1 Plan

We define local complete intersections, which will form a nice class of morphisms that we will find to be a suitable class to look at. To define it, we need the notion of a regular immersion, which may be thought of as being the closed embeddings obtained from iteratively cutting out codimension-one subspaces. We then take a detour to linear algebra and define the quasi-coherent analogue of the linear-algebraic exterior algebra of a module. We use this notion to spice up our differentials from degree 1 to degree k, as well as to define the canonical sheaf, a fundamentally important line bundle obtained from the cotangent bundle.

With these definitions in place, we may state the Grothendieck Duality Theorem, a generalization of the classical Serre Duality, which essentially says that for local complete intersections the canonical sheaf is the unique sheaf inducing a certain perfect pairing on cohomology. We will not aim to prove this result; instead, it's perhaps more useful to take a look at a historical motivation behind this result, which is mostly absent in Liu's book. To do this, we revisit the old Serre Duality from the perspective of derived categories, where the generalization to Grothendieck Duality becomes more transparent — in a relative sense at least.

General remark. For simplicity, *all* our schemes are assumed to be locally Noetherian!

2 Regular Immersions

Definition 6.3.1. Let A be a ring. A sequence $a_1, \ldots, a_n$ of elements of A is called a regular sequence if a_1 is not a zero-divisor of A, and if for all $i > 1$, a_i is not a zero-divisor of $A/(a_1, \ldots, a_{i-1})$. If A is a Noetherian local ring (which is the only case we will care about) then regularity is independent of the order — see Exercise 6.3.1.

Definition 6.3.4. Now let $\pi : X \hookrightarrow Y$ be an immersion, and let x be a point in X. We say that π is a regular immersion of codimension n at x if the kernel of the natural map $\mathscr{O}_{Y,\pi(x)} \to \mathscr{O}_{X,x}$ is generated by a regular sequence of n elements in $\mathscr{O}_{Y,\pi(x)}$. We say π is a regular immersion if the property holds at every point of X.

Remark. Liu defines an immersion to be an open immersion followed by a closed immersion, and not the other way around, though if your world is locally Noetherian, the two notions are equivalent.

Remark. The geometric way to think about this notion is as follows. If $X \hookrightarrow Y$ is a closed immersion, then the associated ideal sheaf may be thought of as being the functions which cut out X from the bigger scheme Y. If the ideal sheaf is locally generated by a non-zero-divisor, we call the closed subscheme an effective Cartier divisor, which we'll see again in the next lecture. Intuitively, you should think of them as being the codimension-one subspaces in the way that

your intuition actually thinks of 'codimension-one stuff'. (So for instance, the closed subscheme $k[x, y]/(x)$ cut out from $k[x, y]/(x^2)$ is formally of codimension one, but intuitively you don't lose a dimension, so it's not a Cartier divisor.) A regular subscheme $X \hookrightarrow Y$ can then be thought of as being obtained from Y by taking an effective Cartier divisor on Y, then taking another effective Cartier divisor on that, and so on.

Example. Let's connect the algebraic definition with the geometric one. Say we have a closed immersion $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$. If the closed immersion is regular, then I is obtained from a regular sequence in A, and conversely, if I is obtained from such a regular sequence, then the immersion is regular.

Proof: Suppose I is obtained from a regular sequence. Now pick a point $[\mathfrak{p}]$ in $\operatorname{Spec} A/I$ with image $[\mathfrak{q}]$ in $\operatorname{Spec} A$. The natural map $A_{\mathfrak{q}} \to (A/I)_{\mathfrak{p}} = A_{\mathfrak{q}}/I_{\mathfrak{q}}$ should be obtained from a regular sequence. This should be clear; $I_{\mathfrak{q}}$ is obtained the same regular sequence that I is, and this sequence is stll regular in $A_{\mathfrak{q}}$.

Conversely, suppose that $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$ is a regular immersion. Then the map $A_{\mathfrak{q}} \to A_{\mathfrak{q}}/I_{\mathfrak{q}}$ is obtained from a regular sequence in $A_{\mathfrak{q}}$, say $(a_1/d_1, \ldots, a_n/d_n)$, with $d_i \notin \mathfrak{q}$. Clearly the regular sequence $(a_1, \ldots, a_n)$ still generates $I_{\mathfrak{q}}$, and we easily see that, as elements of A, they generate I as well. $\square$

Example. Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Recall that by Krull's Hauptidealsatz, $\dim A \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$, and that we call A a regular ring if equality holds. A regular scheme is then a (locally Noetherian) scheme whose local rings are regular. This notion is not unrelated to the one we introduced above. We claim that a closed immersion of regular schemes is a regular immersion.

Proof: The condition we want to prove is local so we may reduce to the case where we have an immersion of the form $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$, and where both $\operatorname{Spec} A$ and $\operatorname{Spec} A/I$ are regular schemes. Now take a point $[\mathfrak{p}]$ in $\operatorname{Spec} A/I$ and its image $[\mathfrak{q}]$ in $\operatorname{Spec} A$. Our goal is basically to show that the natural map $A_{\mathfrak{q}} \to (A/I)_{\mathfrak{p}} = A_{\mathfrak{q}}/I_{\mathfrak{q}}$ is obtained from a regular sequence. Since our schemes are regular, we may assume that $A_{\mathfrak{q}}$ and $A_{\mathfrak{q}}/I_{\mathfrak{q}}$ are regular rings.

Denote by $\mathfrak{m}$ the maximal ideal of $A_{\mathfrak{q}}$. Then $\mathfrak{m}/\mathfrak{m}^2$ is a k-vector space (k being the residue field of $A_{\mathfrak{q}}$); give it a basis $a_1, \ldots, a_n$. It's easy to see that $I_{\mathfrak{q}}/(I_{\mathfrak{q}} \cap \mathfrak{m}^2)$ is a k-vector subspace of $\mathfrak{m}/\mathfrak{m}^2$, so we may assume it has a subbasis $a_1, \ldots, a_r$ with $r \leq n$. A reasonable guess would then be to say that $A_{\mathfrak{q}}/I_{\mathfrak{q}}$ is obtained as $A_{\mathfrak{q}}/(a_1, \ldots, a_r)$, which would prove our result. We have a natural quotient map $A/(a_1, \ldots, a_r) \to A/I$ which we claim to be injective. The proof of this claim uses regularity of $A_{\mathfrak{q}}$ and $A_{\mathfrak{q}}/I_{\mathfrak{q}}$. By regularity, $A_{\mathfrak{q}}$ has Krull dimension n, and so $A_{\mathfrak{q}}/(a_1, \ldots, a_r)$ has dimension $n - r$; on the other hand, $A_{\mathfrak{q}}/I_{\mathfrak{q}}$ has dimension $n - r$ as well, because its residue field has basis $a_{r+1}, \ldots, a_n$. Since regular local rings are integral domains (Proposition 4.2.11), it now suffices to show the following:

Lemma. A surjective ring homomorphism between two integral domains of the same Krull dimension is necessarily injective as well.

Proof of Lemma: A surjective ring homomorphism $f : B \to A$ is precisely a closed immersion $\operatorname{Spec} f : \operatorname{Spec} A \hookrightarrow \operatorname{Spec} B$. The scheme-theoretic image of this map is $V(\operatorname{Ker} f)$. On the other hand, since the dimensions coincide, the scheme-theoretic image of $\operatorname{Spec} f$ must be all of $\operatorname{Spec} B$. Hence $V(\operatorname{Ker} f) = \operatorname{Spec} B$ and so $\operatorname{Ker} f = (0) = \{0\}$. $\qquad\qquad\square$

Example. Take the closed subscheme of $\mathbb{C}[x, y]/(xy)$ cut out by only the y-axis. Both intuitively and algebraically this does not give a regular immersion: on the one hand the y-axis is not intuitively codimension-one in the union of the x- and the y-axis; on the other hand, x is clearly a zero-divisor of $\mathbb{C}[x, y]/(xy)$.

Example. Here's a similar but more complicated example: take the closed subscheme of $\operatorname{Spec} \mathbb{C}[x, y, z]$ cut out by the equations xy, xz, and yz (i.e., the three coordinate axes of your original 3-space). Intuition tells us that this does *not* give a regular immersion: imagine cutting out xy from $\operatorname{Spec} \mathbb{C}[x, y, z]$, and then cutting out say yz from that. Clearly you don't lose a dimension everywhere! And indeed, when we look at it algebraically we run into problems, because (yz) is a zero-divisor in $\mathbb{C}[x, y, z]/(xy)$. (Rigorously proving this subscheme is not regular would be more difficult, because you'd have to verify that there are no other regular sequence from which the three coordinate axes can be obtained. This example is still sufficiently simple that it can be proved from the definitions without too much effort, but it certainly calls for a different characterization of regular immersions that is more useful for proving something is not regular.)

Definition 6.3.7. Let $\pi : X \hookrightarrow Y$ be an immersion. Decompose it into a closed immersion $i : X \to V$ followed by an open immersion $j : V \to Y$. Denote by $\mathscr{I}$ the ideal sheaf associated with the closed immersion. The <u>conormal sheaf</u> $\mathscr{C}_{X/Y}$ of X in Y is then defined to be $i^*(\mathscr{I}/\mathscr{I}^2)$; its dual $C^\vee_{X/Y} = \mathcal{H}om(\mathscr{C}_{X/Y}, \mathscr{O}_X)$ is the <u>normal sheaf</u>. Of course, we'd like this definition to be independent of the choice of decomposition of π. Take another decomposition $\pi = j' \circ i'$, with $i' : X \to V'$ a closed immersion and $j' : V' \to Y$ an open immersion. Both V and V' are basically just open subsets of Y. Clearly X would still be mapped into $V \cap V'$ so we may assume that $V \subseteq V'$ and that i and i' are practically the same map. In this situation equality of the two sheaves is locally trivial to prove.

Remark. How should we think of these objects? We should think of them as being analogous to the (co-)normal bundle you'd define in differential geometry. To motivate this analogy, consider the following situation:
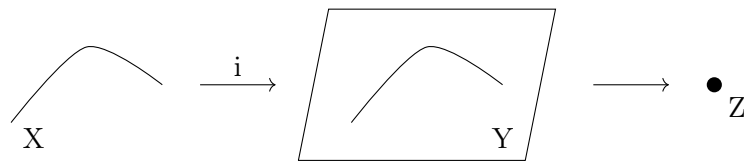
Imagine a point x in X. Its tangent space in X injects into its tangent space in Y; the cokernel of this injection is precisely the normal bundle at x in Y. Intuitively, we thus have an exact sequence of the form

$$0 \longrightarrow \mathscr{T}_X \longrightarrow i^* \mathscr{T}_Y \longrightarrow \mathscr{N}_{X/Y} \longrightarrow 0.$$

Now dualize this sequence and fill in $\mathscr{C}_{X/Y} = i^*(\mathscr{I}/\mathscr{I}^2)$ to get

$$0 \longrightarrow i^*(\mathscr{I}/\mathscr{I}^2) \longrightarrow i^* \Omega^1_{Y/Z} \longrightarrow \Omega^1_{X/Z} \longrightarrow 0. \tag{1}$$

This is precisely the long exact sequence we obtained in the previous lecture, except now we have a '$0 \to$' term on the left, which has to do with our intuition being smooth — see Proposition 6.3.13, which we'll treat in a moment. (N.B. This motivation is from Vakil's *Foundations of Algebraic Geometry*, paragraph 21.2.13.)

With this notion now more-or-less justified, we now verify that in the regular world, the conormal bundle is as nice as we'd expect it to be. This fact will be important in our construction of the so-called canonical sheaf.

Corollary 6.3.8. Let $\pi : X \to Y$ be a regular immersion of codimension n between (locally Noetherian) schemes. Then $\mathscr{C}_{X/Y}$ is a locally free sheaf on X of rank n.

Proof: The book refers to the following algebraic lemma.

Lemma 6.3.6. Let A be a ring, and I an ideal generated by a regular sequence $(a_1, \ldots, a_n)$. The image of the a_i in I/I^2 form a basis of I/I^2 over A/I. In particular, I/I^2 is a free A/I-module of rank n.

Its proof is not particularly interesting, so we'll omit it. Instead, let's look at why we can reduce to this lemma. Decompose π into a closed immersion $i : X \to V$ followed by an open immersion $V \to Y$. The map i is again regular by Exercise 6.3.3. Pick an affine open $\operatorname{Spec} A$ of V, and write its pre-image in X as $\operatorname{Spec} A/I$. Our goal is basically to show that I/I^2 is a free A/I-module. We proved somewhere above that regularity of an immersion of the form $\operatorname{Spec} A/I \to \operatorname{Spec} A$ implies that I is obtained from a regular sequence. Thus the hypothesis of the lemma is satisfied and the result follows. $\qquad \square$

Proposition 6.3.11 (d). The property of being a regular immersion is stable under flat base change. That is, if $f : X \to Y$ be a regular immersion of codimension n, and $Y' \to Y$ is a flat morphism, then the map $X \times_Y Y' \to Y'$ is also a regular immersion of codimension n.

Proof: We may reduce to the local situation, and as such we may consider the situation where we have a Cartesian diagram

$$
\begin{array}{ccc}
\operatorname{Spec} B/I & \longrightarrow & \operatorname{Spec} B \\
\downarrow & & \downarrow{\scriptstyle\varphi} \\
\operatorname{Spec} A/I & \longhookrightarrow & \operatorname{Spec} A,
\end{array}
$$

with $\operatorname{Spec} B \to \operatorname{Spec} A$ a flat morphism. Let's say I is obtained from A by a regular sequence $(a_1, \ldots, a_n)$. Then its image in B is obtained from B by a sequence $\big(\varphi^{\#}(a_1), \ldots, \varphi^{\#}(a_n)\big)$, with $\varphi^{\#}$ the (flat) ring homomorphism corresponding to the morphism φ in the diagram above. It remains to be shown that this sequence is flat. This is Lemma 6.3.10 in the book, which relies on induction on n. Let me explain why the induction is justified. The inductive basis amounts to showing that, given a flat ring map $f : A \to B$, and a not a zero-divisor in A, $f(a)$ is not a zero-divisor in B. Suppose that $\big(\varphi^{\#}(a_1), \ldots, \varphi^{\#}(a_k)\big)$ is known to be a regular sequence for some k. The next step would be to show that $\varphi^{\#}(a_{k+1})$ is not a zero-divisor in $B/\big(\varphi^{\#}(a_1), \ldots, \varphi^{\#}(a_k)\big)$. Now $\varphi^{\#}(a_{k+1})$ is also the image of $a_{k+1} + (a_1, \ldots, a_k)$ under the induced map $A/(a_1, \ldots, a_k) \to B/\big(\varphi^{\#}(a_1), \ldots, \varphi^{\#}(a_k)\big)$. Thus if we show that the induced map is flat, this is the exact same situation as that in the inductive basis.

Claim. If $f : A \to B$ is a flat ring map, and I is an ideal of A, then the induced map $A/I \to B/(f(I))$ is flat as well.

Proof of claim: By Theorem 1.2.4, for every ideal J of A, the map $J \otimes_A B \to JB$ is an isomorphism. We wish to show something similar for the induced map. Pick an ideal J' of A/I, and write it as J/I for some ideal J of A. Note that

$$
\begin{aligned}
J' \otimes_{A/I} B/IB &\cong J/I \otimes_{A/I} B/IB \\
&\cong J \otimes_{A/I} A/I \otimes_A B \\
&\cong J/I \otimes_A B \\
&\cong (J/I)B \cong J'(B/IB),
\end{aligned}
$$

as desired. $\qquad\square$

Remark. Exercise 6.3.2 provides us with a partial converse, provided $Y' \to Y$ is not just flat but faithfully flat. You shouldn't expect this converse to hold when dropping faithfulness. Any non-trivial localization morphism $R \to S^{-1}R$ is flat but not faithfully flat, so pick a morphism $X \to \operatorname{Spec} R$ which looks like a regular immersion near the localization $\operatorname{Spec} S^{-1}R$, but not globally, and you have your counterexample.

Proposition 6.3.11 (a). If $f : X \to Y$ and $g : Y \to Z$ are regular immersions, then so is $g \circ f$, and we have a canonical exact sequence

$$
0 \longrightarrow f^*\mathscr{C}_{Y/Z} \longrightarrow \mathscr{C}_{X/Z} \xrightarrow{\ \alpha\ } \mathscr{C}_{X/Y} \longrightarrow 0.
$$

Proof sketch: We prove the first part in Exercise 6.3.3. As for the exact sequence, we need only show left-exactness, because the rest of the sequence exists by construction of conormal sheaves. The proof in the book uses the fact that sheaves are locally free and coherent, and refers to several statements in previous chapters that we did not treat.

Finally, we mention without proof the following proposition. It generalizes the earlier example where we showed that closed immersions of regular schemes are regular, and it also formalizes Equation (1) which we based on intuition alone.

Proposition 6.3.13. Let X and Y be smooth schemes over a scheme S. Any immersion $\pi : X \to Y$ of S-schemes is a regular immersion, and we have a canonical exact sequence

$$0 \longrightarrow \mathscr{C}_{X/Y} \longrightarrow \pi^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow 0.$$

3 Local Complete Intersections

We have seen several beautiful characterizations of smooth morphisms; perhaps regrettably, smoothness is a rather exceptional property, and most morphisms encountered in the wild will not be this nice. It is therefore desirable to study a more general class of morphisms than that of the smooth morphisms. The class of morphisms that we will consider is that of the local complete intersection morphisms, which will over time prove itself to be a suitable class to look at, in part because it ends up in the statement of the main theorem that we are working towards.

Definition 6.3.17. Let $f : X \to Y$ be a finite-type morphism, and let x be a point of X. We say f is a <u>local complete intersection at x</u> if there exists a neighbourhood U of x such that $f|_U$ factors as a composition of $i : U \hookrightarrow Z$ and $g : Z \to Y$, where i is a regular immersion, and g is a smooth morphism. We say f is a <u>local complete intersection</u> if it is a local complete intersection at all of its points.

Proposition 6.3.20 (b) and (c). Local complete intersections are stable under composition and flat base change.

Proof: First note that being of finite type is known to be stable under composition and base change, so we need not worry about that. Let's start with the first claim.

Let $f : X \to Y$ and $g : Y \to Z$ be two local complete intersections. We may locally write $g \circ f = g_1 \circ g_2 \circ f_1 \circ f_2$ with f_2, g_2 regular immersions and f_1, g_1 smooth morphisms. If we manage to show that $h = g_2 \circ f_1$ is a local complete intersection, then locally we may write it has $g_1 \circ h_1 \circ h_2 \circ g_2$ with h_1 a regular immersion, and h_2 a smooth morphism. Since compositions of smooth morphisms and regular immersions are again smooth resp. regular, it would follow that $g \circ f$ is indeed a local complete intersection.

Thus it suffices to show the following. If $f : X \to Y$ is a smooth morphism, and $g : Y \to Z$ is a regular immersion, then $g \circ f$ is a local complete intersection. From this point on, it is more feasible to use the following property of smoothness.

Lemma (Remark 6.3.19). Let $\pi : X \to Y$ be a smooth morphism. Then π locally decomposes into a regular immersion into $\mathbb{A}_Y^n$ followed by the projection $\mathbb{A}_Y^n \to Y$.

Proof of Lemma: The difficult part is knowing that "$\mathbb{A}_Y^n$" means the fibred product of the morphisms $Y \to \operatorname{Spec} \mathbb{Z}$ and $\mathbb{A}_{\mathbb{Z}}^n \to \operatorname{Spec} \mathbb{Z}$. The map π looks locally like a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$. Since f is smooth, it is in particular of finite type, so we may assume B is a finitely generated A-algebra, hence we may write $B = A[x_1, \ldots, x_n]/I$ for some ideal I. Thus there exists a canonical map $\operatorname{Spec} B \to \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]$ and the resulting diagram

$$
\begin{array}{ccc}
\operatorname{Spec} B & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n] & \longrightarrow & \operatorname{Spec} \mathbb{Z}
\end{array}
$$

commutes, inducing a map from $\operatorname{Spec} B$ to the fibred product $\mathbb{A}_Y^n$. It remains to be shown that this resulting map $\operatorname{Spec} B \to \mathbb{A}_Y^n$ is in fact a regular immersion. Practically by construction the map $\operatorname{Spec} B \to \mathbb{A}_Y^n$ is a closed immersion. Proposition 6.3.13 then immediately tells us the immersion is regular: both $\operatorname{Spec} B$ and $\mathbb{A}_Y^n$ can be regarded as a smooth Y-scheme, and the map $\operatorname{Spec} B \to \mathbb{A}_Y^n$ is then a morphism of Y-schemes.

The second part of the claim is immediate: smoothness is stable under base change, and regular immersions are stable under base change as well by Proposition 6.3.11 (d), which we proved earlier. $\qquad \square$

Corollary 6.3.22. Let $f : X \to Y$ be a local complete intersection, and let's say f decomposes (globally!) as an immersion $i : X \to Z$ followed by a smooth morphism $g : Z \to Y$. Then i is a regular immersion. Moreover, if f is a regular immersion, then we have a canonical exact sequence

$$
0 \longrightarrow \mathscr{C}_{X/Y} \longrightarrow \mathscr{C}_{X/Z} \longrightarrow i^* \Omega^1_{Z/Y} \longrightarrow 0.
$$

Proof: We may assume that i is a closed immersion. Define $W = X \times_Y Z$, so that we have the diagram

$$
\begin{array}{ccc}
W & \xrightarrow{\;p_Z\;} & Z \\
p_X \big\uparrow\!\!\big\downarrow \pi & \nearrow i & \big\downarrow g \\
X & \xrightarrow[\;f\;]{} & Y,
\end{array}
$$

where π is the map induced by the universal property of W. By Exercise 3.3.6, π is a closed immersion, and in fact it's an immersion of smooth X-schemes because $p_X : W \to X$ and $\operatorname{Id} : X \to X$ are both smooth, hence by Proposition 6.3.13, π is a regular immersion. The

map i is thus a local complete intersection (being locally decomposed by the map π followed by the smooth map p_Z). As said in the previous proof, smooth maps $U \to V$ locally decompose into a regular immersion $U \to \mathbb{A}_V^n$ followed by the projection $\mathbb{A}_V^n \to V$. Regular immersions being closed under composition, i locally decomposes into a regular immersion $X \to \mathbb{A}_Z^n$ followed by the usual projection $\mathbb{A}_Z^n \to Z$. By Exercise 6.3.2 (c), it follows that i is in fact a regular immersion.

Now for the second claim. By Proposition 6.3.11 (a) we have an exact sequence

$$0 \longrightarrow \pi^*\mathscr{C}_{W/Z} \longrightarrow \mathscr{C}_{X/Z} \longrightarrow \mathscr{C}_{X/W} \longrightarrow 0.$$

We need only show that $\pi^*\mathscr{C}_{W/Z} \cong \mathscr{C}_{X/Y}$ and $\mathscr{C}_{X/W} \cong i^*\Omega_{Z/Y}^1$. The first isomorphism follows by noting that $\mathscr{C}_{W/Z} \cong p^*\mathscr{C}_{X/Y}$ because we have a pullback diagram (see Proposition 6.1.24), and that $p \circ \pi = \mathrm{Id}$; as for the second isomorphism, apply Lemma 6.3.13 to the morphism $\pi : X \to W$ viewed as a morphism of X-schemes to obtain $\mathscr{C}_{X/W} \cong \pi^*\Omega_{W/X}^1$, then note that $\Omega_{W/X}^1 \cong q^*\Omega_{Z/Y}^1$, again because we live in a pullback diagram, and conclude by noting that $q \circ \pi = i$. $\qquad\square$

4 The Determinant Bundle

Definition. Recall the following notions of linear algebra. If M is a module over a ring A, we define the <u>tensor algebra</u> $T(M)$ to be the direct sum

$$T(M) = A \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \cdots = \bigoplus_{n=0}^{\infty} T^n(M).$$

Its additive structure is as one would expect; multiplication is defined by the canonical isomorphism $T^k(M) \otimes T^\ell(M) \to T^{k+\ell}(M)$, which is then extended by linearity to all of $T(M)$. Note that it is non-commutative. The <u>exterior algebra</u>

$$\Lambda(M) = \bigoplus_{n=0}^{\infty} \Lambda^n(M)$$

is then defined to be the quotient of $T(M)$ by the ideal generated by all elements of the form $x \otimes x$ for all $x \in M$. Expanding $(a + b) \otimes (a + b)$, we see that $a \otimes b = -b \otimes a$ in $\Lambda^2(M)$, hence the exterior algebra is a skew-commutative A-algebra.

Example. Let M be a free module of rank n, say with basis $e_1, \ldots, e_n$. Then $T^k(M)$ is free of rank k^n, and the simple tensors $e_{i_1} \otimes \cdots \otimes e_{i_k}$ form a basis of $T^k(M)$. The tensor algebra $T(M)$ is isomorphic to the free non-commutative algebra $R\langle x_1, \ldots, x_n \rangle$ — in other words, the algebra of polynomials with non-commuting variables. The isomorphism $R\langle x_1, \ldots, x_n \rangle \to T(M)$ is found by sending the monomial $x_{i_1} \cdots x_{i_k}$ to $e_{i_1} \otimes \cdots \otimes e_{i_k}$, and linearly extending. The exterior algebra $\Lambda(M)$ is then $R\langle x_1, \ldots, x_n \rangle / I$ where I is the ideal generated by all x_i^2 and all $x_i x_j + x_j x_i$. We

may thus write $\Lambda(M) = \{e_1, \ldots, e_n : e_i \wedge e_i = 0 \text{ and } e_i \wedge e_j = -e_j \wedge e_i\}$. It follows that $\Lambda^k(M)$ is free of rank $\binom{n}{k}$; in particular, $\Lambda^n(M)$ is a free of rank 1 over A; it is called the <u>determinant</u> of M, and is denoted $\det M$.

Remark. Here's the reason for calling it this way. A morphism $f : M \to N$ of A-modules induces a morphism $\Lambda^k(M) \to \Lambda^k(N)$ by sending $m_1 \otimes \cdots \otimes m_k$ to $\varphi(m_1) \otimes \cdots \otimes \varphi(m_k)$. Taking A a field, M an n-dimensional A-vector space, and $N = M$, the induced linear map $\Lambda^n(M) \to \Lambda^n(M)$ is just a map $A \to A$ defined by multiplication by some scalar; this scalar is precisely the determinant of the transformation f.

Definition. The above concepts are instantly carried over to quasi-coherent sheaves over a scheme; indeed, quasi-coherent sheaves have natural notions of direct sums and tensor products, hence if $\mathscr{F}$ is a quasi-coherent sheaf, we may define the quasi-coherent sheaves $T^k(\mathscr{F})$ and $\Lambda^k(\mathscr{F})$; if $\mathscr{F}$ is locally free of rank n, then by the above discussion $T^k(\mathscr{F})$ and $\Lambda^k(\mathscr{F})$ are locally free of some rank as well; in particular, $\Lambda^n(\mathscr{F})$ is locally free of rank 1, and as such may be thought of as a line bundle; we will henceforth refer to it as the <u>determinant bundle</u>, and denote it by $\det \mathscr{F}$.

Corollary 6.4.2 (a) and (b). Given a morphism of schemes $\pi : X \to Y$, if $\mathscr{F}$ is a finite-rank locally free sheaf on Y, then $\det \pi^* \mathscr{F} \cong \pi^* \det \mathscr{F}$. Next let $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$ be an exact sequence of finite-rank locally free sheaves then $\det \mathscr{F} \cong \det \mathscr{E} \otimes_{\mathscr{O}_X} \det \mathscr{G}$.

Proof: As the second claim is proved in the book, we prove the first one. We may cover X with affine opens $\operatorname{Spec} A_i$ and Y with affine opens $\operatorname{Spec} B_i$ such that $\pi(\operatorname{Spec} A_i) \subseteq \operatorname{Spec} B_i$. Since $\mathscr{F}$ is locally free, we may furthermore assume that $\mathscr{F}|_{\operatorname{Spec} B_i} \cong \widetilde{N_i}$ for some free finite-rank B_i-module N_i. Now remember that on affine opens, the pullback has an easy description: we have $\pi^* \mathscr{F}|_{\operatorname{Spec} A_i} \cong (N_i \otimes_{B_i} A_i)^{\sim}$. We introduce an algebraic lemma to construct our isomorphism locally.

Lemma 6.4.1 (a). Let B be an A-algebra, and M a free A-module of finite rank. We have an isomorphism $(\det M) \otimes_A B \xrightarrow{\sim} \det(M \otimes_A B)$.

Proof of Lemma: Pick a basis $(e_1, \ldots, e_r)$ of M over A, let $e_i' = e_1 \otimes 1 \in M \otimes_A B$. The homomorphism is then defined by $(e_1 \wedge \cdots \wedge e_r) \otimes b \mapsto (e_1' \wedge \cdots \wedge e_r')b$. Since the inverse map (simply defined by taking the arrow in the other direction) is clearly a module homomorphism as well, we have an isomorphism. $\qquad\square$

Proof of Corollary 6.4.2, continued: We apply the above lemma to our situation to find the desired isomorphism locally. It remains to be checked whether this isomorphism 'patches together' on overlaps. Take affine opens $\operatorname{Spec} A_i$ and $\operatorname{Spec} A_j$ in X such that $\pi(\operatorname{Spec} A_i) \subseteq \operatorname{Spec} B_i$, $\pi(\operatorname{Spec} A_j) \subseteq \operatorname{Spec} B_j$. If they have non-trivial overlap, we should check that the restricted local isomorphisms coincide. The only non-trivial choice that we made in the construction of the lemma was a choice of basis, so if we show that the isomorphism is the same if we pick a

different basis we are done. But this is really not that hard to see: had we picked some other basis $(f_1, \ldots, f_r)$ of M over A, we would've sent $(f_1 \wedge \cdots \wedge f_r) \otimes b$ to $(f_1' \wedge \cdots \wedge f_r')b$ with $f_i' = f_i \otimes 1$. Then $e_1 \wedge \cdots \wedge e_r$ would've been $\lambda(f_1 \wedge \cdots \wedge f_r)$ for some λ. By linearity, this is sent to $\lambda(f_1' \wedge \cdots \wedge f_r')b$, which is precisely $(e_1' \wedge \cdots \wedge e_r')b$. $\qquad\square$

5   The Canonical Sheaf

Definition 6.4.3. If $\pi : X \to Y$ is a smooth morphism of finite type between locally Noetherian schemes, then by Proposition 6.2.5 the sheaf $\Omega^1_{X/Y}$ is locally free; hence we have a natural notion of a determinant bundle $\det \Omega^1_{X/Y}$ in this situation. There are two ways to generalize this to a larger class of morphisms. Here's the first way. If $\pi : X \to Y$ is an arbitrary morphism of schemes, then $\Omega^1_{X/Y}$ is quasi-coherent, so we can still define the sheaf $\Lambda^k \Omega^1_{X/Y}$ for $k > 1$; we call it the <u>sheaf of relative differentials of degree k</u> and denote it (of course) by $\Omega^k_{X/Y}$.

Definition 6.4.7. The other generalization is more subtle. Let $\pi : X \to Y$ be a quasi-projective local complete intersection. Recall that quasi-projectivity means that we can factor our map π into a finite-type open immersion $X \to W$ and a projective morphism $W \to Y$ (Definition 3.3.35). The projective morphism can in turn by definition be factored into a closed immersion $W \to \mathbb{P}^n_Y$ followed by the canonical projection $\mathbb{P}^n_Y \to Y$, where $\mathbb{P}^n_Y$ is just the fibred product of $\mathbb{P}^n_{\mathbb{Z}}$ and Y over $\operatorname{Spec} \mathbb{Z}$. Put in other words, we may (not necessarily uniquely) decompose π into an immersion $i : X \to Z$ followed by a smooth map $Z \to Y$. We define the <u>canonical sheaf</u> of π to be the invertible sheaf

$$\omega_{X/Y} = \det \mathscr{C}^{\vee}_{X/Z} \otimes_{\mathscr{O}_X} i^* \big(\det \Omega^1_{Z/Y} \big).$$

Remark. You can interpret '$\det \mathscr{C}^{\vee}_{X/Z}$' as both $\det \big(\mathscr{C}^{\vee}_{X/Z} \big)$ or $(\det \mathscr{C}_{X/Z})^{\vee}$; they are canonically isomorphic.

Lemma 6.4.5. The above definition is independent of the choice of the decomposition of π up to isomorphism.

Proof: Decompose π into two different ways, say

with i, i' immersions and g, g' smooth morphisms. Because π is assumed to be a local complete intersection, Corollary 6.3.22 tells us that i, i' are in fact regular immersions. Let $W = Z \times_Y Z'$,

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so that we get a diagram

$$
\begin{array}{ccccc}
 & & X & & \\
 & \overset{i}{\swarrow} \quad \downarrow{\scriptstyle h} \quad \overset{i'}{\searrow} & & & \\
Z & \overset{p}{\longleftarrow} \quad W \quad \overset{p'}{\longrightarrow} & Z', \\
 & \overset{g}{\searrow} \quad \downarrow \quad \overset{g'}{\swarrow} & & & \\
 & & Y & &
\end{array}
$$

By Corollary 6.3.22 again, we have the canonical exact sequences

$$ 0 \longrightarrow \mathscr{C}_{X/Z} \longrightarrow \mathscr{C}_{X/W} \longrightarrow h^*\Omega^1_{W/Z} \longrightarrow 0 $$

as well as

$$ 0 \longrightarrow \mathscr{C}_{X/Z'} \longrightarrow \mathscr{C}_{X/W} \longrightarrow h^*\Omega^1_{W/Z'} \longrightarrow 0. $$

Apply Corollary 6.4.2 (a) and (b) to the above exact sequences to find that

$$ \det \mathscr{C}_{X/W} \cong \det \mathscr{C}_{X/Z} \otimes h^* \det \Omega^1_{W/Z} \cong \det \mathscr{C}_{X/Z'} \otimes \det h^*\Omega^1_{W/Z'}. \tag{2} $$

Now, Proposition 6.1.24 characterized the behaviour of sheaves of relative differentials under base change; applying it to our situation gives us $\Omega^1_{W/Z} \cong (p')^*\Omega^1_{Z'/Y}$, and applying it once more gives us $h^*\Omega^1_{W/Z} \cong (i')^*\Omega^1_{Z'/Y}$. By symmetry, we also have $h^*\Omega^1_{W/Z} \cong i^*\Omega^1_{Z/Y}$. Plugging these isomorphisms into Equation (2) gives us the isomorphism

$$ \det \mathscr{C}_{X/Z} \otimes i^* \det \Omega^1_{Z/Y} \cong \det \mathscr{C}_{X/Z'} \otimes (i')^* \det \Omega^1_{Z'/Y}, $$

hence also

$$ \det \mathscr{C}^\vee_{X/Z} \otimes i^* \det \Omega^1_{Z/Y} \cong \det \mathscr{C}^\vee_{X/Z'} \otimes (i')^* \det \Omega^1_{Z'/Y}. \qquad \square $$

Notice that if $\pi : X \to Y$ is smooth, then the canonical sheaf $\omega_{X/Y}$ simply becomes $\det \Omega^1_{Z/Y}$ as we can take the required decomposition of π to be $\pi \circ \mathrm{Id}$. Our goal will be to try and compare the two sheaves $\omega_{X/Y}$ and $\Lambda^k \Omega^1_{X/Y}$ for $k > 1$ if π is a more general kind of morphism.

6 Grothendieck Duality I — Introduction

Warning. This section is rather sketchy, so if you like your results precise and accurate, some statements presented here might make you vomit.

In principle, we are now ready for the statement of Grothendieck Duality. Unfortunately, the result is rather abstract, and Liu gives little motivation or intuition leading up to the result. So before giving the actual statement we make a detour into the world of derived categories; we will see that from a derived perspective, Grothendieck Duality is a natural generalization of a result that we are already familiar with, namely Serre Duality. The following statement is Theorem 18.5.1 in Vakil's *Foundations of Algebraic Geometry:*

Theorem (Serre Duality). Let X be a nice (smooth, projective, other things) variety over a field k of topological dimension n. Then there exists an invertible sheaf ω (a <u>dualizing sheaf</u>) on X such that we have a functorial isomorphism

$$H^i(X, \mathscr{F}) \cong H^{n-i}(X, \mathscr{F}^\vee \otimes \omega)^\vee,$$

for all i, and all finite-rank locally free sheaves $\mathscr{F}$. (By 'functorial', we mean that we have a natural isomorphism of covariant functors $H^i(X, \cdot) \xrightarrow{\sim} H^{n-i}(X, \cdot^\vee \otimes \omega)^\vee$.)

The above statement has a number of horribly severe restrictions, and anyone who even remotely cares about general statements (which presumably includes everyone who is reading this) should wonder whether we can do similar things for (i) more general schemes over (ii) more general base schemes for (iii) more general sheaves. That's exactly what we will concern ourselves with.

We start out by (informally) introducing derived categories. They are basically just a fancy construction used to make working with derived functors more natural. The idea is as follows. Start with an abelian category $\mathscr{A}$, and denote by $D(\mathscr{A})$ the (as of yet mysterious) derived category. Given a left-exact functor $F : \mathscr{A} \to \mathscr{B}$, we want to capture all right-derived functors $R^i F$ into a single functor $\mathbf{R}F : D(\mathscr{A}) \to D(\mathscr{B})$. Since derived functors are obtained via injective or projective resolutions, it makes sense that we want the objects of $D(\mathscr{A})$ to be the (co-)chain complexes of $\mathscr{A}$, and indeed that's the case.

What about the morphisms? The choice of morphisms in $D(\mathscr{A})$ is based on the observation that calculation of (co-)homology of (co-)chain complexes over $\mathscr{A}$ is invariant under homotopy of chain complexes. Since things up to (co-)homology are more relevant right now, we want any two homotopy-equivalent morphisms in $\text{Ch}(\mathscr{A})$ identified in $D(\mathscr{A})$. One more thing we want is the following. Recall that quasi-isomorphisms are morphisms which induce isomorphisms on cohomology. In the derived world we would like such morphisms to be isomorphisms, because we only care about things up to homotopy-equivalence right now. Thus it makes sense that we want to add formal inverses of quasi-isomorphisms to $D(\mathscr{A})$.

To summarize, the <u>derived category</u> $D(\mathscr{A})$ of $\mathscr{A}$ has as its objects the chain complexes over $\mathscr{A}$, and has as its morphisms the morphisms of chain complexes modulo homotopy-equivalence, together with formal inverses of quasi-isomorphisms. Here's a first glimpse of their power:

Proposition. We have inclusions of $\mathscr{A}$ into $D(\mathscr{A})$: given an object B in $\mathscr{A}$, denote by $B[i]$ the cochain complex in $D(\mathscr{A})$ with a B on the i-th spot. We then have

$$\text{Hom}_{D(\mathscr{A})}(A[0], B[i]) \cong \text{Ext}^i(A, B),$$

the right-hand side being the usual Ext functor that you are familiar with.

Pseudo-proof: The object $B[i]$ is the shift of an injective resolution in $D(\mathscr{A})$, hence the Hom in $D(\mathscr{A})$ is just a way of computing derived functors of Hom. $\qquad\square$

That's all we need to take a second look at Serre Duality! Recall that if X is a nice scheme, then we have a line bundle ω such that we have isomorphism

$$H^i(X, \mathscr{F}) \cong H^{n-i}(X, \mathscr{F}^\vee \otimes \omega)^\vee.$$

An alternative (and on first sight less attractive) way to state the result would be to say that

$$\mathrm{Hom}_k\big(H^i(X, \mathscr{F}), k\big) \cong H^{n-i}(X, \mathscr{H}om(\mathscr{F}, \omega)),$$

for all locally free sheaves $\mathscr{F}$. A slightly more general statement would be to say that, for all k-vector spaces V,

$$\mathrm{Hom}_k\big(H^i(X, \mathscr{F}), V\big) \cong H^{n-i}(X, \mathscr{H}om(\mathscr{F}, V \otimes \omega)) \cong \mathscr{E}xt^{n-i}(\mathscr{F}, V \otimes \omega),$$

where $\mathscr{E}xt$ is the Ext-functor in the (abelian) category of quasi-coherent sheaves on X. To get to the statement we want, we make three observations about the above isomorphism.

- The generalization from k to V may not appear to be interesting, but from a relative point of view it gives us a big clue. Indeed remember that X is a k-scheme, i.e. equipped with a morphism $f : X \to \mathrm{Spec}\, k$, and note that the V are actually the finite-rank sheaves on $\mathrm{Spec}\, k$. The line bundle ω which appears on the right-hand side may as well be thought of as some kind of functor sending a finite-rank sheaf V on $\mathrm{Spec}\, k$ to the sheaf $V \otimes \omega$.
- Remember that the cohomology groups $H^i(X, \mathscr{F})$ are really just defined as the right-derived functors of the pushforward functor f_*. Hence in the derived world, they are represented by a single functor $\mathbf{R}f_*(\mathscr{F})$.
- By the proposition we just 'proved', the $\mathscr{E}xt$'s are represented by the Hom in the derived world.

We can now give the derived version of Serre Duality. If X is a scheme, denote by $D(X) = D(\mathrm{QCoh}_X)$ the derived category of quasi-coherent sheaves on X. Then we find:

Theorem (Serre Duality revisited). Let X be a nice k-scheme with structural morphism f, and let $D(X)$ be the derived category of quasi-coherent sheaves on X. Then there exists a functor $f^! : D(\mathrm{Spec}\, k) \to D(X)$ such that

$$\mathrm{Hom}_{D(\mathrm{Spec}\, k)}(\mathbf{R}f_*(\mathscr{F}), V) \cong \mathrm{Hom}_{D(X)}\big(\mathscr{F}, f^!(V)\big).$$

The functor $f^! : D(\mathrm{Spec}\, k) \to D(X)$ in the above notation is the <u>exceptional inverse image functor</u>, and it arises from the functor $V \mapsto V \otimes \omega$ which we saw in the previous isomorphism. Put in other words, Serre Duality is nothing but the existence of an adjunction $\mathbf{R}f_* \dashv f^!$.

The idea that led to Grothendieck Duality is now the simple observation that suddenly, both sides of the equation still make sense if we replace the morphism $X \to \mathrm{Spec}\, k$, on which

many restrictions rested, by much more general morphisms of schemes $X \to Y$. Grothendieck Duality, in its general form, essentially tells us what the functor $f^!$ should be like in more general situations. Unfortunately, this description is a lot more complicated than what is described here. In fact in certain general cases the functor $f^!$, which lives in the derived world, cannot even be translated back into a functor in the non-derived world. In the next section we will look at what the functor looks like in the case we have a flat projective local complete intersection between locally Noetherian sheaves. We will see that we need not worry about derived formalisms in this case.

7   Grothendieck Duality II — Formal Statement

Let $\pi : X \to Y$ be a separated and quasi-compact morphism between (locally Noetherian) schemes. Let $\mathscr{F}$ and $\mathscr{G}$ be quasi-coherent sheaves on X. For any affine open subset V of Y, each homomorphism $\phi : \mathscr{F}|_{\pi^{-1}(V)} \to \mathscr{G}|_{\pi^{-1}(V)}$ induces a homomorphism $H^r\big(\pi^{-1}(V), \mathscr{F}|_{\pi^{-1}(V)}\big) \to H^r\big(\pi^{-1}(V), \mathscr{G}|_{\pi^{-1}(V)}\big)$ on the level of cohomology. This gives rise to a canonical bilinear map

$$\pi_* \mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \times R^r \pi_* \mathscr{F} \to R^r \pi_* \mathscr{G},$$

where $R^r \pi_*$ is the r-th right derived functor of the pushforward functor π_*. In fact, Liu practically takes this fact as the definition of $R^r \pi_*$:

Proposition 5.2.28. Let $\pi : X \to Y$ be a separated and quasi-compact morphism of schemes. Let $\mathscr{F}$ be a quasi-coherent sheaf on X. For every $r \geq 0$ there exists a unique quasi-coherent sheaf $R^r \pi_* \mathscr{F}$ on Y such that for every affine open subset V of Y, we have $R^r \pi_* \mathscr{F}(V) = H^r\big(\pi^{-1}(V), \mathscr{F}|_{\pi^{-1}(V)}\big)$.

Definition 6.4.18. The canonical bilinear map induces in turn a homomorphism

$$\pi_* \mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \to \mathcal{H}om_{\mathscr{O}_Y}(R^r f_* \mathscr{F}, R^r f_* \mathscr{G}).$$

We define the _r-dualizing sheaf_ of π to be a quasi-coherent sheaf ω on X, together with a homomorphism of $\mathscr{O}_Y$-modules $\mathrm{tr} : R^r \pi_* \omega \to \mathscr{O}_Y$ (called the _trace map_), such that for any quasi-coherent sheaf $\mathscr{F}$ on X, the natural bilinear map $\pi_* \mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \omega) \times R^r \pi_* \mathscr{F} \to R^r \pi_* \omega \xrightarrow{\mathrm{tr}} \mathscr{O}_Y$ induces an isomorphism

$$\pi_* \mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \omega) \xrightarrow{\sim} \mathcal{H}om_{\mathscr{O}_Y}(R^r \pi_* \mathscr{F}, \mathscr{O}_Y).$$

Example. In Exercise 6.4.8, we treat a very special case: if $f : \mathrm{Spec}\, A \to \mathrm{Spec}\, B$ is a morphism of integral scehemes, such that the ring map $f^{\#} : B \to A$ is a finite extension of integral domains, and such that the extension $L \mid K$ ($\mathrm{Frac}(A) = L$ and $\mathrm{Frac}(B) = K$) is a finite separable extension, then the r-dualizing sheaf for $r = 0$ is given by the quasi-coherent sheaf $\widetilde{W}$ on $\mathrm{Spec}\, A$, where W is the _codifferent_ $\{x \in L : \mathrm{Tr}_{L|K}(xA) \subseteq B\}$.

Theorem 6.4.32 (Grothendieck Duality). Let $\pi : X \to Y$ be a flat projective local complete intersection of relative dimension r — that is, such that $\dim_x X_{\pi(x)} = r$ for all x in X. Then the r-dualizing sheaf ω exists and is unique, and it is isomorphic to the canonical sheaf $\omega_{X/Y}$.

In particular, if π happens to be a smooth morphism, then $\omega_{X/Y}$ is, essentially by definition, just $\det \Omega^1_{X/Y}$, and so $\omega \cong \det \Omega^1_{X/Y} = \Omega^r_{X/Y}$. This means that even in the smooth case, the theorem gives more information than the original Serre Duality, as we now have an explicit description of the dualizing sheaf in terms of the sheaf of differentials.

8 Some exercises

6.3.1. Let $(A, \mathfrak{m})$ be a Noetherian local ring, and let $(a_1, \ldots, a_n)$ be a regular sequence of $\mathfrak{m}$. Then $\left(a_{\sigma(1)}, \ldots, a_{\sigma(n)}\right)$ is again a regular sequence for any permutation σ.

Solution: Since the symmetric group S_n is generated by all transpositions, it suffices to check that the result holds for transpositions. In other words, if we have a sequence of the form $(a_1, \ldots, a_i, a_{i+1}, \ldots, a_n)$ which is regular, it suffices to check that $(a_1, \ldots, a_{i+1}, a_i, \ldots, a_n)$ is again regular. It suffices to verify that $(a_{i+1}, a_i, a_{i+2}, \ldots, a_n)$ is regular in $A/(a_1, \ldots, a_{i-1})$, since the first part of the sequence is already known to be regular. Similarly, $(a_{i+2}, \ldots, a_n)$ is already known to be regular on $A/(a_1, \ldots, a_{i+1})$ by assumption, so all we need to do is show that (a_{i+1}, a_i) is regular on $A/(a_1, \ldots, a_{i-1})$. To summarize, we have thus far reduced the exercise to the following:

Claim. Let R be a Noetherian ring, and let a, b be two elements of R such that a is not a zero-divisor of R, and b is not a zero-divisor of $R/(a)$. Then b is not a zero-divisor on R, and a is not a zero-divisor on $R/(b)$.

Proof of claim: Start with the first statement. Suppose that $bx = 0$ for some $x \in R$. Since b is a non-zero-divisor of $R/(a)$, we have $x \in (a)$, say $x = ax_1$, hence $bax_1 = 0$. Now a is a non-zero-divisor of R hence $bx_1 = 0$. Rinse and repeat. $x = ax_1$, $x_1 = ax_2$, and so on. It follows that $x \in \mathfrak{m}^i$ for each $i \geq 1$, where $\mathfrak{m}$ is the unique maximal ideal of R. By the Krull Intersection Theorem, x must be 0, as desired.

Now for the second statement. Suppose to the contrary that a is a zero-divisor on $R/(b)$. Write $ax = bx'$ for some $x, x' \in R$ with $x \notin (b)$. Since b is assumed to be a non-zero-divisor on $R/(a)$, the fact that $bx' \in (a)$ implies that $x' \in (a)$, say $x' = ax''$, so that we have $ax = bax''$. By assumption, a is a non-zero-divisor of R, so we have $x = bx''$, hence $x \in (b)$, which is a contradiction. $\qquad\qquad\square$

Remark. Vakil's *Foundations of Algebraic Geometry* proves the same result in Theorem 8.4.6 in a different way, by an unnecessary but attractive application of spectral sequences.

6.3.2. (c) Let the notation be as in Exercise 6.3.1. Let T be a variable, and B the localization of $A[T]$ at the maximal ideal $(\mathfrak{m}, T)$. If an ideal J of A is such that $(J, T) \subseteq B$ is generated by a regular sequence, then J is generated by a regular sequence in A. Show the same result in more than one variable.

Remark. I will only explain how this result is used in the proof of Corollary 6.3.22. We had an immersion $i : X \to Z$ which decomposes as a regular immersion $\pi : X \to W$, followed by a smooth map $p_Z : W \to Z$. Smooth maps in turn locally decompose into a regular immersion $U \to \mathbb{A}_Z^n$ followed by the usual projection $\mathbb{A}_Z^n \to Z$ (with U an open subscheme of W). Thus we can locally decompose i into a regular immersion $i : X \to \mathbb{A}_Z^n$ followed by the projection $\mathbb{A}_Z^n \to Z$. Locally, we get the following situation:

$$X \supseteq \operatorname{Spec} B \xrightarrow{\text{regular}} \operatorname{Spec} A[x_1, \ldots, x_n] \longrightarrow \operatorname{Spec} A \subseteq Z$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$\operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n] \longrightarrow \operatorname{Spec} \mathbb{Z}.$$

We want to show the composition $\operatorname{Spec} B \to \operatorname{Spec} A$ is also regular. Pick a point $[\mathfrak{p}] \in \operatorname{Spec} B$ and their images $[\mathfrak{q}] \in \operatorname{Spec} A$ and $[(\mathfrak{q}, f_1, \ldots, f_k)] \in \operatorname{Spec} A[x_1, \ldots, x_n]$. We have maps of Noetherian local rings $A_\mathfrak{q} \to A[x_1, \ldots, x_n]_{(\mathfrak{q}, f_1, \ldots, f_k)} \cong A_\mathfrak{q}[x_1, \ldots, x_n]_{(f_1, \ldots, f_k)} \to B_\mathfrak{q}$. Since $\operatorname{Spec} B \to \operatorname{Spec} A[x_1, \ldots, x_n]$ is regular, it means that we can find an ideal I of $A_\mathfrak{q}[x_1, \ldots, x_n]_{(f_1, \ldots, f_k)}$ generated by a regular sequence such that $B = A_\mathfrak{q}[x_1, \ldots, x_n]_{(f_1, \ldots, f_k)}/I$. We now apply Exercise 6.3.2 (c) (in the case of more than one variable) to this situation to conclude that $I \cap A_\mathfrak{q}$ is generated by a regular sequence in $A_\mathfrak{q}$, as desired. $\qquad\square$

6.3.3. Let $f : X \to Y$ and $g : Y \to Z$ be immersions of locally Noetherian schemes. If f and $g \circ f$ are regular immersions, then so is g. Give an example where $g \circ f$ and g are regular immersions, without f being one.

Solution: We start with the counterexample. Our intuition comes in handy here. Let Z be affine two-space, Y the union of the x- and the y-axis, and X just the x-axis. Clearly, X does not lie in Y as a codimension-one subspace, yet both X and Y lie in Z as a codimension-one subspace. Algebraically, this gives us the morphisms $k[x, y] \to k[x, y]/(xy)$ and $k[x, y]/(xy) \to k[x, y]/(x)$.

We now treat the first part. Due to time constraints, I did not end up completing it — what follows is a suggestion. Since the condition is local, we may consider the following algebraic statement. Let A, B, and C be local Noetherian rings, say with maximal ideals $\mathfrak{m}_A, \mathfrak{m}_B, \mathfrak{m}_C$, and let $g : C \to B$ and $f : B \to A$ be local ring homomorphisms such that $f \circ g$ and f are surjective morphisms obtained from a regular sequence in C and in B, respectively. We want to show g is obtained in the same way. First note that A, B, C have the same residue fields, because the induced maps $B/\mathfrak{m}_B \to A/\mathfrak{m}_A$ and $C/\mathfrak{m}_C \to A/\mathfrak{m}_A$ are surjective maps of fields. Next note that g and f induce surjective maps of k-vector spaces $\mathfrak{m}_C/\mathfrak{m}_C^2 \xrightarrow{g_k} \mathfrak{m}_B/\mathfrak{m}_B^2 \xrightarrow{f_k} \mathfrak{m}_A/\mathfrak{m}_A^2$. Give $\mathfrak{m}_C/\mathfrak{m}_C^2$ a basis $(b_1, \ldots, b_n)$, $\mathfrak{m}_B/\mathfrak{m}_B^2$ a basis $(g_k(b_1), \ldots, g_k(b_m))$, and $\mathfrak{m}_A/\mathfrak{m}_A^2$ a

basis $(f_k g_k(b_1), \ldots, f_k g_k(b_\ell))$, with $\ell \leq m \leq n$. Choose representing elements $(c_1, \ldots, c_n) \in \mathfrak{m}_C$, $(g(c_1), \ldots, g(c_m)) \in \mathfrak{m}_B$, and $(fg(c_1), \ldots, fg(c_\ell)) \in \mathfrak{m}_A$. I expect that the map $C \to A$ is then obtained by modding out the sequence $(c_{\ell+1}, \ldots, c_n)$, and that the sequence $B \to A$ is obtained by modding out the sequence $(g(c_{\ell+1}), \ldots, g(c_m))$. It would then hopefully follow that the map $C \to B$ is obtained by modding out the sequence $(c_{m+1}, \ldots, c_n)$, which is a regular sequence, essentially by construction.

Remark. It is also true that the composition of two regular immersions is again regular. This is stated in Proposition 6.3.11 (a), but not proved, as Liu deems it too trivial a result to bother with. Here's a proof. Let $f : X \to Y$ and $g : Y \to Z$ be regular immersions of locally Noetherian schemes. Pick a point $x \in X$. Then we know that $\mathscr{O}_{Z,gf(x)} \to \mathscr{O}_{Y,f(x)}$ and $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ are both obtained from regular sequences — say, the former is obtained from a sequence $(c_1, \ldots, c_r)$ in $\mathscr{O}_{Z,gf(x)}$, and the latter from $(b_{r+1}, \ldots, b_{r+s})$ in $\mathscr{O}_{Y,f(x)}$. Choose pre-images c_i of b_i (which exist because the morphisms are surjective). Then the composition $\mathscr{O}_{Z,gf(x)} \to \mathscr{O}_{X,x}$ is obtained from modding out the ideal generated by the sequence $(c_1, \ldots, c_{r+s})$. It remains to be shown that this sequence is regular. Clearly for $i \leq r$, c_i is a non-zero-divisor in $\mathscr{O}_{Z,gf(x)}/(c_1, \ldots, c_{i-1})$. Similarly if $r > i$ then c_i is a non-zero-divisor in $\mathscr{O}_{Y,f(x)}/(c_{r+1}, \ldots, c_i)$. But since $\mathscr{O}_{Y,f(x)}$ is essentially just $\mathscr{O}_{Z,gf(x)}/(c_1, \ldots, c_r)$, it follows that c_i is a non-zero-divisor in $\mathscr{O}_{Z,gf(x)}/(c_1, \ldots, c_{i-1})$, as desired. $\qquad\square$

(A variant of) 6.4.8. Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes such that the corresponding map of rings $f^\# : B \to A$ is a finite extension of integral domains, with fraction fields $L = \operatorname{Frac}(A)$ and $K = \operatorname{Frac}(B)$, such that that $L \mid K$ is a finite separable extension. The codifferent is the A-module $W_{B/A} = \{x \in L : \operatorname{Tr}_{L|K}(xA) \subseteq B\}$, where $\operatorname{Tr}_{L|K}$ is the field trace.[1] On $\operatorname{Spec} A$, we may define the quasi-coherent sheaf $\widetilde{W}_{B/A}$, which we claim is the r-dualizing sheaf for $r = 0$.

Proof: We first need a trace map $R^0 f_* \widetilde{W}_{B/A} = f_* \widetilde{W}_{B/A} \to \widetilde{B}$, in other words, we need a map of B-modules $W_{B/A} \to B$. There's an obvious choice here — just take the field trace map $\operatorname{Tr}_{L|K}$. (I presume this is where the name 'trace map' comes from.) Now take any quasi-coherent sheaf $\widetilde{N}$ on $\operatorname{Spec} A$; that is, take any A-module N. The natural bilinear map we're interested in is now given by

$$\operatorname{Hom}_A(N, W_{A/B}) \times N \to W_{A/B} \xrightarrow{\operatorname{Tr}_{L|K}} B,$$

which induces a B-module morphism

$$\operatorname{Hom}_A(N, W_{A/B}) \to \operatorname{Hom}_B(N, B).$$

By definition of the dualizing sheaf, we want this map to be an isomorphism. We start with injectivity. Suppose $f \in \operatorname{Hom}_A(N, W_{A/B})$ is a map such that its image in $\operatorname{Hom}_{N/B}$ is the zero

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[1] Let $L \mid K$ be a finite field extension. Then L may be viewed as a vector space over K. multiplication by an element $\alpha \in L$ is a K-linear transformation, which may be represented by a matrix. The trace $\operatorname{Tr}_{L|K}(\alpha)$ is then the trace of this matrix.

map; that is, suppose that $\text{Tr}_{L|K}(f(x)) = 0$ for all $x \in N$. Then we show that f is the zero map. Fix some $x \in N$, and let a be an element of A. Since f is A-linear, we have

$$\text{Tr}_{L|K}(f(ax)) = \text{Tr}_{L|K}(af(x)) = 0 \quad \text{for all} \quad a \in A.$$

Since $L \mid K$ is separable, the trace form is non-degenerate (see, for instance, Proposition 1.2.8 in Jürgen Neukirch's *Algebraic Number Theory*), hence $f(x) = 0$. Since x was an arbitrary element of N, if follows that $f = 0$, as desired. As for surjectivity, let $g : N \to B$ be a B-module homomorphism, then just note that we can write g as $g = \text{Tr}_{L|K} \circ f^{\#} \circ g$; the map $f^{\#} \circ g$ is then the desired A-module homomorphism in the pre-image. $\qquad \square$