the section 1 of $\mathcal{O}(D_A)$ becoming a section p of $\mathcal{O}(1) \otimes ({\Omega^2_{P/S}}^{-1}) \simeq \mathcal{O}(3) \otimes (\overset{2}{\wedge} a_*\mathcal{O}(1))^{-1}$.

It is important in all of this that we are not looking at binary cubic forms, i.e. at elements of some $\mathrm{Sym}^3(V)$ (on which $GL(V)$ acts via its quotient by μ_3) but at elements of $\mathrm{Sym}^3(V) \otimes (\overset{2}{\wedge} V)^\vee$ (on which the action is faithful).

Sincerely,

P. Deligne