

June 1, 2006

Dear Edixhoven,

Here is a direct construction. It is more easily expressed in the language of Artin stack, and my main ingredient is that the Artin stack of cubic algebras is smooth.

Proof of “smooth”: any cubic algebra over a field  $k$  is a quotient of  $k[X, Y]$ , and for any cubic algebra  $A$  (over a basis  $S$ ) the space of  $x, y \in A$  giving an embedding into  $\mathbb{A}_S^2$  is smooth (over  $S$ ) being open in the affine space  $A \times A$  (over  $S$ ). The scheme of  $(A, x, y)$  is hence smooth over the Artin stack, and maps onto the Artin stack. It hence suffices to prove that it is smooth. It is  $\text{Hilb}_3(\mathbb{A}^2)$ , and one invokes the smoothness of Hilbert schemes of 0-dimensional subspaces of a surface.

Functor  $c) \rightarrow a), b)$ . We have to construct  $V, p$  on the stack. Outside of  $A$ 's of the form  $k.1 + m, m^2 = 0$ , it is already done. What remains is of codimension  $\geq 2$  (in fact: 4). As, outside of this bad locus,  $V$  is just  $(A/1)^\vee$ , the extension of  $V$  by  $j_*$  is just  $(A/1)^\vee$ . The section  $p$  extends uniquely. If from those extended  $(V, p)$  we use the functor  $a, b \rightarrow c)$  to get  $A'$ , we have  $A \xrightarrow{\sim} A'$  outside of  $\text{codim} > 2$ , and hence everywhere.

This gives a right sided inverse to the functor  $a, b \rightarrow c)$ . To see it is a left inverse as well, one repeats the argument starting from the stack of  $(V, p)$ , for which smoothness is trivial. Applying the functors  $a, b \rightarrow c \rightarrow a, b)$ , we get  $V', p'$  and outside of  $\text{codim} \geq 2$  (in fact 4 again, of course), we have  $V, p \xrightarrow{\sim} V', p'$ , which extends.

Without stacks. You could express all of this without stack, by looking at the scheme  $M$  of cubic algebra given with a basis of the underlying vector space (or module). It is  $\text{GL}(3)$ -equivariant, and  $M$  is smooth over  $\text{Spec}(\mathbb{Z})$ , by the same Hilbert scheme argument as before. In fact, the stack is just  $[M: \text{GL}(3)]$ . An open subset of  $M$  corresponds to etale algebras:  $\text{GL}(3)/S_6$ , of relative dimension 9 over  $\text{Spec}(\mathbb{Z})$ . The bad locus is of relative dimension 5 [one has to give 1 and a supplementary plane]. Hence the codimension 4 previously claimed. You get this time an equivariant  $(V, p)$  over  $M$ , and this is the same as a functor  $c) \rightarrow a), b)$ .

All the best,

P. Deligne

P.S. The duality between  $A/1$  and  $V$  in the Gorenstein case can be understood as follows. We have  $D = \text{Spec}(A) \hookrightarrow P$ , and  $\mathcal{O}(1) = \Omega_P^1(D)$ . The dualizing sheaf of  $D$  is the restriction to  $D$  of  $\Omega_P^1(D)$ , and the duality is

$$\int_D a.\text{restriction to } D \text{ of } \omega \text{ in } \Omega_P^1(D)$$

or, equivalently

$$\mathcal{R}es_D((\text{extension } \tilde{a} \text{ of } a \text{ to a neighborhood}).\omega).$$

P.P.S. For rank  $n$  algebra, the same argument shows that the Artin stack of such algebras is non smooth only on the complement of the open subset of algebras for which the module  $\Omega^1$  is generated by two elements.